Normal Multimodal Logics

Laurent Catach
IBM Paris Scientific Center
3-5 Place Vendôme, 75001 Paris, France

Abstract

This paper studies what we call *normal multimodal logics*, which are general modal systems with an arbitrary set of normal modal operators. We emphasize the importance of *non-simple* systems, for which some *interaction axioms* are considered. A list of such acceptable axioms is proposed, among which the *induction* axiom has a special behavior. The class of multimodal logics that can be built with these axioms generalizes many existing modal, temporal, dynamic and epistemic systems, and could also suggest new formalizations using modal logics. The main result is a general *determination theorem* for these multimodal systems, which establishes a correspondence between our axioms and conditions over Kripke frames; this should avoid the need for showing determination each time a new system is considered.

1 Introduction

1.1 Presentation

During the last decade, it has been widely shown how modal logics provide suitable tools for various theoretical formalizations in computer science. In fact, many modal systems can be found in the literature, and there are a number of areas where such logics are used. Most popular readings of the modal formula $\Box\alpha$ are, for example, *"α is necessarily true"* (standard modal logic), *"α will always be true"* (temporal logic), *"X knows that α"* or *"X believes that α"* (epistemic logic), or *"after executing some program a, α will be true"* (dynamic logic), etc.

In general, only *one type* of modality is considered, i.e. only one aspect (time, knowledge, programs, ...) is treated at a time. But relatively few attempts have been made to employ all these systems *simultaneously*; on the other hand, if modal logics are to be of any practical interest, and especially in AI, it seems very natural to ask whether these different modelizations can be "put together", so we could talk about *necessity, time, knowledge, belief, actions, plans, deterministic programs, concurrent programs, obligations, conditionals, etc,* **within the same language**.

Thus, attempting to define a rigorous and unified framework for such systems, which can be called *multimodal logics* (an abbreviation for *multiple modal logics*), is the initial motivation for our work. Therefore, our first task is to define syntactic, axiomatic and semantic bases for these systems. However, a very desirable feature of multimodal systems lies in their ability to represent some *interrelations* between the different aspects (i.e. between modalities $\Box_1, \Box_2$ ...), such as the well-known *"If X knows that α, then X believes that α"* of epistemic logic. Therefore, some questions that naturally arise in considering multimodal logics are:

1. *Which combinations of modal systems should be examined?*
2. *What kinds of interactions between these systems make sense? Should they be specified semantically or axiomatically?*
3. *Can we develop a systematic approach to these multimodal systems, and extend standard techniques developed for traditional modal logic?*

It is without the scope of this paper to provide appropriate answers to points (1) and (2), since it depends very much on the intended formalizations, and moreover these questions may be subject to philosophical discussions. The only thing we can say is that some particular combinations, such as *knowledge and belief*, or *knowledge and time*, are certainly of primary interest, especially to AI.

So we will focus on point (3), and try to follow a systematic approach, as in [Chellas, 1980] for standard modal logic. Though multimodal logics could be entirely defined by their semantics, in a model-theoretic way, as in [Thomason, 1984] or [Halpern and Shoham, 1986] (and this approach seems particularly relevant when *time* is considered), we prefer a more axiomatic approach. To begin with, we propose a first class of *interaction axioms* $G^{a,b,c,d}$, with some examples. Then, a general determination theorem is given for the normal multimodal systems generated by these axioms; the proof uses an extension of the *canonical model* method for modal logics. To handle induction, however, this method fails, and we have to use the Fischer-Ladner filtration method, as will be indicated.

1.2 Expressiveness: examples

One main feature of multimodal languages is their ability to express *complex modalities*, obtained by composing modal operators of different types or, more generally, by using *formal operations* over modalities. For example, *"Bob knows it will be the* *case that"* or *"Bob knows it is impossible for Alice to believe that"* are such complex modalities.

