A Logic for Hypothetical Reasoning

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Abstract

This paper shows that classical logic is inappropriate for hypothetical reasoning and develops an alternative logic for this purpose. The paper focuses on a form of hypothetical reasoning which appears computationally tractable. Specifically, Horn-clause logic is augmented with rules, called *embedded implications*, which can hypothetically add atomic formulas to a rulebase. By introducing the notion of *rulebase independence*, it is shown that these rules can express hypothetical queries which classical logic cannot. By adopting methods from modal logic, these rules are then shown to be intuitionistic. In particular, they form a subset of intuitionistic logic having semantic properties similar to those of Horn-clause logic.

1 Introduction

Several researchers in the logic-programming community have pointed out the utility of augmenting Prolog with the ability to hypothetically add facts to a rulebase. Miller, for instance, has shown how such rules can structure the runtime environment of a logic program [Miller, 1986]. Warren and Manchanda have also proposed such logics for reasoning about database updates [Warren, 1984; Manchanda, 1988]. The legal domain, in particular, has inspired much work into this kind of hypothetical reasoning. Gabbay, for example, has reported a need to augment Prolog with hypothetical rules in order to encode the British Nationality Act. The act contains rules such as, "You are eligible for citizenship if your father would be eligible if he were still alive" [Gabbay and Reyle, 1984]. Also, McCarty has developed a wide class of hypothetical rules for the purpose of constructing computer-based legal consultation systems, especially systems for reasoning about contract law and corporate tax law [McCarty, 1988a; McCarty, 1981].

Although hypothetical reasoning is in general complex [Ginsberg, 1986], these systems focus on a form of hypothetical reasoning which appears computationally tractable. In particular, they augment Horn-clause logic with rules called *embedded implications*. These are rules of the form $A \leftarrow (B \leftarrow C)$, which means, "$A$ is true if adding C to the rulebase causes B to be true."

The formal properties of embedded implications are still being explored. Gabbay, for instance, has shown that hypothetical embedded-implications have an intuitionistic semantics [Gabbay, 1985], and Miller has developed an intuitionistic fixpoint semantics for the predicate case [Miller, 1986]. McCarty has extended this work to a larger class of formulas and established interesting semantic results [McCarty, 1988a]. Bonner has shown that query processing in such systems is PSPACE-complete in the function-free predicate case (EXPTIME-complete when hypothetical deletions are allowed) [Bonner, 1988a].

This paper continues this line of theoretical research in two ways. First, it formulates a precise sense in which classical logic is incapable of expressing hypothetical queries and rules. Specifically, queries are required to be *rulebase independent*; that is, a query should not have to be reformulated if the rulebase changes. An example is then given of a simple, hypothetical query which cannot be expressed in classical logic in a rulebase-independent way.

Second, this paper presents a new development of the intuitionistic semantics of embedded-implications. In particular, instead of developing fixpoint semantics, we apply techniques of modal logic to construct a *canonical model.* This provides a new perspective, and hopefully new insight, into the intuitionistic basis of hypothetical reasoning. It is shown, in particular, that hypothetical embedded-implications are a subset of intuitionistic logic with semantic properties similar to those of Horn clauses. Indeed, classical Horn-clauses are a special case of intuitionistic embedded-implications both proof-theoretically and semantically.

This paper is an overview of [Bonner, 1988b], to which the interested reader is referred for details and proofs.

2 Examples

This section gives examples of hypothetical queries and rules. They focus on a rulebase representing the policy and regulations of a university. For instance, the atomic formula $take(s, c)$ means that student s has taken course c, and $grad(s)$ means that s is eligible for graduation. The rulebase contains facts such as $take(tony, cs250)$, and rules such as

$$grad(s) \leftarrow take(s, cs250), take(s, his101)$$

The notation $R \vdash \psi$ means that query ψ is true when applied to rulebase R. For example, consider the query, "Retrieve those students who could graduate if they took (at most) one more course." This query can be formalized at the meta-level as follows:[1] "Retrieve those s such that

$$\exists c\ [R \cup \{take(s, c)\} \vdash grad(s)] \qquad (1)$$

In our logic of hypotheticals, this query is represented by the expression $\exists c\ [grad(s) \leftarrow take(s, c)]$. This is an object-

[1] See [Kowalski, 1979] for a description of meta-level and object-level reasoning.

level expression $\psi(s)$ such that $R \vdash \psi(s)$ iff condition (1) is satisfied.

