Constraint-Theorems on the Prototypification of Shape

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ABSTRACT

Mathematical results are presented that strongly constrain the prototypification of complex shape. Such shape requires *local* prototypification in two senses: (1) prototypification occurs *in parallel* at different parts of the figure, and (2) prototypification varies *differentially* (smoothly) across an individual part. With respect to (1), we present a theorem that states that every Hoffman-Richards codon has a unique Brady Smooth Local Symmetry. The theorem solves the issue of defining units for parallel decomposition, for it implies that a codon is the *minimal* unit with respect to the existence of prototypification via symmetry, and is *maximal* with respect to prototypification via *non-ambiguous* symmetry. Concerning issue (2) above, a further theorem is offered that severely limits the possible shapes that result from the *sequential* application of prototypifying operations to smoothly varying deformation. This second result explains why considerably fewer prototype classes exist than one would otherwise expect.

I. INTRODUCTION

It has usually been assumed that, in human cognition, the prototypification of an object (e.g. a shape) occurs in a single step (e.g. Rosch, 1978). However, in a number of papers (Leyton, 1984, 1985, 1986a, 1986b, 1986c, 1986d), I have argued that prototypification is decomposed into a sequence of well-defined psychologically-manageable stages; that is, an object is assigned a backward history of successively greater stages of prototypification.

In the present paper, theorems are offered that allow us to extend this decompositional analysis to the prototypification of *complex* shape. The prototypification of such shape requires stages that are *local,* in two senses: (1) prototypification occurs in parallel, at different regions of the figure; and (2) prototypification removes deformation that differentially (smoothly) varies over an individual region. Our theorems constrain local prototypification in these two senses; that is: (1) they help to establish an optimal decomposition with respect to parallel prototypification, and (2) they strongly constrain the possible results of the *sequential* decomposition of differential prototypification.

Let us, however, first give an example of sequential prototypification for *simple* shape, in order to identify more clearly what needs to be extended to handle complex shape. In a converging set of experiments (e.g. Leyton, 1984, 1985, 1986a, 1986b, 1986c, 1986d) I found that, when subjects are presented with a rotated parallelogram, Fig 1a, they reference it to a non-rotated one, Fig 1b, which they then reference to a rectangle, Fig 1c, which they then reference to a square, Fig 1d.

We can characterize these results in the following way. Let us assume that the relationship between the ultimate prototype, the square, and the first figure, the rotated parallelogram, is given by a linear transformation, T. Any non-singular linear transformation (without reflection) can be represented as a product of three more primitive linear transformations, thus: $T =$ *Stretch* $\times$ *Shear* $\times$ *Rotation*. This decomposition of a linear transformation will be called its *Iwasawa* decomposition. The three transformations, comprising the decomposition, characterize the three stages in Fig 1. That is, working from right to left, Fig 1d $\rightarrow$ Fig 1c is a stretch, Fig 1c $\rightarrow$ Fig 1b is a shear, and Fig 1b $\rightarrow$ Fig 1a is a rotation. Therefore, the experimental results indicate that this decomposition is psychologically salient and follows in a specific order.

For later usage, in this paper, it is worth having a matrix representation of the decomposition, thus:

$$\text{Stretch} \times \text{Shear} \times \text{Rotation} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Observe that (1) in the first matrix, the eigenvalues λ_i represent the amounts of stretch along the directions of the eigenvectors of that matrix; (2) in the middle matrix, μ is the amount of movement in the x-direction of the y-basis vector; and (3) in the last matrix, θ is the extent of rotation. This decomposition is reminiscent of the Gram-Schmidt orthogonalization (Hoffman & Kunze, 1961), where a set of linearly independent vectors (i.e. a non-singular matrix) is transformed into an orthonormal set (i.e. a rotation matrix) by first shearing the set and then stretching it. Although the order is different in the Gram-Schmidt process, this does not ultimately matter, because the subgroup of stretches group-theoretically normalizes the subgroup of shears (Lang, 1975), and therefore an equivalent representation can be found with the required ordering.

