# FINGERPRINTS THEOREMS

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*Abstract.* We prove that the scale map of the zero-crossings of almost all signals filtered by a gaussian of variable size determines the signal uniquely, up to a constant scaling. Exceptions are signals that are antisymmetric about all their zeros (for instance infinitely periodic gratings). Our proof provides a method for reconstructing almost all signals from knowledge of how the zero-crossing contours of the signal, filtered by a gaussian filter, change with the size of the filter. The proof assumes that the filtered signal can be represented as a polynomial of finite, albeit possibly very high, order. The result applies to zero- and level-crossings of signals filtered by gaussian filters. The theorem is extended to two dimensions, that is to images. These results imply that extrema (for instance of derivatives) at different scales are a complete representation of a signal.

## 1. Introduction

Images are often described in terms of "edges", that are usually associated with the zeros of some differential operator. For instance, zero-crossings in images convolved with the laplacian of a gaussian have been extensively used as the basis representation for later processes such as stereopsis and motion (Marr, 1982). In a similar way, sophisticated processing of 1-D signals requires that a symbolic description must first be obtained, in terms of *changes* in the signal. These descriptions must be concise and, at the same time, they must capture the meaningful information contained in the signal. It is clearly important, therefore, to characterize in which sense the information in an image or a signal is captured by extrema of derivatives.

Ideally, one would like to establish a unique correspondence between the changes of intensity in the image and the physical surfaces and edges which generate them through the imaging process. This goal is extremely difficult to achieve in general, although it remains one of the primary objectives of a comprehensive theory of early visual processing.

A more restricted class of results, that does not exploit the constraints dictated by the signal or image generation process, has been suggested by work on zero-crossings of images filtered with the laplacian of a gaussian. Logan (1977) had shown that the zero-crossings of a 1-D signal ideally bandpass with a bandwidth of less than an octave determine uniquely the filtered signal (up to scaling). The theorem has been extended—only in the special case of oriented bandpass filters—to 2-D images (Poggio, et al., 1982; Marr, et al., 1979) but it cannot be used for gaussian filtered signals or images, since they are not ideally bandpass. Nevertheless, Marr et al. (1979) conjectured that the zero-crossings maps, obtained by filtering the image with the second derivative of gaussians of variable size, are very rich in information about the signal itself (see also Marr and Poggio, 1977; Grimson, 1981; Marr and Hildreth, 1980; Marr, 1982; for multiscale representations see also Crowley, 1982 and Rosenfeld, 1982 also for more references).

More recently, Witkin (1983) (see also Stansfield, 1980) introduced a scale-space description of zero-crossings, which gives the position of the zero-crossing across a continuum of scales, i.e., sizes of the gaussian filter (parameterized by the $\sigma$ of the gaussian). The signal—or the result of applying to the signal a linear (differential) operator—is convolved with a gaussian filter over a continuum of sizes of the filter. Zero- or level- crossings of the (filtered) signal are contours on the $x-\sigma$ plane (and surfaces in the $x, y, \sigma$ space). The appearance of the *scale map* of the zero-crossing is suggestive of a *fingerprint*. Witkin has proposed that this concise map can be effectively used to obtain a rich and qualitative description of the signal. Furthermore, it has been proved in 1-D (Babaud et al, 1983; Yuille and Poggio, 1983a) and 2-D (Yuille and Poggio, 1983a) (J. Koenderink, pers. comm., 1984) has now obtained similar results exploiting properties of the diffusion equation.) that the gaussian filter is the only filter with a "nice" scaling behavior, i.e., a simple behavior of zero-crossing across scales, with several attractive properties for further processing. In this paper, we prove a stronger *completeness* property: the map of the zero-crossing across scales determines the signal uniquely for almost all signals (in the absence of noise). The scale maps obtained by gaussian filters are true *fingerprints* of the signal. Our proof is constructive. It shows how the original signal can be reconstructed by information from the zero-crossing contours across scales. It is important to emphasize that our result applies to level-crossings of any arbitrary linear (differential) operator of the gaussian, since it applies to functions that obey the diffusion equation. These results were originally reported in Yuille and Poggio (1983b). The proof is constructive and applies in both 1-D and 2-D. Reconstruction of the signal is of course not the goal of early signal processing. Symbolic primitives must be extracted from the signals and used for later processing. Our results imply that scale-space fingerprints are complete primitives, that capture the whole information in the signal and characterize it uniquely. Subsequent processes can therefore work on this more compact representation instead of the original signal.