To give a very simple example, let us consider a bi-modal epistemic system L, with two agents *Alice* and *Bob*, and two belief operators K_1 and K_2:

$K_1\alpha$: *"Alice believes α"*

$K_2\alpha$: *"Bob believes α"*

Suppose that Alice and Bob have, as in real life, different ways of reasoning about their beliefs; for example, Alice may be good at both positive and negative introspection, whereas Bob never performs any kind of introspection. With the traditional epistemic approach, K_1 is then a KD45 modal operator, whereas K_2 is simply a KD operator. Suppose also that the following assertion holds:

"Alice believes everything Bob believes",

for example if Alice is a little bit naive, or if she is deeply in love with Bob (despite his lack of introspection capabilities!). Then, we would like the axiom scheme $\vdash(K_2\alpha \rightarrow K_1\alpha)$ to hold in our system L. In short, in our terminology the resulting bi-modal system will be *not homogeneous* (since K_1 is of type KD45 and K_2 of type KD) and *with interactions* (since the above axiom links K_1 and K_2).

Other examples of interaction principles can be given in considering:

- knowledge and belief:
  "If X knows that Y knows that α, then X knows that α"
  "If X believes α, then X believes that he knows α"
- knowledge and time ([Halpern and Vardi, 1986]):
  "If X knows that in the next state α will be true, then in the next state he will know that α is true"
- belief and time ([Lehmann and Kraus, 1986]):
  "If X believes that tomorrow α will be true, then he believes that tomorrow he will still believe that α is true"

As we will see, our results apply to such interaction axioms.

1.3 Related work

Dynamic logic ([Parikh, 1981], [Harel, 1984]) and process logic ([Harel, Kozen and Parikh, 1981]) already use families of modal operators, denoted by [*a*], where *a* represents a program. Also, epistemic logics ([Halpern and Moses, 1985], [Halpern, 1986]) provide modal languages with several operators $K_1, K_2 \ldots K_n$. Both are, in fact, multimodal logics; but both make the two following important restrictions:

- they are *homogeneous* systems, which means that every modal operator ([*a*] or K_i) belongs to the *same* system of traditional modal logic (e.g. T, S4, S5, ...)
- they form systems *without any interactions*, which means that, roughly speaking, each modal operator [*a*] or K_i is totally independent (axiomatically or semantically) of the others.

Temporal logics, in both their linear or branching-time versions, can also be viewed as special cases of multimodal logics, since many operators are involved in the language. In fact, these operators (some of which being *not* normal) are linked by very special connections, generally indicated by the semantics; for example, operators $\bigcirc$ ("next") and $\Box$ ("always") of linear-time logic simply interact by a transitive-closure correspondence.

Beside these well-known types of logics, some other multimodal systems have been explored; the reader is referred to [Cohen, 1960], [Rennie, 1970], [Fariñas, 1983], [Thomason, 1984], [Fariñas and Orlowska, 1985], [Lucas and Lavendhomme, 1985], [Lehmann and Kraus, 1986], [Halpern and Vardi, 1986], [Halpern and Shoham, 1986], [Fischer and Immerman, 1987]. Most of them fall within the scope of the multimodal systems we consider here, as the reader may verify.

2 Formal syntax and semantics

2.1 Language

A propositional multimodal language $\mathscr{L}$ is determined by a set Φ_0 of propositional variables p, q ..., a set Σ_0 of *atomic parameters A, B* ..., "$\cup$" (union) and ";" (composition) operations over parameters, the boolean connectives $\neg \wedge \vee \rightarrow \leftrightarrow$ and, finally, the "[]" construct for modal operators. We also distinguish an element λ in Σ_0 to be the *neutral element* for the composition of parameters, i.e. to be the *identity* parameter. The set Σ of all *abstract parameters* is built from Σ_0 and the "$\cup$" and ";" operations, and the set Φ of all formulas is built from Φ_0, the boolean connectives, and the rule "if *a* is a parameter and α a formula, $[a]\alpha$ is a formula". As usual, $\langle a \rangle \equiv_{Def} \neg[a]\neg\alpha$.

The set $\mathbf{OPS}_0$ of *atomic modal operators* contains operators $[A]$ and $\langle A \rangle$ for $A \in \Sigma_0$; in the following, $\Box \in \mathbf{OPS}_0$ means that $\Box = [A]$ with $A \in \Sigma_0$, and $\mathscr{L}(\Box)$ will designate the sub-language of $\mathscr{L}$ built from $\{A\}$ instead of Σ_0. We also define $\partial = [\lambda] = \langle \lambda \rangle$ as being the identity operator.

