Conditional Logics of Normality as Modal Systems

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Abstract

Recently, conditional logics have been developed for application to problems in default reasoning. We present a uniform framework for the development and investigation of conditional logics to represent and reason with "normality", and demonstrate these logics to be equivalent to extensions of the modal system S4. We also show that two conditional logics, recently proposed to reason with default knowledge, are equivalent to fragments of two logics developed in this framework.

Introduction

It is widely acknowledged that commonsense reasoning is nonmonotonic, or defeasible. Given a certain body of knowledge, the facts one infers may not be accepted if this knowledge is augmented with new information. One reason for this defeasibility is that we often reason by *default*, or jump to conclusions by assuming that the state of affairs represented by our knowledge is in some sense typical or normal. Default reasoning lies at the heart of a theory of AI, and much effort has been expended in developing formalisms to represent and reason with "default" knowledge (see (Reiter 1987) for a survey).

Recently, the use of conditional logics in nonmonotonic reasoning has been explored (see, e.g., (Bell 1990; Boutilier 1988; Delgrande 1987; Delgrande 1988; Lehmann 1989; Kraus, Lehmann and Magidor 1990; Nute 1984)). Conditional logics were originally developed to account for properties of conditional statements in natural language. These logics consist of the classical propositional logic (CPL) augmented with a conditional connective, often written $>$. This additional connective is necessitated as it is generally agreed that the material conditional does not adequately reflect linguistic usage of "if-then" constructs. Logics for subjunctive conditionals have been widely studied (e.g. (Stalnaker 1968; Lewis 1973)) and are frequently based on possible worlds semantics which follow (roughly) the suggestion of Stalnaker (1968): determine the truth of a conditional in a certain situation by evaluating the truth of the consequent in the most similar situation in which the antecedent is true. These logics possess a number of properties which are not only intuitively valid of subjunctives, but also reasonable for an account of "default rules". For instance, strengthening and transitivity are not generally valid for the conditional connective:

Str From $B > C$, infer $(A \wedge B) > C$

Tran From $A > B$ and $B > C$, infer $A > C$.

One cannot infer that a wet match would light, given that a match would; neither can one infer that penguins fly, given that penguins are birds and birds fly.

Writing the conditional connective as $\Rightarrow$ (to distinguish it from the subjunctive interpretation), we will interpret a sentence $A \Rightarrow B$ is as meaning "In the most normal course of events in which A holds, B holds as well", or "A normally implies B". Rather than evaluating the truth of the consequent in the most *similar* situation where the antecedent holds, we intend to evaluate it in the most *normal* situation. Arguably, much of our default knowledge can be interpreted as being of this form.

In (Boutilier 1988), the conditional logic E was presented as an extension of Delgrande's (1987) logic N, and was investigated as a basis for default reasoning. There a connection was shown to exist between E and the modal system S4.3. In this note, we will develop this connection further, between conditional logics of normality (CLNs) and modal logic. In the next section, we will provide a uniform framework for exploring CLNs and discuss several such logics. In particular, we will show these logics to be equivalent to extensions of the modal system S4 (KT4). This contrasts with the analysis of subjunctives discussed above, for as Lewis (1973) points out, $>$ cannot be defined in terms of the standard unary modal operator $\Box$ and truth-functional connectives. In the following section, we will show that two conditional logics for default reasoning recently presented in the literature are equivalent to fragments of logics developed here (those fragments without nested occurrences of the conditional connective), and hence to fragments of S4-systems. We will conclude by discussing some advantages of viewing CLNs in the manner proposed. Complete proofs of theorems can be found in (Boutilier 1989).

