# PATHOLOGY ON GAME TREES: A SUMMARY OF RESULTS*

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## ABSTRACT

Game trees are widely used as models of various decision-making situations. Empirical results with game-playing computer programs have led to the general belief that searching deeper on a game tree improves the quality of a decision. The surprising result of the research summarized in this paper is that there is an infinite class of game trees for which increasing the search depth does not improve the decision quality, but instead makes the decision more and more random.

## I INTRODUCTION

Many decision-making processes are naturally modeled as perfect information games between two players [3, 7]. Such games are generally represented as trees whose paths represent various courses the game might take. In artificial intelligence, the well-known minimax procedure [2, 7] is generally used to choose moves on such trees.

If a correct decision is to be guaranteed using minimaxing, substantial portions of the game tree must be searched, even when using tree-pruning techniques such as alpha-beta [2, 7]. This is physically impossible for large game trees. However, good results have been obtained by searching the tree to some limited depth, estimating the minimax values of the nodes at that depth using a heuristic evaluation function, and computing the minimax values for shallower nodes as if the estimated values were correct [2, 7]. There is almost universal agreement that when this is done, increasing the search depth increases the quality of the decision. This has been dramatically illustrated with game-playing computer programs [1, 8, 9], but such results are purely empirical.

The author has developed a mathematical theory modeling the effects of search depth on the probability of making a correct decision. This research has produced the surprising result that there is an infinite class of game trees for which as long as the search does not reach the end of the tree (in which case the best possible decision could be guaranteed), deeper search does not improve the decision quality, but instead makes the decision more and more random. For example,

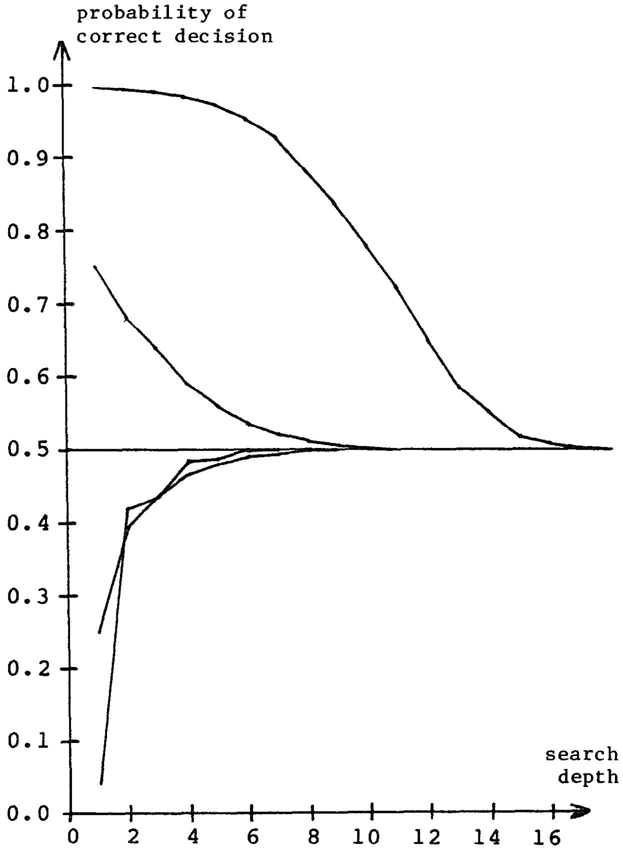


FIGURE 1.--Probability of correct decision as a function of search depth on the game tree G(1,1), for five different evaluation functions. On G(1,1), the probability of correct decision is 0.5 if the choice is made at random. For each of the five functions, this value is approached as the search depth increases.

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Figure 1 illustrates how the probability of correct decision varies.

Section 2 of this paper summarizes the mathematical model used in this research, Section 3 presents the main result, and Section 4 contains concluding remarks.

## II THE MATHEMATICAL MODEL

Let G be a game tree for a game between two players named Max and Min. Nodes where it is Max's or Min's move are called max and min nodes, respectively. Assume that G has no draws (this restriction can easily be removed, but it simplifies the mathematics). Then if G is finite, every node of G is a forced win for either Max or Min. Such nodes are called "+" nodes and "-" nodes, respectively. If G is infinite, not every node need be a "+" or "-" node, but the "+" and "-" labeling can easily be extended to all nodes of G in a way which is consistent with all finite truncations of G.

Correct decisions for Max and Min are moves leading to "+" and "-" nodes, respectively. "+" max nodes (which we call S nodes) may have both "+" and "-" children; "+" min nodes (T nodes) have only "+" children; "-" min nodes (U nodes) may have both "+" and "-" children; and "-" max nodes (V nodes) have only "-" children. Thus it is only at the S and U nodes that it makes a difference what decision is made. These nodes are called <u>critical</u> nodes.

An evaluation function on G may be any mapping e from the nodes of G into a set of numbers indicating how good the positions are estimated to be. For computer implementation, the range of e must be finite. We take this finite set to be $\{0,1,\ldots,r\}$, where r is an integer.