Having introduced hypothetical queries, we can also use them in the premises of rules. Such rules turn our query language into a logic for building rulebases. For example, suppose the university wishes to enact the following rule:

> "If a student is within one course of graduation, and he is not eligible for primary aid, then he is eligible for secondary aid."

This hypothetical rule can be represented by the following two rules:[2]

$$aid2(s) \leftarrow \sim aid1(s),\ grad1(s).$$
$$grad1(s) \leftarrow \exists c\ [grad(s,c) \leftarrow take(s,c)]$$

Here, $aid1(s)$ and $aid2(s)$ mean that student s is eligible for primary and secondary aid, resp. $grad1(s)$ means that s is within one course of graduation.

3 Expressibility

This section defines a new notion of expressibility which applies to rulebase systems. It centers on the idea that queries should be independent of the rulebase. Using this definition, a hypothetical query is constructed which cannot be expressed in classical logic.

3.1 Rulebase Independence

The term "rulebase query" is a generalization of "database query" and refers to a question that is posed to a system containing a large number of rules as well as facts. In general, there are two aspects to such a query: (i) a question that the user has in mind, and (ii) an expression which he constructs to represent it. "Query formulation" is the task of constructing this expression.

This section argues that query formulation should be independent of the rulebase. In particular,

- A user should be able to construct an expression to represent a query without a detailed knowledge the rulebase.
- If the rulebase is updated, the expression representing a query should not have to change.

Similar constraints exist in database systems; e.g., a user should be able to formulate a database query without knowing the contents of the database.

Rulebase independence is important for a variety of reasons. Firstly, it permits casual use of the rulebase. That is, one does not have to be an expert in the contents of the rulebase in order to formulate queries. Secondly, it increases reliability. If a user's knowledge of the rulebase is inaccurate, then his queries may be formulated incorrectly. Thirdly, it makes changes less expensive. If queries are *not* rulebase independent, then changes in the rulebase can propagate throughout the user community. For example, utility programs which query the rulebase and libraries of commonly used queries would have to be updated whenever the rulebase is changed. Finally, the notion of rulebase independence is important not only to the formulation of queries but also to the construction of rulebases.

[2]The first rule uses negation-by-failure [Kowalski, 1979].

Indeed, the premise of a rule is nothing more than a rulebase query itself. Thus, keeping rule premises independent of the rest of the rulebase has all of the advantages just listed for queries: it makes them easier to formulate and more reliable, and it prevents small changes from propagating throughout the rulebase.

Rulebase independence is captured in the following definition:

> A rulebase query is expressible in a language if and only if it is possible to construct a single expression ψ which returns the correct answer for all rulebases R.

3.2 Classical Logic

Because of the properties of material implication, classical logic cannot represent some hypothetical queries in a rulebase-independent way. For example, consider the query: "If one of B_1 or B_2 were added to the rulebase, would C become true?" To represent this query, we need an expression ψ such that for all rulebases R,

$$R \models \psi \quad \textit{iff} \quad R \cup \{B_1\} \models C \quad \textit{or} \quad R \cup \{B_2\} \models C$$

The obvious candidate for ψ is the expression $(C \leftarrow B_1) \vee (C \leftarrow B_2)$. Indeed, in intuitionistic logic, this expression does represent the query in a rulebase independent way. In classical logic, however, material implication leads to the following equivalence:

$$\begin{aligned} \psi &\equiv (C \leftarrow B_1) \vee (C \leftarrow B_2) \\ &\equiv (C \vee \sim B_1) \vee (C \vee \sim B_2) \\ &\equiv C \vee \sim B_1 \vee \sim B_2 \\ &\equiv C \leftarrow B_1,\ B_2 \end{aligned}$$

Hence, $R \models \psi$ iff $R \cup \{B_1, B_2\} \models C$, by the deduction theorem. Classically, then, ψ does not represent the above query. Is there is some other expression which does? The following theorem (proven in [Bonner, 1988b]) assures us that there is not.