The purpose of the present paper is to extend the above simple use of the Iwasawa decomposition to analyze the prototypification of complex shape. The two main problems for

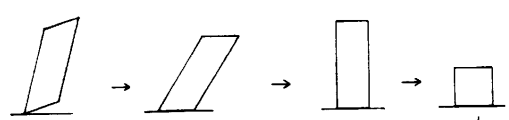

Figure 1. One of the successive reference phenomena discovered in Leyton (1984, 1985, 1986a).

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such an investigation is that (1) the decomposition, as used so far, has been applied *globally,* and (2) the decomposition is that of *linear transformations.* For example, consider the seal shown in Fig 2 (third row, second column). To prototypify this shape, one would want to apply different operations to different regions of the figure (the head, the back, etc.). Again, given an individual region, the simple use of a linear transformation would not usually have the desired effect; e.g. a linear transformation would not straighten the arched back. Thus, the problem is that, with complex shape, one requires local prototypification in *two* senses: prototypification (1) applies to the subparts and (2) varies differentially. The purpose of this paper is to present a set of mathematical results that yield solutions to these two problems.

II. HOW TRANSFORMATIONS ACT ON PROTOTYPE STRUCTURE

It can be assumed that the prototypicality ranking of a shape corresponds to the latter's degree of symmetry. However, although this introduces into consideration the crucial factor of the symmetries of the shape, a much tighter relationship between the shape symmetries and the deforming transformations has been proposed in Leyton (1984), and has been corroborated using a considerable number of empirical studies in several areas of perceptual organization (Leyton, 1984, 1985, 1986a, 1986b, 1986c, 1986d). The relationship is summarized in:

INTERACTION LAW (Leyton, 1984): *The symmetry axes of the prototype are interpreted as eigenspaces of the most allowable transformations.*

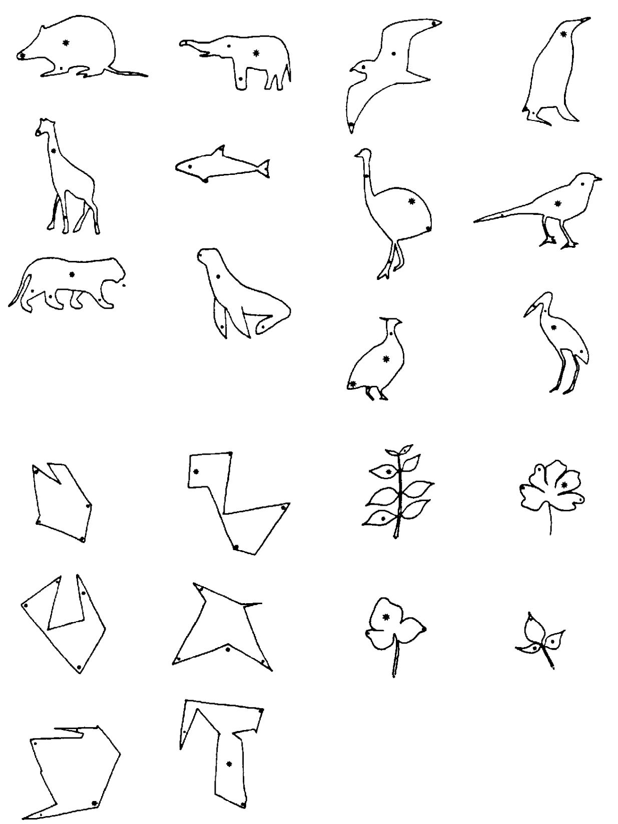

Figure 2. The twenty-two complex natural and abstract shapes in which human subjects converted local symmetry axes into local eigenvectors.

An eigenspace is a linear subspace (e.g. a line through the origin) that maps to itself under a linear transformation. Visually, an eigenspace-line or eigenvector is interpreted as a direction of flexibility. As an illustration of the Interaction Law, observe in Fig 1 that the salient symmetry axes of the square (i.e. the side-bisectors) become the eigenspace-lines in the transition of the square (Fig 1d) to the rectangle (Fig 1c).