Conditional Logics of Normality

In this section, we will present a possible worlds semantics for CLNs. The sentence $A \Rightarrow B$ is intended to represent "A normally implies B". We will take this to mean that B holds at the most normal (or least exceptional) worlds at which A holds. The concept of normality will be represented by an accessibility relation R between possible worlds. World v will be accessible to world w if v represents a state of affairs which is at least as unexceptional (or normal) as that represented by w. There are some restrictions which should be placed on any R intended to be interpreted in this manner. We will insist that R be reflexive, so that a state of affairs is no more exceptional than itself, and transitive. Given these restrictions on R, we can be more precise about the meaning of $A \Rightarrow B$. If $A \Rightarrow B$ is true at some world w, then in those least exceptional worlds (as seen by w) where A holds, B holds as well. This means, at any more normal state of affairs, either A is necessarily false at all less exceptional worlds, or there exists a less exceptional world where A and B hold, and $A \supset B$ holds at all worlds more normal than that one. In the language of modal logic, $A \Rightarrow B$ holds iff the following does:

$$\Box(\Box\neg A \vee \Diamond(A \wedge \Box(A \supset B))).$$

This seems to capture the notion of least-exceptional A-worlds. We will now formalize these ideas.

The language of CLNs (denoted $\mathbf{L}_C$) is formed from a denumerable set $\mathbf{P}$ of propositional variables, together with the connectives $\neg$, $\supset$ and $\Rightarrow$. The connectives $\wedge$, $\vee$ and $\equiv$ are defined in terms of these in the usual way, and we define $\alpha \not\Rightarrow \beta$ as $\neg(\alpha \Rightarrow \beta)$. As is customary (e.g. (Stalnaker 1968)), $\Box\alpha$ is defined to be $\neg\alpha \Rightarrow \alpha$, and $\Diamond\alpha$ is $\neg(\alpha \Rightarrow \neg\alpha)$.

Definition A *CT4-model* is a triple $M = \langle W, R, \varphi\rangle$, where W is a set (of possible worlds), R is a reflexive, transitive binary relation on W (the accessibility relation), and φ maps $\mathbf{P}$ into 2^W ($\varphi(A)$ is the set of worlds where A holds).

Definition Let $M = \langle W, R, \varphi\rangle$ be a CT4-model, with $w \in W$. The truth of a formula α at w in M (where $M \models_w \alpha$ means α is true at w) is defined inductively as:

1. $M \models_w \alpha$ iff $w \in \varphi(\alpha)$ for atomic sentence α.
2. $M \models_w \neg\alpha$ iff $M \not\models_w \alpha$.
3. $M \models_w \alpha \supset \beta$ iff $M \models_w \beta$ or $M \not\models_w \alpha$.
4. $M \models_w \alpha \Rightarrow \beta$ iff for each w_1 such that wRw_1 either
   (a) there is some w_2 such that w_1Rw_2, $M \models_{w_2} \alpha$, and for each w_3 such that w_2Rw_3, $M \not\models_{w_3} \alpha$ or $M \models_{w_3} \beta$; or
   (b) for every w_2 such that w_1Rw_2, $M \not\models_{w_2} \alpha$.

It is easy to verify that the connectives $\Box$ and $\Diamond$, introduced by definition, have the following familiar truth conditions:

1. $M \models_w \Box\alpha$ iff $M \models_{w_1} \alpha$ for each w_1 such that wRw_1.
2. $M \models_w \Diamond\alpha$ iff $M \models_{w_1} \alpha$ for some w_1 such that wRw_1.

$\Box\alpha$ can be interpreted as "In all less exceptional worlds α holds", and $\Diamond\alpha$ as "In some less exceptional world α holds".

Definition A CT4-model $M = \langle W, R, \varphi\rangle$ *satisfies* a sentence α (written $M \models \alpha$) iff $M \models_w \alpha$ for each $w \in W$. A sentence α is *CT4-valid* (written $\models_{CT4} \alpha$) just when $M \models \alpha$ for every CT4-model M.

We will now define the logic CT4. Since the modal system S4 is characterized by the class of reflexive, transitive models, we will base our axiomatization on a standard one for S4[1]. Completeness will follow quite easily from the completeness of S4 and the interdefinability of $\Rightarrow$ and $\Box$.

Definition The *conditional logic* CT4 is the smallest $S \subseteq \mathbf{L}_C$ such that S contains CPL and the following axioms, and is closed under the following rules of inference:

K $\Box(A \supset B) \supset (\Box A \supset \Box B)$
T $\Box A \supset A$
4 $\Box A \supset \Box\Box A$
C $(A \Rightarrow B) \equiv \Box(\Box\neg A \vee \Diamond(A \wedge \Box(A \supset B)))$
Nes From A infer $\neg A \Rightarrow A$.
MP From $A \supset B$ and A infer B.
US From A infer A', where A' is a substitution instance of A.