Theorem 1 *Classically, there is no expression ψ such that for any set R of propositional Horn clauses,*[3]

$$R \models \psi \quad \textit{iff} \quad R \cup \{B_1\} \models C \quad \textit{or} \quad R \cup \{B_2\} \models C$$

These ideas extend to the construction of rulebases. In particular, suppose that R is a rulebase not containing the atom A, and suppose we wish to add new rules R_o to R so that for all R,

$$R \cup R_o \vdash A \quad \textit{iff} \quad R \cup \{B_1\} \vdash C \quad \textit{or} \quad R \cup \{B_2\} \vdash C$$

Classically, there is no set of formulas R_o (Horn or otherwise) which satisfies this condition. For if R_o existed, then we could contradict theorem 1 using $\psi = A \leftarrow R_o$. As the next section shows, however, R_o can be constructed using hypothetical embedded-implications. In particular,

$$R_o = \{A \leftarrow (C \leftarrow B_1),\ \ A \leftarrow (C \leftarrow B_2)\}$$

[3]This theorem remains true if R is restricted to Horn clauses built from a given set $\mathcal{L}$ of atoms.

4 Hypothetical Inference

Because classical logic is inadequate for hypothetical inference, new inference mechanisms are needed, i.e., proof procedures for embedded implications. Such procedures have been developed by several researchers [Gabbay and Reyle, 1984; Miller, 1986; McCarty, 1988b], and this section defines a simplified version of them. This propositional version retains the essential properties of the more elaborate systems while admitting a clean theoretical analysis.

Definition 1 *A Horn rule is an expression of the form* $B \leftarrow B_1, B_2, ..., B_k$ *where* $k \geq 0$ *and B and each* B_i *are atomic.*

Definition 2 *An embedded implication is an expression of the form* $B \leftarrow \phi_1, \phi_2, ..., \phi_k$ *where* $k \geq 0$ *and each* ϕ_i *is a Horn rule.*

Note that Horn rules include atomic formulas as a special case, and embedded implications include Horn rules as a special case.

Definition 3 *Suppose R is a set of embedded implications. If B and* B_i *are atomic, then*

1. $R \vdash B$ *if* $B \in R$
2. $R \vdash B$ *if* $B \leftarrow \phi_1, ..., \phi_k$ *is a rule in R and* $R \vdash \phi_i$ *for each i.*
3. $R \vdash B \leftarrow B_1, ..., B_k$ *if* $R \cup \{B_1, ..., B_k\} \vdash B$

If R is a set of Horn rules, then this inference system is equivalent to classical Horn-clause logic. However, if R contains arbitrary embedded implications, then these inference rules do not have a classical semantics. That is, although they are clearly sound with respect to classical logic, they are not complete. To see this, consider the rulebase $\{A \leftarrow (B \leftarrow C),\ D \leftarrow A,\ D \leftarrow C\}$. Classically, D can be inferred from these three rules[4], but it is a simple exercise to see that D cannot be inferred using the above inference rules.[5]

5 Intuitionistic Logic

The rules of hypothetical inference defined above are non classical. Indeed, they were introduced precisely to overcome the shortcomings of classical logic described in section 3.2. The question thus arises as to the nature of their semantics. In fact, they form a subset of intuitionistic logic. This section provides a brief development of intuitionistic logic adapted from [Fitting, 1969] and [McCarty, 1988a].