Now let us examine the validity of the Interaction Law with respect to complex shape. In order to investigate the local prototypification of complex shape, I gave human subjects, under experimentally controlled testing conditions, the twenty-two outlines of complex natural and abstract shapes shown in Fig 2. The subjects were asked to give, at each of four points in each shape, the direction of perceived maximal flexibility of the region local to the point. The results were that the subjects chose a local symmetry axis at each point. Thus they converted local symmetry axes into local eigenspaces. (The statistical significance was considerable: $n = 12$, 88 choices per subject; expected mean = 44; actual mean = 77.58; $p < 0.0005$, one tailed).

These results therefore lead us to the conclusion that the Interaction Law is valid in complex shape, and that it applies locally. The usefulness of the conclusion is that it gives us an indication as to the nature of local prototypification - for one can assume that prototypification occurs along lines of maximal flexibility.

However, the Interaction Law (i.e. symmetry axes are converted into eigenspaces) requires, as input, a symmetry analysis; and since we are using the Interaction Law locally, what we require is a *local* symmetry analysis.

The symmetry analysis we shall use is the Smoothed Local Symmetry (SLS) of Brady (1983). It can be regarded as a natural means of describing the local structure of a contour, because it is yielded by the set of reflectional symmetries between tangent vectors. For example, the bold curved line, in Fig 3, shows a segment of contour. Points A and B are paired because their tangent vectors, $\mathbf{t}_A$ and $\mathbf{t}_B$, are symmetric about some vector $\mathbf{t}$. The dotted line, which is the locus of the mid-points P of the chords AB, is taken to be the symmetry axis. Let us now investigate the relationship between subparts and the SLS, as follows.

III. THE SYMMETRY STRUCTURE OF PARTS

We begin by investigating the relationship between the Brady SLS and the part-analysis provided by Hoffman & Richards (1985) and Richards & Hoffman (1985). These latter researchers have put forward compelling evidence that contours are perceptually partitioned at points of negative curvature extrema; i.e. points of maximal "indentation". For example, if one were to partition the contour of a face at such points, the resulting segments would be the chin, the lips and the nose. In

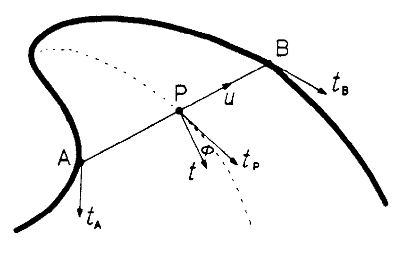

Figure 3. The Brady SLS

fact, Hoffman's and Richards' basic primitive is a segment whose endpoints are curvature minima - and they call such a segment, a *codon*. Thus, they define a *codon representation* of a contour to be the sequence of codons obtained by traversing the contour. Codon representations have two important advantages: (1) Any contour has a unique representation as a codon string; and (2) there are only five types of non-trivial codon, where a *type* corresponds to a unique sequence of singularities. Fig 4 shows the five types. The dots along the codon represent the curvature singularities (minima, maxima, and zeros) that define the particular codon-type.

The question we now ask is whether codons are related to the Interaction Law and therefore to the problem of local prototypification. A theorem, which I proposed and proved in Leyton (1986e), is crucial in answering this question.

SYMMETRY-CURVATURE DUALITY THEOREM **(Leyton, 1986e):** *Any segment of smooth planar curve, bounded by two consecutive curvature extrema of the same type (either both maxima or both minima), has a unique SLS symmetry axis, and the axis terminates at the curvature extremum of the opposite type (respectively, minimum or maximum).*

COROLLARY: *The SLS of a codon is unique, and terminates at the point of maximal curvature on the codon.*

It should be observed that the above theorem relates two previously unrelated branches of perceptual research: (1) Symmetry research, starting with the Gestalt movement and going up to modern AI symmetry extraction programs; and (2) Curvature research, starting with Attneave's (1954) work on information maximization at extrema and going up to, for example, a recent formalization of Attneave's results by Resnikoff (1987).

It is also worth observing that the theorem defines what is a *minimal* local region to consider with respect to symmetry, in the following sense:

Observe that any codon is itself built from a number of examples of only one primitive subpart. Each subpart is a *spiral*. (A spiral is a curve with monotonically changing curvature of the same sign.) In Fig 4, any curve-segment, bounded by two adjacent dots (singularities) is a spiral. Thus, each codon is a sequence of two, three or four spirals. Any smooth curve can therefore be represented as a string of spirals. We shall call this representation the *s-code* of the curve.