To capture finite sets $\mathbf{OPS}_0 = \{\Box_1, \Box_2 \ldots \Box_n\}$, we let $\Sigma_0 = \{1, 2 \ \ldots \ n\}$, writing $\Box_i$ instead of $[i]$; usual epistemic systems can be obtained in this way. To get traditional modal logic, we simply let $\Sigma_0 = \{1\}$. To get dynamic logic, we just add the "*" (iteration) and "?" (test construction) operations to the language. Further extensions, as proposed in [Berman, 1979], could also be incorporated.

Depending on the desired interpretations, some subsets of parameters in Σ_0 can be used to represent various concepts such as *rational agents*, *programs*, *actions* or even *space* or *time dimensions*. Also, a fundamental remark is that Kleene operations ";" and "$\cup$", along with ∂, *are always definable with regard to modal operators*, using:

- $(\partial)\alpha \equiv_{Def} \alpha$
- $(\Box_1; \ldots ;\Box_n)\alpha \equiv_{Def} \Box_1 \ldots \Box_n\alpha$
- $(\Box_1 \cup \ldots \cup \Box_n)\alpha \equiv_{Def} \Box_1\alpha \wedge \ldots \wedge \Box_n\alpha$

Thus, we can always assume the presence of these operations in the language and it allows us, for example, to consider that formulas $\Diamond\Diamond\alpha$ or $(\Box_1\Box_2\alpha \wedge \Box_3\alpha \wedge \alpha)$ can always be represented by $\langle 1;1 \rangle\alpha$ and $[(1;2) \cup 3 \cup \lambda]\alpha$ respectively.

2.2 Systems of multimodal logics

If L denotes a multimodal language $\mathscr{L}$ with an axiomatization, i.e. a set $\mathcal{Ax}$ of axioms and inference rules, then, for $\Box \in \mathbf{OPS}_0$, we define L($\Box$) to be the set $\{\alpha \in \mathscr{L}(\Box) \,/\, \alpha$ is a theorem of L$\}$ and $\mathcal{Ax}(\Box)$ to be the subset of $\mathcal{Ax}$ containing the axioms and rules of $\mathcal{Ax}$ which are either non-modal or involve only formulas belonging to $\mathscr{L}(\Box)$.

The first problem is to know whether L($\Box$) can indeed be viewed as a "sub-system" of L, i.e. to know whether L($\Box$) can be axiomatized, and whether $\mathcal{Ax}(\Box)$ can be used for this purpose. Conversely, can we just "put together" separate axiomatizations for each sub-system L($\Box$), to get an axiomatization of L? A notion of *separability* is therefore needed:

Definition:

An axiomatization $\mathcal{Ax}$ is said to be *separable* if, for each $\Box \in \mathbf{OPS}_0$, L($\Box$) = TH($\mathcal{Ax}(\Box)$), where TH($\mathcal{Ax}(\Box)$) is the set of theorems generated by the axiom and inference rules of $\mathcal{Ax}(\Box)$.

An example of *non-separable* axiomatization is:

$$\Box_1(\alpha \rightarrow \beta) \rightarrow (\Box_1\alpha \rightarrow \Box_1\beta)$$
$$\Box_1\alpha \rightarrow \alpha$$
$$\Box_2(\alpha \rightarrow \beta) \rightarrow (\Box_2\alpha \rightarrow \Box_2\beta)$$
$$\Box_2\alpha \rightarrow \Box_1\alpha$$

since $\Box_2\alpha \rightarrow \alpha$ is in L($\Box_2$) but is not derivable from $\mathcal{Ax}(\Box_2)$, which only contains axiom (3). This point will not be fully examined here (see [Catach, 1988]).

Using separable axiomatizations, we will take *each* sub-system L($\Box$) as being at least *normal*, so we always have axioms $\Box\alpha \leftrightarrow \neg\Diamond\neg\alpha$ and **K**. $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$, and also the rule of necessitation **RN**. "if $\vdash\alpha$ then $\vdash\Box\alpha$" for atomic modal operators. Using definitions of ";" and "U", it can easily be shown that this also holds for *all* operators [*a*], $a \in \Sigma$. Such multimodal systems can be called *normal*. Note that *classical* sub-systems ([Chellas, 1980]) can also be considered ([Catach, 1988]).

The important point is that the sub-systems L($\Box$) may be normal systems of different types, as in the Alice-and-Bob example. If all the sub-systems L($\Box$) are identical to a given system L_0 of traditional modal logic, we say that L is a *homogeneous* multimodal system, based on L_0.