Definition 4 *Suppose* $\mathcal{L}$ *is a finite or countably infinite set of propositional atoms. A substate is a subset of* $\mathcal{L}$*, and an intuitionistic structure is a set of substates. Furthermore, if* s_1 *and* s_2 *are substates, then* $s_1 \leq s_2$ *iff* $s_1 \subseteq s_2$*.*

Note that an intuitionistic structure is really a Kripke structure (M, R, π), where M is the set of substates, the access relation R is the subset relation, and the truth-assignment function π is given by

$$\pi(A) = \{s \mid s \in M \ \ and \ \ A \in s\}$$

Since R and π are trivial, we do not make them explicit.

Definition 5 *(Satisfaction) Suppose* ψ *is a formula, M is an intuitionistic structure, and s is a substate of M. Then* $s, M \models \psi$ *is read, "M satisfies* ψ *at s," and is defined recursively as follows:*

- *If A is atomic, then* $s, M \models A$ *iff* $A \in s$
- $s, M \models \psi_1 \wedge \psi_2$ *iff* $s, M \models \psi_1$ *and* $s, M \models \psi_2$
- $s, M \models \psi_1 \vee \psi_2$ *iff* $s, M \models \psi_1$ *or* $s, M \models \psi_2$
- $s, M \models \sim \psi$ *iff* $r, M \not\models \psi$ *for all* $r \geq s$ *in M*
- $s, M \models \psi_2 \leftarrow \psi_1$ *iff* $r, M \models \psi_1 \Rightarrow r, M \models \psi_2$ *for all* $r \geq s$ *in M*

Note that unlike classical logic, intuitionistic implication is not defined in terms of disjunction and negation. Rather, it has an independent semantic definition. This is why intuitionistic logic does not give rise to the problems mentioned in section 3.2.

Definition 6 *(Models)* $M \models \psi$ *iff* $s, M \models \psi$ *for all substates s of M. In this case we say, "M satisfies* ψ*," or "M is a model of* ψ*".*

Definition 7 *(Entailment) Suppose* ψ_1 *and* ψ_2 *are formulas. Then* $\psi_1 \models \psi_2$ *iff every model of* ψ_1 *is also a model of* ψ_2*.*

6 Semantics

Several researchers have developed fixpoint semantics for inference rules like those of section 4. Miller, for instance, has developed fixpoints semantics for such rules based on intuitionistic and minimal logic [Miller, 1986]. McCarty has considered a larger class of formulas involving negation and embedded universal quantifiers[6], developing an intuitionistic fixpoint semantics and establishing interesting semantic results [McCarty, 1988a]. Recently, Manchanda has considered hypothetical deletions as well as additions, developing a fixpoint semantics based on dynamic logic [Manchanda, 1988]. Indeed, the use of fixpoint semantics to establish completeness results has been common in logic programming since the seminal work of [Apt and Van Emden, 1982] and [Van Emden and Kowalski, 1976].

This section presents an alternative development based on completeness techniques used in modal logic. The aim is to add a new perspective, and hopefully gain new insight into the intuitionistic basis of hypothetical reasoning. To this end, we introduce an intuitionistic structure called the *canonical Kripke model.* This structure, defined proof-theoretically, provides the necessary link between inference and semantics. It also plays a central semantic role, analogous to that of the unique minimal model in Horn-clause logic.

Finally, we compare the intuitionistic semantics of embedded implications to the classical semantics of Horn clauses, outlining the close relationship that exists between them. In particular, some well-known properties of Horn clauses are shown to be a special case of properties of the canonical Kripke model.

[4]To see this, note that the two Horn rules are equivalent to $D \leftarrow (A \vee C)$, and the embedded implication is equivalent to $(A \vee C) \wedge (A \vee \sim B)$.

[5]because there are no rules for inferring B or C.