The importance of basing one's representation on spirals arises from a theorem I proved in Leyton (1986e), which states that an SLS cannot be constructed on a spiral. (We are assuming that the curve's normals cannot change sides). That is, an SLS cannot be constructed on a single unit of the s-code. Furthermore, it is easy to show that an SLS cannot be constructed on any adjacent pair of units in the s-code where the pair both have increasing or both have decreasing curvature. Thus, to allow a symmetry axis to appear, the s-code needs to contain two consecutive spirals where one spiral has increasing and the other has decreasing curvature. But, any such pair is either a codon, or part of a codon that must exist in the curve at that point. Thus, the appearance of symmetry requires that, minimally, the curve must contain a codon.

Figure 4. The five non-trivial codons.

The second thing to observe is that the codons are *maximal* local regions with respect to symmetry uniqueness. That is, as soon as one continues a curve past either end-point minimum of a codon, an extra symmetry axis must appear terminating at that minimum.

The uniqueness of the symmetry axis *within* the codon follows from the fact that, at any pair of SLS points A and B, as in Fig 3, a unique circle can be drawn that is tangential to the curve at A and B. It is shown in Leyton (1986e) that: (1) given any point A on a codon, there is at most one circle that is tangential to A and some other point B on the codon, and (2) this circle is not tangential to a third point on the codon. This result proves the uniqueness of the symmetry-point associated with A, and hence the uniqueness of the symmetry axis within the codon.

The above considerations therefore reveal that there are a set of properties (uniqueness, maximality, minimality, etc) that make the relationship between smooth local symmetries and codons significant.

IV. DIFFERENTIAL PROTOTYPIFICATION

Having seen how the symmetry analysis interacts with the local structure, where local means *subpart*, we shall now look at how the symmetry analysis interacts with the local structure, where local means *differential*. We require a differential characterization of the SLS such that prototypification via the Iwasawa decomposition (Stretch $\times$ Shear $\times$ Rotation) becomes both possible and meaningful. It turns out that the latter requirements strongly constrain the type of characterization that is allowable, as follows:

It is clear that, at any point P along the SLS-axis, two vectors characterize the SLS structure: (1) $\frac{h}{2}\mathbf{u}$ where $\mathbf{u}$ is the unit cross-section vector based at P (as shown in Fig 3) and $\frac{h}{2}$ is the scalar measuring half the cross-section, (2) $\mathbf{t}_P$ which is the unit tangent to the SLS axis. This pair of local vectors defines a local frame, $\mathbf{F}$, which varies as $\mathbf{F}$ moves along the curved axis. What we need to do is to characterize $\mathbf{F}$ as the consequence of transformations, T, on some other frame, $\mathbf{E}$, such that when T is factorized (that is $\mathbf{F}$ becomes $\mathbf{E}$), the resultant shapes are regarded as psychologically more prototypical. Thus we have to decide how to choose $\mathbf{E}$.

Two candidates for $\mathbf{E}$ seem obvious: (1) $\mathbf{E}_1$, which has, as basis vectors, the tangent $\mathbf{t}_P$ to the symmetry line and the normal $\mathbf{n}_P$ to that line; and (2) $\mathbf{E}_2$ which has, as basis vectors, the unit vector $\mathbf{t}$ (about which $\mathbf{t}_A$ and $\mathbf{t}_B$ are symmetrical) and the unit vector $\mathbf{u}$ which is normal to $\mathbf{t}$ and lies along the cross-section. The linear transformations $\mathbf{E}_1 \rightarrow \mathbf{F}$ and $\mathbf{E}_2 \rightarrow \mathbf{F}$ each comprise stretch and shear. Furthermore, when $\mathbf{F}$ propagates along the axis, it undergoes rotation. Observe that, if $\mathbf{E}_1$ is used, rotation is conveniently described as rotation of the axis-tangent $\mathbf{t}_P$, whereas if $\mathbf{E}_2$ is used then rotation is conveniently described as rotation of the cross-section.