2.3. Axioms

In addition to axioms for λ and Kleene operations:

$$[\lambda]\alpha \leftrightarrow \alpha$$
$$[a;b]\alpha \leftrightarrow [a][b]\alpha$$
$$[a \cup b]\alpha \leftrightarrow ([a]\alpha \wedge [b]\alpha)$$

our class of multimodal logics is obtained by considering systems axiomatized by *any finite number* of axioms schemes of the following type:

$$\langle a \rangle [b]\alpha \rightarrow [c]\langle d \rangle \alpha$$

where *a, b, c, d* denote arbitrary parameters. If we refer to axiom $\mathbf{G}^{k,l,m,n}$. $\Diamond^k\Box^l\alpha \rightarrow \Box^m\Diamond^n\alpha$ of modal logic ([Chellas, 1980]), our axiom will be noted $\mathbf{G}^{a,b,c,d}$ and called the *"a,b,c,d-incestuality"* axiom. Note that $\mathbf{G}^{a,b,c,d}$ is equivalent to $\mathbf{G}^{c,d,a,b}$.

The fact that *a, b, c, d* may be *complex* parameters (i.e. built from atomic ones, using ";" and "U") make axioms $\mathbf{G}^{a,b,c,d}$ very general. In particular, $\mathbf{G}^{a,b,c,d}$ covers $\mathbf{G}^{k,l,m,n}$, and therefore covers the traditional **D, T, B, 4, 5** axioms of modal logic ([Chellas, 1980]). For example, if $a=b=\lambda$ and $c=d=A$, we get the symmetry axiom **B** for $\Box = [A]$. Consequently, each normal sub-system L($\Box$) can be any of the fifteen well-known modal systems generated by **D, T, B, 4** and **5**, e.g. KD, KT, KT4 (S4), KTB4 (S5), KD45, etc.

If the axiomatization $\mathcal{Ax}$ of L consists *only* in the superposition of all the axiomatizations $\mathcal{Ax}(\Box)$ of the sub-systems L($\Box$), we say that L is a **simple** multimodal logic, and $\mathcal{Ax}$ is separable. If $\Sigma_0 = \{1, 2 \ldots n\}$, examples of non-simple systems can be given by considering the following $\mathbf{G}^{a,b,c,d}$ interaction axioms:

- $\Box_2\alpha \rightarrow \Box_1\alpha$ (*inclusion*)
- $\Box_3\alpha \rightarrow (\Box_2\alpha \rightarrow \Box_1\alpha)$ (*relative inclusion*)
- $\Box_2\alpha \leftrightarrow \Box_1\alpha$ (*equivalence*)
- $\alpha \rightarrow \Box_1\Diamond_2\alpha$ (*semi-adjunction*)
- $\Box_2\alpha \rightarrow \Diamond_1\alpha$ (*common seriality*)
- $\Box_1\Box_2\alpha \rightarrow \Box_2\Box_1\alpha$ (*semi-commutativity*)
- $\Box_3\alpha \leftrightarrow (\Box_1\alpha \wedge \Box_2\alpha)$ (*union*)
- $\Box_3\alpha \leftrightarrow \Box_1\Box_2\alpha$ (*composition*)

Finally, we also consider the following pair:

$$[b]\alpha \rightarrow ([a]\alpha \wedge [a][b]\alpha)$$
$$[b](\alpha \rightarrow [a]\alpha) \rightarrow ([a]\alpha \rightarrow [b]\alpha)$$

called the *a,b-induction* axioms. Taking $b=a^*$, we get the Segerberg axioms for PDL ([Kozen and Parikh, 1981], [Harel, 1984]). Taking $a = 1 \cup 2 \cup \ldots \cup n$, we get the axiom for *common operators C* or *D* of epistemic logics ([Halpern and Moses, 1985], [Lehmann and Kraus, 1986]). We can also get the induction axioms of temporal logics in this way. Note that the first one, written $[b]\alpha \rightarrow [a \cup (a;b)]\alpha$, is of type $\mathbf{G}^{a,b,c,d}$.

Thus, the above axiom schemes cover many existing systems of modal or multimodal logic. Also, they make the generation of a large class of new ones possible; our Alice-and-Bob story provides such an example.