[6]A quantifier is *embedded* if it appears in the premise of a rule, as in $A(x) \leftarrow \forall y\ [B(x,y) \leftarrow C(x,y)]$

6.1 Soundness and Completeness

To show that the hypothetical inference rules of section 4 are intuitionistic, one must prove that they are sound and complete with respect to intuitionistic semantics. In particular, one must prove the following two theorems:

Theorem 2 *(Soundness) If R is a set of embedded implications and ϕ is a Horn rule, then if $R \vdash \phi$ then $R \models \phi$.*

Theorem 3 *(Completeness) If R is a set of embedded implications and ϕ is a Horn rule, then if $R \models \phi$ then $R \vdash \phi$.*

Soundness is straightforward and follows from modus ponens and the deduction theorem.

Proving completeness is more complex. The approach taken here is an adaptation of techniques used in modal logic and centers on the notion of a canonical model [Chellas, 1980]. In particular, given a set of embedded implications R,

1. Construct an intuitionistic structure M_R, called the *canonical Kripke model* of R.
2. Show that $M_R \models R$.
3. Show that if $M_R \models A$ then $R \vdash A$, for every atom A.

Thus, if $R \models A$, then A is true in all models of R. In particular, it is true in M_R, and so by point 3, $R \vdash A$. This establishes the completeness of *atomic* inference; i.e., $R \models A \Rightarrow R \vdash A$. By the deduction theorem, however, it follows that $R \models \phi \Rightarrow R \vdash \phi$ for any Horn-rule ϕ. This would prove theorem 3.

The central question, however, is how to construct the canonical Kripke model M_R. We define it to be the range of a proof-theoretic operator cl_R. This operator thus provides the essential link between hypothetical inference and intuitionistic semantics.

Definition 8 *If R is a set of embedded implications constructed from the atoms in $\mathcal{L}$, and s is a (possibly infinite) subset of $\mathcal{L}$, then $cl_R(s)$ is the atomic closure of R and s and is defined as follows:*

$$cl_R(s) = \{A \in \mathcal{L} \mid R \cup s \vdash A\}$$

Definition 9 *If R is a set of embedded implications constructed from the atoms in $\mathcal{L}$, then M_R is the canonical Kripke model of R and is defined as follows:*

$$M_R = \{cl_R(s) \mid s \subseteq \mathcal{L}\}$$

This definition establishes point 1 above. Points 2 and 3 follow from these definitions in a straightforward way (details may be found in [Bonner, 1988b]). Thus, the rules of hypothetical inference defined in section 4 are sound and complete with respect to intuitionistic semantics. In other words, this restricted form of hypothetical reasoning is intuitionistic reasoning.

6.2 Semantic Properties

Although the inference system of section 4 is intuitionistic, it is not equivalent to the full intuitionistic logic. Disjunctions, for instance, cannot be expressed. This section describes the semantic properties of this subset of intuitionistic logic. In particular, it is shown that hypothetical embedded-implications have properties similar to those of Horn clauses. These properties are sometimes sighted as the basis for the computational attractiveness of Prolog [Makowsky, 1986; McCarty, 1988a], suggesting that embedded implications may also be attractive as a logic programming language.

The first results are fundamental properties of the canonical Kripke model.

Theorem 4 *M_R has a unique minimal substate s_o. That is, if $s \in M_R$ then $s_o \leq s$.*

Theorem 5 *M_R is the unique maximal model of R. That is, if $M \models R$ then $M \subseteq M_R$*

Theorem 4 follows immediately from the definition of M_R by setting $s_o = cl_R(\{\})$. It can be generalized, however, to the ***substate intersection property***:[7]

Theorem 6 *The intersection of a collection of substates in M_R is itself a substate in M_R. That is, if $M \subseteq M_R$ then $\cap M \in M_R$.*

The existence of a unique maximal model, having the substate intersection property was first established by McCarty in [McCarty, 1988a]. Indeed, McCarty defines an intuitionistic structure called K^* which he shows is the unique maximal model of a set of embedded implications. Theorem 5 assures us that McCarty's K^* is identical to our M_R, at least in the propositional case. Conceptually, however, the main difference between them is that K^* is defined semantically whereas M_R is defined proof-theoretically.