Our results have theoretical interest in that they answer the question as to what information is conveyed by the zero- and level-crossings of multiscale gaussian filtered signals. From a point of view of applications, the results in themselves do not justify the use of the fingerprint representation. *Completeness* of a representation (connected with Nishihara's *sensitivity*) is not sufficient (Nishihara, 1981). A good representation must,

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in addition, be *robust* (i.e. *stable* in Nishihara's terms) against photometric and geometric distortions (the general point of view argument). It should also possibly be *compact* for the given class of signals. Most importantly it should make *explicit* the information that is required by later processes.

## 2. Assumptions and results

We consider the zero-crossings of a signal $I(x)$, space-scale filtered with the second derivative of a gaussian, as a function of $x$, $\sigma$. Let $E$ be defined by

$$E(x,\sigma) = I * G$$

$$E(x,\sigma) = I(x) * [G(x,\sigma)] = \int I(\varsigma)\frac{1}{\sigma}\frac{d^2}{dx^2}exp^{-\frac{(x-\varsigma)^2}{2\sigma^2}}d\varsigma. \qquad [2.1]$$

Notice that $E(x,\sigma)$ obeys the diffusion equation in $x$ and $\sigma$:

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{\sigma}\frac{\partial E}{\partial \sigma}. \qquad [2.2]$$

We restrict ourselves to images, or signals, P such that $E$ can be expressed as a finite Taylor series of arbitrarily high order and such that $E$ is not antisymmetric about all its zeros. Observe that any filtered image can be approximated arbitrarily well in this way, because of the classical Weierstrass approximation theorem, except for those functions antisymmetric about all their zeros. This class of functions is discussed in detail in a forthcoming paper (Yuille and Poggio, 1984a) where it is shown that additional information about the gradient of the function on the zero-crossings is sufficient to determine the signal. Note that, for a finite order polynomial, functions antisymmetric about all their zeros only have one zero-crossing contour.

We will show that the local behavior of the zero-crossing curves (defined by $E(x,\sigma) = 0$) on the $x - \sigma$ plane determines the image. Our reconstruction scheme provides the image $I$ in terms of Hermite polynomials. The proof of this result can be generalized to 2-D and extended to zero- and level-crossings of linear (differential) operators. More precisely we have proven the following theorem:

**Theorem 1**: *The derivatives (including the zero-order derivative) of the zero-crossings contours defined by $E(x,\sigma) = 0$, at two distinct points at the same scale, uniquely determine a signal of class P up to a constant scaling (except on a set of measure zero).*

Note that the theorem does not apply to signals that do not have at least two distinct zero-crossings contours. Yuille and Poggio (1983b) have extended Theorem 1 to the two dimensional case:

**Theorem 2**: *Derivatives of the zero-crossings contours, defined by $E(x,y,\sigma) = 0$, at two distinct points at the same scale, uniquely determine an image of class P up to a scaling factor (except on a set of measure zero).*

These theorems break down when all the zero-crossing contours are independent of scale (i.e. the contours go straight up in the scale-space fingerprint). This is a rare, though interesting, special case and is discussed in detail in a future paper (Yuille and Poggio 1984a). It can only occur for functions which are antisymmetric about all their zeros, such as sinusoidal functions, and for odd polynomials with only one real zero.