2.4 Notations

The standard notations of Lemmon can be easily extended to multimodal systems, by indexing the sub-systems L($\Box$). For example, our Alice-and-Bob system will be noted $(KD45)_1(KD)_2K_{1,2}$, if $K_{1,2}$ denotes $\Box_2\alpha \rightarrow \Box_1\alpha$. Similarly, $(KD45)_1^{(n)}(KD)_2^{(m)}$ would designate a simple multimodal system with *n* operators of type KD45 and *m* operators of type KD. Note that, for any integer $n>0$, $K^{(n)}$ is the smallest normal *n*-modal system (simple or not).

2.5 Models

Kripke semantics easily extends to multimodal logics. A *multi-(relational) frame* is a pair $F = \langle W, \mathscr{R} \rangle$, where W is a set of possible worlds and $\mathscr{R}$ is a set of binary relations over W; in that case, F is said to be the *join* of the frames $<W,R>$ with $R \in \mathscr{R}$. If $\mathcal{C}_1$, $\mathcal{C}_2 \ldots \mathcal{C}_n$ is a family of classes of frames, the *join* of these classes is the class $\mathcal{C}$ of multi-frames $<W,\{R_1,R_2 \ldots R_n\}>$ such that $<W,R_i>$ belongs to $\mathcal{C}_i$ for each $1 \leq i \leq n$.

If L is a normal multimodal system, $F = <W,\mathscr{R}>$ is said to be a multi-frame *for* L if there exists a mapping ρ from Σ to $\mathscr{R}$ satisfying:

- $\rho(\lambda) = i$
- $\rho(a \cup b) = \rho(a) \cup \rho(b)$
- $\rho(a;b) = \rho(a) \mid \rho(b)$

where $i = \{(w,w) \mid w \in W\}$ is the *identity* (or *diagonal*) relation over W, and "$\cup$" and "$|$" denote the usual union and composition of binary relations.

Multi-models $M = <W,\mathscr{R},V>$ are defined, as expected, by introducing an assignment function V from $W \times \Phi_0$ to $\{0,1\}$. Truth of formulas in worlds of multi-models, written $(M,w) \models \alpha$, is defined inductively as usual; thus, for every parameter a and formula α, we have:

$(M,w) \models [a]\alpha$ *iff* $(M,w') \models \alpha$ for every w′ such that $(w,w') \in \rho(a)$

Satisfiability and validity in multi-models, multi-frames and classes of multi-frames, for formulas or sets of formulas, are defined in the usual way. We omit details. In the following, multi-frames and multi-models are defined directly as $<W,\rho>$ and $<W,\rho,V>$ respectively.

3 Determination

We use the usual operations "$\subseteq$" (inclusion), "$^{-1}$" (converse), "$\cup$" (union), "$|$" (composition) and "$^{+}$" (transitive closure) over binary relations. If $<W,\rho>$ is a multi-frame for L, and if *a, b, c, d* are parameters in Σ, we define *a,b,c,d-incestuality* as being the following property:

if $(w,w') \in \rho(a)$ and $(w,w'') \in \rho(c)$, then there exists w''' such that $(w',w''') \in \rho(b)$ and $(w'',w''') \in \rho(d)$.

Formally, this yields $\rho(a)^{-1} \mid \rho(c) \subseteq \rho(b) \mid \rho(d)^{-1}$, which can be pictured as follows:

$$\begin{tikzcd} & w' \arrow[dr, "\rho(b)"] & \\ w \arrow[ur, "\rho(a)"] \arrow[dr, "\rho(c)"'] & & w''' \\ & w'' \arrow[ur, "\rho(d)"'] & \end{tikzcd}$$

Theorem:

Let L be a normal multimodal system built from a finite set of axioms $\mathbf{G}^{a,b,c,d}$. Then L is determined by the class of multi-frames having the corresponding *a,b,c,d*-incestual properties.

As usual, soundness is easily obtained. To show completeness, we use the *proper canonical (multi-) frame* $F^c = <W^c,\rho^c>$ of L, defined as follows:

1. W^c is the set of all maximal consistent sets of L,
2. for every ∇, $\nabla' \in W^c$ and $a \in \Sigma$, $\rho^c(a)$ is defined by:
   $(\nabla,\nabla') \in \rho^c(a)$ *iff* $\{\alpha \mid [a]\alpha \in \nabla\} \subseteq \nabla'$

and we show that (i) F^c is a frame for L, and (ii) if $\mathbf{G}^{a,b,c,d}$ is an axiom of L, then F^c is an *a,b,c,d*-incestual frame. See [Catach, 1988] for a detailed proof.