Definition A sentence α is *provable in* CT4 (written $\vdash_{CT4} \alpha$) iff $\alpha \in$ CT4. α is *derivable* from a set $\Gamma \subseteq \mathbf{L}_C$ (written $\Gamma \vdash_{CT4} \alpha$) if there is some finite subset $\{\alpha_1, \ldots, \alpha_n\}$ of Γ such that $\vdash_{CT4} (\alpha_1 \wedge \ldots \wedge \alpha_n) \supset \alpha$.

Theorem 1 *The system* CT4 *is characterized by the class of CT4-models; that is,* $\vdash_{CT4} \alpha$ *iff* $\models_{CT4} \alpha$.

The connection between CT4 and S4 is now quite clear. The semantics of CT4 is based on the same class of models as that of S4, and the axiomatic basis of CT4 is merely an adaptation of one for S4 (plus the "characteristic" conditional axiom **C**). In fact, in a very strong sense, these two logics are equivalent. We can translate sentences from $\mathbf{L}_C$ into the language of modal systems, $\mathbf{L}_M$, and conversely, as follows:

Definition For $\alpha \in \mathbf{L}_C$, the *translation* of α into $\mathbf{L}_M$ (denoted α°) is defined inductively as follows:

1. α, if α is atomic.
2. $\neg\beta^\circ$, if α has the form $\neg\beta$.
3. $\beta^\circ \supset \gamma^\circ$, if α has the form $\beta \supset \gamma$.

[1] See, e.g., (Hughes and Cresswell 1984). We use the abbreviation $\Box$ in the axiomatization for clarity.

4. $\Box(\Box\neg\beta^\circ \vee \Diamond(\beta^\circ \wedge \Box(\beta^\circ \supset \gamma^\circ)))$, if α has the form $\beta \Rightarrow \gamma$.

Definition For $\alpha \in \mathbf{L}_M$, the *translation* of α into $\mathbf{L}_C$ (denoted α^*) is defined inductively as follows:

1. α, if α is atomic.
2. $\neg\beta^*$, if α has the form $\neg\beta$.
3. $\beta^* \supset \gamma^*$, if α has the form $\beta \supset \gamma$.
4. $\neg\beta^* \Rightarrow \beta^*$, if α has the form $\Box\beta$.

These mappings induce isomorphisms between the Lindenbaum algebras of the logics CT4 and S4, and each induces the inverse of the other (see (Boutilier 1989)). So reasoning done with one logic can just as easily be done with the other. In this sense, the logics are equivalent. In fact, they can be viewed as definitional variants of each other.

Theorem 2 *$\vdash_{CT4} \alpha \equiv (\alpha^\circ)^*$ and $\vdash_{S4} \alpha \equiv (\alpha^*)^\circ$. Also, $\vdash_{CT4} \alpha \supset \beta$ iff $\vdash_{S4} \alpha^\circ \supset \beta^\circ$. In other words,* CT4 *and* S4 *are equivalent.*

CT4 does capture many of the properties expected of a logic of normality. For instance, each of the following sentences or rules (see, e.g., (Delgrande 1987; Lehmann 1989)) is valid in CT4:

ID $A \Rightarrow A$

CC $((A \Rightarrow B) \wedge (A \Rightarrow C)) \supset (A \Rightarrow (B \wedge C))$

RT $(A \Rightarrow B) \supset (((A \wedge B) \Rightarrow C) \supset (A \Rightarrow C))$

CC′ $((A \Rightarrow C) \wedge (B \Rightarrow C)) \supset ((A \vee B) \Rightarrow C)$

RCM From $\Box(B \supset C)$, infer $(A \Rightarrow B) \supset (A \Rightarrow C)$

CM From $A \Rightarrow B$ and $A \Rightarrow C$, infer $(A \wedge B) \Rightarrow C$

Furthermore, neither of **Str** or **Tran** is valid, and the connective $\Rightarrow$ is exception-allowing, as the following set is consistent in CT4.