The theorems do not directly address the *stability* of this reconstruction scheme. The first question concerns stability of the reconstruction of the *filtered* function $E(x,\sigma)$ at the $\sigma$ where the derivatives are taken. Note that our result relies only on two points on the zero-crossing contours. Exploitation of the whole zero-crossings contours should make the reconstruction considerably more robust. The second question is about the stability of the recovery of the unfiltered signal $I(x)$ from $E(x,\sigma)$. This is equivalent to inverting the diffusion equation, which is numerically unstable since it is a classically ill-posed problem. Reconstruction is, however, possible with an error depending on the signal to noise behavior (see Yuille and Poggio, 1983b).

### 2.1. Outline of the 1-D Proof

We summarize here the 1-D proof from a slightly different point of view that clarifies its bare structure.

The proof starts by taking derivatives along the zero crossing contours at a certain point. Such derivatives split into combinations of $x$ and $t$ derivatives (where $t = \sigma^2/2$). Because the filter is assumed to be gaussian, however, derivatives can be expressed in terms of $x$ derivatives. This is a key point: since the filtered signal $E(x,t)$ satisfies the diffusion equation, the $t$ derivatives can be expressed in terms of the $x$ derivatives simply by $E_t = E_{xx}$. The next stage is to find the $x$ derivatives of $E(x,t)$ up to an arbitrary degree $n$ from such derivatives along the zero crossing contours in the $x - t$ plane. We show that this can be done by using 2 points on 2 contours. (It is possible that one point is sufficient, but we are as yet unable to prove this.) Since $E(x,t)$ is entire analytic, because of the gaussian filtering, it can be represented by a Taylor series expansion in $x$. Since we know the values of the $n$ derivatives of $E(x,t)$ with respect to $x$, we know its Taylor series expansion and hence $E(x,t)$. The unfiltered signal $I(x)$, ($E(x,t) = I(x) * G(x,t)$) can then be recovered in the ideal noiseless case by deblurring the gaussian. A particularly simple way of doing this is provided by a property of the function $\phi_n$ in which we will expand the function $F$: the coefficients of an expansion of $I(x)$ in terms of $\phi_n$ are equal to the coefficients of the Taylor series expansion of $E(x,t)$. In the presence of noise, the recovery of $I(x)$ from $E(x,t)$ is obviously unstable, since it is a classically ill-posed problem. It is limited by $S/N$ ratio since high spatial frequencies in the signal are masked by the noise for increasing $t$. (For instance, if $F(x) = \sum a_\mu e^{i\mu x}$, the filtered signal is $E(x,t) = \sum a_\mu e^{i\mu x}e^{-\mu^2 t}$.) Note that since the zero-crossing contours are available at all scales a reconstruction scheme that exploits more than 2 points will be significantly more robust. As one would expect, the reconstruction of the unfiltered signal is therefore affected by noise. The reconstruction of the filtered signal $E(x,t)$ is likely to be considerably more robust. We plan to study theoretically and with computer simulations the noise sensitivity of the reconstruction scheme.

## 3. Proof of the Theorem in 1-D

Our proof can be divided into three main steps. The first shows that derivatives at a point on a zero-crossing contour put strong constraints on the "moments" of the Fourier transform of $E(x,\sigma)$ (see eq. 3.1.4). The second relates the "moments" to the coefficients of the expansion of $I(x) = E(x,0)$ in functions related to the Hermite polynomials. Finally the "moments" can be uniquely determined by the derivatives on a second point of a different zero-crossing contour. We outline here only the first part of the proof, which is given in full in Yuille and Poggio (1983b).

### 3.1. The "moments" of the signal are constrained by the zero-crossing contours

Let the Fourier transform of the signal $I(x)$ be $\tilde{I}(\omega)$ and the gaussian filter be $G(x,\sigma) = \frac{1}{\sigma}e^{\frac{-x^2}{2\sigma^2}}$ with Fourier transform $\tilde{G}(\omega) = e^{\frac{-\sigma^2\omega^2}{2}}$.