In order to see how important the choice of basis $\mathbf{E}_1$ or $\mathbf{E}_2$ is, let us consider what happens when the frames $\mathbf{E}_i$ undergo *no*

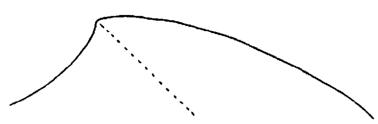

Figure 5. A local frame that does not accord with the Interaction Law can lead to psychologically meaningless results when the frame is prototypified.

rotation. For example, consider again the contour shown in Fig 3. It has a curved symmetry axis. Thus, when the basis $\mathbf{E}_1$ is propagated along the axis, it undergoes rotation. Now let us prototypify by removing rotation. The resultant shape is shown in Fig 5. However, observe that, even though the axis is straight, the shape itself (e.g. as given by the contour) does not seem significantly more prototypical.

It seems therefore that $\mathbf{E}_1$ is an inappropriate basis. Thus let us reject it, as a basis, and investigate what happens when $\mathbf{E}_2$ is the chosen basis.

However, before we do this, it is important to observe that there is a good reason why we could have suspected, in advance, that $\mathbf{E}_1$ would be a bad choice, and why we might believe that $\mathbf{E}_2$ will be more successful. Recall the Interaction Law. It states that, perceptually, symmetry axes provide an appropriate basis of eigenvectors for actions on shape. Observe also that the symmetry axes here are $\mathbf{t}$ and $\mathbf{u}$ ($\mathbf{t}$ is such in the plane of the page, and $\mathbf{u}$ is such in the plane of the cross-section of the implied three-dimensional shape; see Leyton, 1985, 1986c, 1986f, for details). Now, returning to the definition of $\mathbf{E}_1 \rightarrow \mathbf{F}$ and $\mathbf{E}_2 \rightarrow \mathbf{F}$, one finds that it is only in the latter case that the initial basis (i.e. $\mathbf{E}_2$) is a basis of eigenvectors (for example, the cross-section vector, $\mathbf{u}$, is an eigenvector of stretch).

Let us now compute the full linear transformation associated with the basis, $\mathbf{E}_2 = (\mathbf{t}, \mathbf{u})$. Let ϕ be the angle between $\mathbf{t}$ and $\mathbf{t}_P$, and θ be the angle between basis $\mathbf{E}_2 = (\mathbf{t}, \mathbf{u})$ and corresponding fixed basis $(\mathbf{t}_1, \mathbf{u}_1)$ at the beginning of the SLS-axis; i.e. at the beginning of the protrusion. Then the matrix describing the linear transformation $(\mathbf{t}_1, \mathbf{u}_1) \rightarrow (\frac{h}{2}\mathbf{u}, \mathbf{t}_P)$, is given by:

$$\begin{pmatrix} |\mathbf{t}_P|\cos\phi\cos\theta & -|\mathbf{t}_P|\cos\phi\sin\theta \\ |\mathbf{t}_P|\sin\phi\cos\theta+\frac{h}{2}\sin\theta & -|\mathbf{t}_P|\sin\phi\sin\theta+\frac{h}{2}\cos\theta \end{pmatrix}$$

Furthermore, crucially, we can now compute the Iwasawa decomposition. It is

$$\begin{pmatrix} |\mathbf{t}_P|\cos\phi & 0 \\ 0 & \frac{h}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{2|\mathbf{t}_P|\sin\phi}{h} & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Now let us investigate whether the factorization of these transformations, from the shape, results in psychologically salient prototypes.

Consider Fig 6. The top node represents an arbitrary shape characterized by the Iwasawa decomposition. Prototypification occurs by removing the factors; i.e. by progressing downward in the tree. The middle row of nodes of the tree represents the first level of prototypification and the bottom row represents the second level of prototypification. The tree shows all possible factorizations.

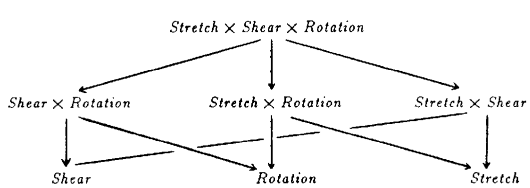

Figure 6. All possible prototypifications via factorizations of the Iwasawa decomposition of a linear transformation.