Ideally, e(g) would equal r if g were a "+" node and 0 if g were a "-" node, but evaluation functions are usually somewhat (and sometimes drastically) in error. Increasing the error means decreasing e(g) if g is a "+" node and increasing e(g) if g is a "-" node. Thus if we assume that the errors made by e are independent and identically distributed, the p.d.f. f for the values e returns on "+" nodes is a mirror image of the p.d.f. h for the values e returns on "-" nodes; i.e., $f(x) = h(r-x)$, $x = 0,1,\ldots,r$. f may be represented by the vector $P = (f(0),f(1),\ldots,f(r))$, which is called the <u>probability vector for e</u>.

## III RESULTS

The probability vector for e induces probability vectors on the minimax values of the nodes of G, and the probability of making a correct decision at any critical node g of G is a function of the probability vectors for the minimax values of the children of g. This probability is thus determined by the structure of the subtree rooted at g, and little can be said about it in general. However, if G has a sufficiently regular structure, the properties of this probability can be analyzed.

Let m and n be positive integers, and let G(m,n) be the unique game tree for which

1. the root is an S node (this choice is arbitrary and the results to follow are independent of it);
2. each critical node has m children of the same sign and n children of opposite sign;

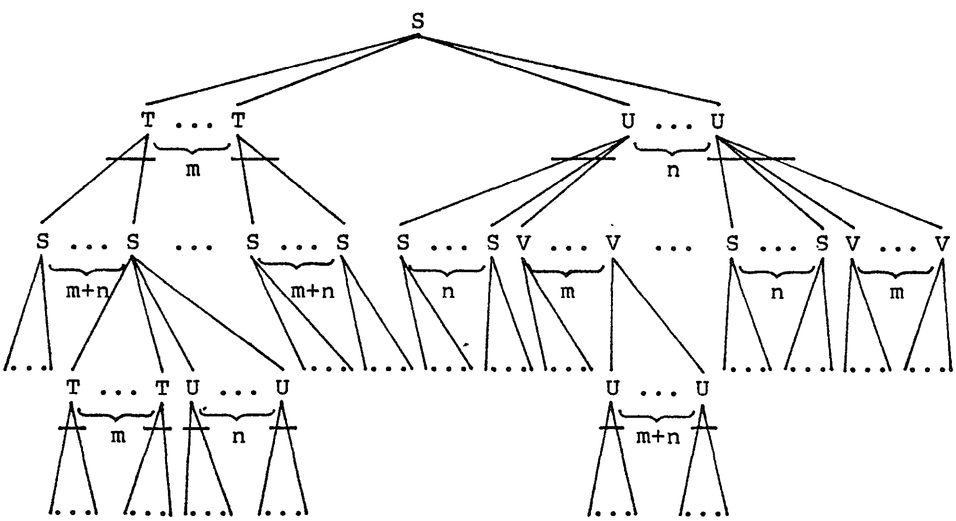


FIGURE 2.--The game tree G(m,n). Min nodes are indicated by the horizontal line segments drawn beneath them.

3. every node has m+n children.
G(m,n) is illustrated in Figure 2.

If moves are chosen at random on G(m,n), the probability that a correct choice is made at a critical node is obviously m/(m+n). If the choice is made using a depth d minimax search and an evaluation function with probability vector P, it is proved [4, 5] that the probability that the decision is correct depends only on m, n, P, and d. We denote this probability by $D_{m,n}(P,d)$. The trees G(m,n) have the following surprising property.

Theorem 1. For almost every* probability vector P and for all but finitely many values of m and n,

$$\lim_{d \to \infty} D_{m,n}(P,d) = m/(m+n).$$

Thus, as the search depth increases, the probability of correct decision converges to what it would be if moves were being chosen at random. This pathological behavior occurs because as the search depth increases it becomes increasingly likely that all children of a critical node receive the same minimax value, whence a choice must be made at random among them.

Figure 1 illustrates Theorem 1 on the game tree G(1,1), using five different values of P. The significance of Theorem 1 for finite games is that infinitely many finite games can be generated by truncating G(m,n) in whatever way desired. Deeper search on these trees will yield increasingly random decisions as long as the search does not reach the end of the tree.

Additional theoretical and experimental results reported elsewhere [4, 5, 6] provide additional information about which of the G(m,n) are pathological and why. Theorem 1 almost certainly extends to a much larger class of game trees, but the irregular structure of most game trees would require a much more complicated proof.

## IV CONCLUSIONS

The author believes that the pathology of the trees G(m,n) indicates an underlying pathological tendency present in most game trees. However, in most games this tendency appears to be overridden by other factors. Pathology does not appear to occur in games such as chess or checkers [1, 8, 9], but it is no longer possible blithely to assume (as has been done in the past) that searching deeper will always result in a better decision.

* A property holds for almost every member of a set if it holds everywhere but on a subset of measure zero. Thus for any continuous p.d.f. on the set, the probability of choosing a member of the set to which the property does not apply is 0.

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