The zero crossings are given by solutions of $E(x,t) = 0$. Using the convolution theorem we can express $E(x,t)$ as

$$E(x,t) = \int e^{-\omega^2 t}e^{i\omega x}\tilde{I}(\omega)d\omega. \qquad [3.1.1]$$

and $t = \sigma^2/2$. The Implicit Function theorem gives curves $x(t)$ which are $C^\infty$ (this is a property of the gaussian filter and of the diffusion equation, see Yuille and Poggio, 1983a,b). Let $\varsigma$ be a parameter of the zero crossing curve. Then

$$\frac{d}{d\varsigma} = \frac{dx}{d\varsigma}\frac{\partial}{\partial x} + \frac{dt}{d\varsigma}\frac{\partial}{\partial t}. \qquad [3.1.2]$$

On the zero-crossing surface, $E = 0$ and $\frac{d^n}{d\varsigma^n}E = 0$ for all integers $n$. Knowledge of the zero crossing curve is equivalent to knowledge of all the derivatives of $x$ and $t$ with respect to $\varsigma$.

We compute the derivatives of $E$ with respect to $\varsigma$ at $(x_o, t_o)$. The first derivative is :

$$\begin{aligned}\frac{d}{d\varsigma}E(x,t) = &\frac{dx}{d\varsigma}\int e^{-\omega^2 t}e^{i\omega x}(i\omega)\tilde{I}(\omega)d\omega \\ &+ \frac{dt}{d\varsigma}\int e^{-\omega^2 t}(-\omega^2)e^{i\omega x}\tilde{I}(\omega)d\omega\end{aligned} \qquad [3.1.3]$$

and is expressed in terms of the first and second moments of the function $e^{-\omega^2 t}e^{i\omega x}I(\omega)$. The moment of order n is defined by:

$$M_n = \int_{-\infty}^{\infty}(i\omega)^n e^{-\omega^2 t}e^{i\omega x}\tilde{I}(\omega)d\omega. \qquad [3.1.4]$$

The second derivative is

$$\begin{aligned}\frac{d^2}{d\varsigma^2}E(x,t) = &\frac{d^2x}{d\varsigma^2}\int e^{-\omega^2 t}e^{i\omega x}(i\omega)\tilde{I}(\omega)d\omega \\ &+ \frac{d^2t}{d\varsigma^2}\int e^{-\omega^2 t}(-\omega^2)e^{i\omega x}\tilde{I}(\omega)d\omega \\ &+ \left(\frac{dx}{d\varsigma}\right)^2\int e^{-\omega^2 t}e^{i\omega x}(-\omega^2)\tilde{I}(\omega)d\omega \\ &+ 2\frac{dx}{d\varsigma}\frac{dt}{d\varsigma}\int e^{-\omega^2 t}(-\omega^2)e^{i\omega x}(i\omega)\tilde{I}(\omega)d\omega \\ &+ \left(\frac{dt}{d\varsigma}\right)^2\int e^{-\omega^2 t}(\omega^4)e^{-i\omega x}\tilde{I}(\omega)d\omega.\end{aligned} \qquad [3.1.5]$$

Since the parametric derivatives along the zero crossing curve are zero, equation [3.1.3] is a homogeneous linear equation in the first two moments. Similarly, [3.1.5] is a homogeneous linear equation in the first four moments. In general, the $n^{th}$ equation, $\frac{d^n}{d\varsigma^n}E(x,t) = 0$, is a homogeneous equation in the first $2n$ moments. We choose our axes such that $x_o = 0$. We can then show that the moments of $e^{-\omega^2 t}I(\omega)$ are the coefficients $a_n$ in the expression of the function $I(x)$ in Hermite polynomials. So we have $n$ equations in the first $2n$ coefficients $a_n$. To determine the $a_n$ uniquely, we need $n$ additional and independent equations which can be provided by considering a neighboring zero crossing curve at $(x_1, t_o)$ (see Yuille and Poggio, 1983b).