Observe now that when the transformations are used *globally,* each node of the figure represents a mathematically realizable shape. For example, starting with a rotated parallelogram at the top node, the middle level yields, from left to right, a rotated rhombus, a rotated rectangle, a parallelogram; and the bottom level yields a rhombus, a rotated square and a rectangle.

The question which concerns us is what shapes are obtained at the nodes when one uses the transformations *locally.* A theorem which I proposed and proved in Leyton (1986f) is crucial in answering this question:

THEOREM (Leyton, 1986f): *Let a shape be locally characterized by the Iwasawa decomposition defined by the coordinate system of eigenvectors that are the symmetry vectors of the SLS (i.e. in accord with the Interaction Law). Then the removal of one of the factor subgroups necessarily involves the removal of one of the other factor subgroups.*

That is, the theorem states that one level of prototypification is mathematically impossible; i.e. a shape with only rotation and shear is impossible, a shape with only shear and stretch is impossible, and a shape with only rotation and stretch is impossible.

Having established that there are mathematically no realizable shapes at the middle level of the tree in Fig 6, let us move on to the bottom level. It is easy to show that no shapes are mathematically possible at the bottom left node. This leaves shapes at only the Rotation and Stretch nodes, on that row. The shapes corresponding to these nodes are, respectively, (1) the *flexed* symmetries such as the worm in Fig 7, and (2) the *global* symmetries such as the goblet in Fig 7. Thus the conclusion is that although the Iwasawa decomposition, using basis $\mathbf{E}_2 = (\mathbf{t}, \mathbf{u})$, disallows one level of prototypification, the prototypes that it does produce are, psychologically, highly salient as prototypes. This contrasts with the use of basis $\mathbf{E}_1$, which allows shapes at other nodes of the hierarchy (e.g. Fig 5 is at node Stretch×Shear), but where the shapes are not significantly prototypical. Thus again we have here a corroboration of the Interaction Law, because the basis $\mathbf{E}_2$ accords with that law, whereas the basis $\mathbf{E}_1$ does not.

V. PROTOTYPIFICATION CONSTRAINTS

In conclusion therefore, the above theorems provide strong differential-geometric constraints on the decomposition of prototypification, as follows:

First, recall that the Symmetry-Curvature Duality Theorem can be regarded as yielding a minimality constraint: the appearance of symmetry implies that the contour must contain at least a codon; e.g. the contour cannot consist only of a spiral, or of two consecutive spirals separated by a zero curvature point, etc. Thus, if we regard prototypes as shapes that have global symmetry structure, and deformed prototypes as shapes that have a local symmetry structure that is the image of the former global structure under deformation, then the minimality condition implies that prototypification cannot take place if one does not have codons.

Figure 7. A flexed symmetry (the worm) and a global symmetry (the goblet).

Recall also that the Duality Theorem can be regarded as yielding the following maximality constraint: a codon is the maximal possible region possessing a unique symmetry axis. This constraint provides two natural means by which prototypification can be decomposed: (1) The codon, being the maximal unit for a unique axis, corresponds to a prototypification *stage,* in the sense that the removal of the codon (e.g. a protrusion), by shrinking, removes the curvature extremum *and* the axis at the same time (by the Duality Theorem). Thus the contour is made more uniform by a factor of one exactly extremum and exactly one axis. It is important to observe that if the Duality Theorem did not constrain the symmetry axis of a codon to terminate at the extremum, the removal of the axis by shrinking would not necessarily remove the extremum and thus yield a more prototypical contour. (2) The codon, being the maximal contour unit for a unique axis, corresponds to a unit that can be manipulated in parallel with other such units. Thus, the Duality Theorem implies a prototypification-decomposition that is both serial and parallel. (We should note that Pizer, Oliver & Bloomberg (1986) have implemented an algorithm that hierarchically orders protrusions in an SAT-based analysis, and obtains psychologically natural results.)

Finally, the theorem of the last section strongly constrains the further decomposition of the prototypification units just defined. It states that, when this decomposition is characterized by the Iwasawa decomposition, prototypification with respect to one of the Iwasawa factors necessarily involves prototypification with respect to one of the others. Thus the theorem tells us why there are only a few shape prototypes; i.e. those that are global symmetries or flexed symmetries.

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