$\{Bird, Bird \Rightarrow Fly, \neg Fly\}$

As well, the sentence

$\Diamond A \wedge A \not\Rightarrow B \wedge A \not\Rightarrow \neg B$

is satisfiable[2], showing that A need not normally indicate an attribute B or its negation; but, the sentence

$\Diamond A \supset ((A \Rightarrow B) \supset (A \not\Rightarrow \neg B))$

is valid, meaning A cannot normally imply both an attribute and its negation (unless A is not possible). An interesting fact is that $A \Rightarrow B \equiv \Box(A \Rightarrow B)$ is a theorem of CT4; if A normally implies B in some state of affairs, then in all less exceptional states, it should continue to hold.

While CT4 captures many aspects of normal implication, there are some theorems, intuitively valid in many circumstances, which fail to hold. For instance, the rule of *Rational Monotony* (Lehmann 1989), and the related axiom **CV** (Delgrande 1987), are not valid in CT4.

[2] Contrast this with Stalnaker's (1968) semantics, which validates the Conditional Law of Excluded Middle $(A > B) \vee (A > \neg B)$.

RM From $A \Rightarrow C$ and $A \wedge B \not\Rightarrow C$, infer $A \Rightarrow \neg B$

CV $(A \not\Rightarrow B) \supset ((A \Rightarrow C) \supset (A \wedge \neg B \Rightarrow C))$

In order to discover conditional logics which validate these schemata, we will allow any system in the language $\mathbf{L}_C$ which extends CT4 to be called a conditional logic of normality.

Definition A *conditional logic of normality* (CLN) is any system $S \subseteq \mathbf{L}_C$ closed under the inference rules **Nes**, **MP** and **US**, such that CT4 $\subseteq S$.

Corollary 1 *Any modal system which extends S4 is equivalent to some CLN, and any CLN is equivalent to some modal system which extends S4.*

Of particular interest is the system CT4D, or CT4 plus the axiom **D**:

D $\Box(\Box A \supset B) \vee \Box(\Box B \supset A)$.

CT4D was first studied as the logic E in (Boutilier 1988), and was presented there as an extension of Delgrande's (1987) system N[3]. Of course, the axiom **D** is one used to extend the modal system S4 into S4.3 (or KT4D), so it is not surprising that CT4D is characterized by the class of connected CT4-models, or that CT4D is equivalent to S4.3.

Definition $M = \langle W, R, \varphi\rangle$ is a *CT4D-model* iff M is a CT4-model and R is a *connected* relation (that is, if uRv and uRw, then either vRw or wRv).

Theorem 3 *$\models_{CT4D} \alpha$ iff $\vdash_{CT4D} \alpha$.*

Theorem 4 CT4D *and* S4.3 *are equivalent.*

Theorem 3 is also proven in (Boutilier 1988), as is theorem 4, which uses the mappings $*$ and $\circ$.

By insisting on a connected relation, we require that any (accessible) situations be comparable. If neither of w_1 or w_2 is more normal than the other, then they must be equally normal (rather than incomparable). CT4D validates both **RM** and **CV**, and enjoys the other properties described above as belonging to CT4. It seems to be a very suitable logic for reasoning with default and prototypical properties. Of course, it is not the only extension of CT4 which merits examination. Many other CLNs extending CT4 (and hence modal systems extending S4) may prove interesting and useful as logics of normality. For instance, consider CT4G, equivalent to S4.2, which is CT4 plus the axiom **G**:

G $\Diamond\Box A \supset \Box\Diamond A$.

This logic, contained in CT4D, fails to validate **RM** but does include a weaker version of it:[4]

[3] CT4D extends N in that it treats sentences properly which have nested occurrences of the conditional connective, and it validates the rule **CM**, which N does not.

[4] $\top$ is the identically true proposition (e.g., any truth functional tautology). $\top \Rightarrow A$ is interpreted as "Normally A".

WRM From $A \Rightarrow C$ and $A \wedge B \not\Rightarrow C$, infer $(A \Rightarrow \neg B) \vee (\top \Rightarrow \neg A)$

The logic CT45 has as an additional axiom **5**:

5 $\Diamond A \supset \Box\Diamond A$.