As an example, the above theorem shows that our Alice-and-Bob system $(KD45)_1(KD)_2K_{1,2}$ is determined, as expected, by the class $\mathcal{C}$ of multi-frames $<W,\{R_1,R_2\}>$ where R_1 is serial, transitive and euclidean, R_2 is serial, and $R_1 \subseteq R_2$. Note that, as far as soundness is concerned, we can give a very general result for *all simple* multimodal logics:

Theorem (Soundness):

Let L be a *simple* multimodal system, such that for every $\Box \in \mathbf{OPS}_0$, $L(\Box)$ is sound with respect to a class $\mathcal{C}_\Box$ of frames. Then L is sound with respect to the *join* $\mathcal{C}$ of the classes $\mathcal{C}_\Box$.

4 Induction

We expect that multimodal systems containing one or more pairs of *a,b*-induction axioms (see 2.3) should be determined by the classes of multi-frames $<W,\rho>$ where $\rho(b) = \rho(a)^{+}$. Soundness can indeed be stated for these multimodal systems, i.e. *a,b*-induction axioms are always valid in multi-frames $<W,\rho>$ where $\rho(b) = \rho(a)^{+}$. But completeness cannot be obtained using the proper canonical model; all we can show is that if an *a,b*-induction holds in L, then $\rho^c(a)^{+} \subseteq \rho^c(b)$, and Segerberg axioms are not strong enough to capture transitive closure, i.e. to show the converse $\rho^c(b) \subseteq \rho^c(a)^{+}$. In fact, there exist (infinite) sets of formulas which are consistent in a system L containing an *a,b*-induction axiom but which cannot be satisfied in any model for which $\rho(b) \subseteq \rho(a)^{+}$. This result was already known for dynamic logics with the "*" operator, using results from dynamic algebras ([Parikh, 1981], [Harel, 1984]).

The easier way to handle induction is to extend the Fischer-Ladner *filtrations method* for dynamic logic ([Fischer and Ladner, 1979], [Harel, 1984]), which does yield completeness results (and also decidability and complexity ones at the same time) for some multimodal systems. But no *general* result can be enounced easily, the problem being that *a,b,c,d*-incestuality is not always preserved during filtration, as in DPDL ([Ben-Ari, Halpern and Pnueli, 1982]). The reader is referred to [Catach, 1988] for a more careful study of multimodal systems with induction axioms.

5 Other topics

- As mentioned above, extending the filtrations method to multimodal logics yields several results, namely the *finite model property*, *decidability*, and even *complexity*. As for the last one, we expect that the complexity of the validity problem should be *PSPACE-complete* for multimodal systems *without* any induction axioms, and

EXPTIME-complete if at least one induction axiom is considered. Complexity should also depend very much on the considered formulas.

- The Lindenbaum algebra associated with a multimodal logic is a boolean algebra with unary operators, in the sense of [Jónsson and Tarski, 1951]. Therefore, studying *multimodal algebras* yields many interesting results, such as an elegant proof of determination in some cases.
- Other types of frames and models can be considered for normal multimodal logics, namely *multidimensional* ones $<W_1 \times W_2 \times \ldots \times W_n, \{R_1, R_2 \ldots R_n\}>$. *Protocols*, as defined in [Fischer and Immerman, 1987], are examples of such models.

Conclusion

This paper presents some formal developments of multimodal logics, which are general modal systems with arbitrary sets of modal operators. A class of axioms, and especially of *interaction* axioms, has been proposed, generating a wide class of systems, for which a general determination theorem has been given. Problems when dealing with induction axioms have also been indicated. Many other aspects of multimodal logics remain to be investigated, as has already been done for standard modal logics; some of them are studied in [Catach, 1988].

Acknowledgements

I am very grateful to David Makinson for helpful remarks and suggestions. I also thank anonymous referees for correcting errors, and Rosalind Greenstein for her help in preparing the paper.