Theorems 4 and 5 have implications for inference. In particular, from 4 it follows that the canonical Kripke model contains all the information necessary to perform hypothetical inference. That is,

Corollary 1 *If ϕ is a Horn rule, then*

$$R \vdash \phi \text{ iff } M_R \models \phi$$

This offers a semantic interpretation of negation-as-failure: ϕ cannot be inferred from R iff ϕ is not true in the maximal model of R. Note the similarity of this to the semantics of failure in Horn-clause logic: an atom cannot be inferred from a set of Horn clauses iff it is not true in the minimal model. In this sense, the canonical Kripke model is an intuitionistic analogue of the unique minimal model of Horn-clause logic.

This analogy can be taken one step further. Because of the unique minimal model, Horn clauses have the attractive property that they entail a disjunction of atoms iff they entail one of the atoms individually. Similarly, as the next corollary shows, a set of embedded implications entails a disjunction of Horn rules iff it entails one of the Horn rules individually.

Corollary 2 *If $\phi_1, ..., \phi_k$ are Horn rules, then $R \models \phi_1 \vee \cdots \vee \phi_k$ iff $R \models \phi_i$ for some i.*

This corollary means that a theorem prover for intuitionistic embedded-implications does not need extensive modification to deal with disjunctive goals. Indeed, the disjuncts are non-interacting and a theorem prover can work on each one separately. This suggests adding the following rule of hypothetical inference to those of section 4:

[7] In particular, the intersection of all substates of M_R is the unique minimal substate.

- *For any collection of Horn rules $\phi_1, ..., \phi_k$, $R \vdash \phi_1 \vee \cdots \vee \phi_k$ if $R \vdash \phi_i$ for some i.*

6.3 Relationship to Horn Logic

The intuitionistic semantics of embedded implications appears to have little relation to the classical semantics of Horn clauses. This would be surprising since Horn rules are a special case of embedded implications both syntactically and proof-theoretically. This section resolves the apparent incompatibility, showing that when R is a set of Horn rules, then its classical and intuitionistic semantics are closely related. Indeed, many of the well-known properties of classical Horn clauses, such as the existence of a unique minimal model, are special cases of the properties of canonical Kripke models.

Firstly, we note that it does not matter whether Horn rules are treated classically or intuitionistically. In both cases, the inference rules of section 4 form a sound and complete inference system. That is,[8]

Corollary 3 *If R is a set of Horn rules, and ϕ is a Horn rule, then $R \models_i \phi$ iff $R \models_c \phi$.*

Secondly, we note that a single classical model can be interpreted as an intuitionistic substate, and that a collection of classical models can be interpreted as an intuitionistic model. This forms the basis of the following theorem.

Corollary 4 *Suppose R is a set of Horn rules. Then M is an intuitionistic model of R iff M is a collection of classical Herbrand models of R. In particular, the canonical Kripke model M_R is the collection of all classical Herbrand models of R.*

Many semantic properties of classical Horn clauses can now be seen as special cases of the properties of canonical Kripke models. For instance, the model intersection property of Horn clauses is a special case of the substate intersection property of canonical Kripke models; and the unique minimal model property is a special case of the unique minimal substate property. Consider also the property that a set of Horn clauses entails a disjunction of atoms iff it entails one of the atoms individually. This is a special case of corollary 2. Finally, in Horn-clause logic, the unique minimal model plays a central semantic role: an atom is entailed by a set of Horn clauses iff it is in the unique minimal model. This property too is a special case of a more general property of canonical Kripke models:

Corollary 5 *If R is a set of embedded implications, and A is an atom, then $R \vdash A$ iff A is in the unique minimal substate of M_R.*

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[8]Here, $\models_i$ stands for intuitionistic entailment and $\models_c$ stands for classical entailment.

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