## 4. Conclusions

We conclude with a brief discussion of a few issues that are raised by this paper and that will require further work.

a)*Stability of the reconstruction.* Although we have not yet rigorously addressed the question of numerical stability of the whole reconstruction scheme, there seem to be various ways for designing a robust reconstruction scheme. The first step to consider is the reconstruction of the *filtered* signal $E(x,t)$. One could exploit the derivatives at $n$ points – at the given $t$ – and then solve the resulting highly constrained linear equations with least squares methods. Alternatively, it may be possible to fit a smooth curve through several points on one contour, and then obtain the derivatives there in terms of this interpolated curve. The same process must be performed on a second separate zero-crossing contour. This scheme provides a rigorous way of proving that instead of derivatives at two points, the location of the whole zero-crossing contour across scales can be used directly to reconstruct the signal.

The second step involves the reconstruction of the unfiltered signal $I(x)$. This reconstruction step is unstable if only one scale is used, but it can be regularized and effectively performed in most situations, especially by using information from zero-crossings at smaller scales.

b) *Degenerate fingerprints.* Our uniqueness result applies to *almost* all signal: a restricted but well known class of signals, with vertical zero-crossings in the scale-space diagram, correspond to nonunique fingerprints. These signals, which will be discussed in a forthcoming paper (Yuille and Poggio, 1984a), and which correspond to functions antisymmetric about all their zeros, do not belong to the class P introduced in Theorem 1 and 2. Interestingly, elements of this class can be distinguished by level-crossing (with a level different from zero) or by knowledge of the gradient (Yuille and Poggio, 1984a).

c) *Extensions.* Our main results apply to zero- and level-crossings of a signal filtered by a gaussian filter of variable size. They also apply to transformations of a signal under a linear space-invariant operator – in particular they apply to the linear derivatives of a signal and to linear combinations of them. In both 1-D and 2-D, local information at just two points is sufficient. In practice, since many derivatives are needed at each point, information about the whole contour, to which the point belongs, is in fact exploited.

d) *Are the fingerprints redundant?* The proof of our theorem implies that two points on the fingerprint contours are sufficient. As we mentioned earlier, several points are probably required to make the reconstruction robust and to ensure the avoidance of a non-generic pair of points. We conjecture, however, that the fingerprints are redundant and that appropriate constraints derived from the process underlying signal generation (the imaging process in the case of images) should be used to characterize how to collapse the fingerprints into more compact representations. Witkin (1983) has already made this point and discussed various heuristic ways to achieve this goal.

e) *Implications of the results.* As we discussed in the introduction, our results imply that the fingerprint representation is a *complete* representation of a signal or an image. Zero- and level-crossings across scales of a filtered signal capture full information about it. These results also suggest a central role for the gaussian in multiscale filtering that assure that zero- and level-crossing indeed contain full information. Note, however, that the fingerprint theorems do not constrain or characterize in any way the differential filter that has to be used. The filter may be just the identity operator, provided of course that enough zero-crossings contours exist. Independent arguments, based on the constraints of the signal formation process, must be exploited to characterize a suitable filter for each class of signals. For images, second derivative operators such as the Laplacian are suggested by work that takes into account the physical properties of objects and of the imaging process (Grimson, 1983; Torre and Poggio, 1984; Yuille, 1983). We plan to explore this approach in the near future.

e) *Zero-crossings and slopes.* A natural question to ask is whether gradient information across scales at the zero-crossings, in addition to their location, can be used to reconstruct the original. Hummel (1984, pers. comm.) has recently shown that this is the case, as one would of course expect in the light of our results (Yuille and Poggio, 1983b; Yuille & Poggio, 1984a). We have been able to simplify and extend the elegant proof by Hummel and obtain the following result (Yuille and Poggio, 1984b): *knowledge of zero-crossing surfaces and magnitude of the $x - t$ gradient over a finite, nonzero interval of the zero-crossing surface is sufficient to determine the image.*

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