References

[Ben-Ari, Halpern and Pnueli, 1982] M.Ben-Ari, J.Halpern, A.Pnueli: "Deterministic Propositional Dynamic Logic", *Journal of Computer and System Sciences* ***25*** (1982), pp.402-417.

[Berman, 1979] F.Berman: "A completeness technique for D-axiomatizable semantics", *Proc. 11° ACM Symp. on Th. of Comp.* (1979), pp.160-166.

[Catach, 1988] L.Catach: "Multimodal logics", Doctorate Thesis (1988), IBM Scientific Center, Paris, France.

[Chellas, 1980] B.F.Chellas: "Modal Logic. An introduction", Cambridge University Press, 1980.

[Cohen, 1960] L.J.Cohen: "A formalization of referentially opaque contexts", *The Journal of Symbolic Logic* ***25 (3)*** (1960).

[Fariñas, 1983] L.Fariñas: "Space as time", *Information Processing Letters* ***17***, Oct. 1983, pp.113-115.

[Fariñas and Orlowska, 1985] L.Fariñas, E.Orlowska: "DAL: a logic for data analysis" *Theoretical Computer Science* ***36*** (1985), pp.251-264.

[Fischer and Immerman, 1987] M.J.Fischer, N.Immerman: "Interpreting Logics of Knowledge in Propositional Dynamic Logic with converse", *Information Processing Letters* ***25*** (1987), pp.175-181.

[Fischer and Ladner, 1979] M.J.Fischer, R.E.Ladner: "Propositional Dynamic Logic of Regular Programs", *Journal of Computer and System Sciences* ***18*** (1979), pp.194-211.

[Halpern, 1986] J.Y.Halpern (ed): "Reasoning about knowledge: an overview", *Proc. Conference on Theoretical Aspects of Reasoning about Knowledge* (1986), Morgan Kauffmann.

[Halpern and Moses, 1985] J.Y.Halpern, Y.Moses: "A guide to the modal logics of knowledge and belief", *Proc. IJCAI 1985,* pp.480-490.

[Halpern and Shoham, 1986] J.Y.Halpern, Y.Shoham: "A propositional modal logic of time intervals", *Proc. 1st Ann. IEEE Symp. on Logic in Comp. Science* (1986), pp.279-292.

[Halpern and Vardi, 1986] J.Y.Halpern, M.Y.Vardi: "The complexity of reasoning about Knowledge and Time: extended abstract", *Proc. 8° ACM Symp. on Th. of Comp.* (1986), pp.304-315.

[Handbook, 1984] "Extensions of classical logic", *Handbook of Philosophical Logic* ***II***, D.Gabbay & F.Guenthner eds, Reidel, 1984.

[Harel, 1984] D.Harel: "Dynamic Logic", in [Handbook, 1984].

[Harel, Kozen and Parikh, 1981] D.Harel, D.Kozen, R.Parikh: "Process logic: expressiveness, decidability, completeness", *Proc. 21° IEEE Symp. on Found. of Comp. Science* 1980, pp.129-142.

[Jónsson and Tarski, 1951] B.Jónsson, A.Tarski: "Boolean algebras with operators I", *American Journal of Mathematics* ***73*** (1951), pp.891-939.

[Kozen and Parikh, 1981] D.Kozen, R.Parikh: "An elementary proof of the completeness of PDL", *Theoretical Computer Science* ***14*** (1981), pp.113-118.

[Lehmann and Kraus, 1986] D.Lehmann, S.Kraus: "Knowledge, Belief and Time", *Lecture Notes in Computer Science* ***226*** (1986), pp.186-195.

[Lucas and Lavendhomme, 1985] Th.Lucas, R.Lavendhomme: "Complétude de certaines logiques bimodales", *Séminaire de l'Institut de Mathématiques de l'Université de Louvain (Belgium),* Report ***69***, May 1985.

[Makinson, 1966] D.Makinson: "On some Completeness Theorems in Modal Logic", *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* ***12*** (1966), pp.379-384.

[Parikh, 1981] R.Parikh: "Propositional dynamic logics of programs: a survey" *Lecture Notes in Computer Science* ***125*** (1981), pp.102-144.

[Rennie, 1970] M.K.Rennie: "Models for multiply modal systems", *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* ***16*** (1970), pp.175-186.

[Thomason, 1984] R.H.Thomason: "Combinations of tense and modality", in [Handbook, 1984].