Equivalent to S5, its application as a CLN appears to be somewhat limited in that normal implication reduces to strict implication in this system. That is,

$$\vdash_{CT45} (\alpha \Rightarrow \beta) \equiv \Box(\alpha \supset \beta).$$

Preferential and Rational Consequence Relations

In this section, we will examine the nonmonotonic consequence relations of Kraus, Lehmann and Magidor (Lehmann 1989; Kraus, Lehmann and Magidor 1990), and show their relationship to particular CLNs. Gabbay (1985) has proposed studying nonmonotonic reasoning systems as consequence relations[5], and this approach has been developed by Besnard (1988) and Lehmann *et al.* (1989; 1990), among others. The language of this approach is that of CPL together with a binary relation symbol $\mid\sim$. For any propositional formulae α and β, $\alpha \mid\sim \beta$ is called a *conditional assertion* and is intended to mean that if α is known, one may sensibly conclude β. In (Lehmann 1989; Kraus, Lehmann and Magidor 1990), a consequence relation is defined as any binary relation R between propositional formulae for which certain properties hold. If the pair $\langle\alpha, \beta\rangle$ is in R, then using this notion of consequence, one may sensibly conclude β given α, and we write $\alpha \mid\sim \beta$. $\alpha \not\mid\sim \beta$ means $\langle\alpha, \beta\rangle$ is not in R. In particular, two types of consequence are studied in (Lehmann 1989; Kraus, Lehmann and Magidor 1990).

Definition (Lehmann 1989) A *preferential consequence relation* is a consequence relation which satisfies the following rules of inference (some of which are renamed):

LLE From $\models_{CPL} \alpha \equiv \beta$ and $\alpha \mid\sim \gamma$, infer $\beta \mid\sim \gamma$

RCM From $\models_{CPL} \alpha \supset \beta$ and $\gamma \mid\sim \alpha$, infer $\gamma \mid\sim \beta$

ID $\alpha \mid\sim \alpha$

And From $\alpha \mid\sim \beta$ and $\alpha \mid\sim \gamma$, infer $\alpha \mid\sim \beta \wedge \gamma$

Or From $\alpha \mid\sim \gamma$ and $\beta \mid\sim \gamma$, infer $\alpha \vee \beta \mid\sim \gamma$

CM From $\alpha \mid\sim \beta$ and $\alpha \mid\sim \gamma$, infer $\alpha \wedge \beta \mid\sim \gamma$

Definition (Lehmann 1989) A *rational consequence relation* is a preferential consequence relation which satisfies the following rule of inference:

RM From $\alpha \mid\sim \gamma$ and $\alpha \wedge \beta \not\mid\sim \gamma$, infer $\alpha \mid\sim \neg\beta$

[5] While the original notions of consequence studied by Tarski and Scott (cf. (Czelakowski and Malinowski 1985)) require relations to be monotonic, Gabbay's (1985) proposal is to generalize these ideas by allowing nonmonotonic consequence relations, and thus provide logical foundations for nonmonotonic reasoning.

Families of models are proposed to characterize these notions of consequence. These models only determine the truth of conditional assertions.

Definition (Lehmann 1989) Let $\langle X, \prec\rangle$ be a poset. $V \subseteq X$ is *smooth* iff for each $v \in V$, either v is minimal in V (that is, there is no $x \in V$ such that $x \prec v$) or there is some element w minimal in V such that $w \prec v$.

Definition (Lehmann 1989) A *preferential model* (P-model) M is a triple $\langle S, \varphi, \prec\rangle$ where S is a set (of possible worlds), φ maps propositional formulae into 2^S so as to respect the interpretation of connectives ($\varphi(\alpha)$ is the set of worlds where α holds, denoted $||\alpha||^M$), and $\prec$ is a strict partial order on S such that for all propositional formulae α, $||\alpha||^M$ is smooth.

Definition (Lehmann 1989) A *ranked model* (R-model) is a preferential model $M = \langle S, \varphi, \prec\rangle$ where the relation $\prec$ is such that there exists a totally ordered set $\langle T, <\rangle$ and a function $f : S \to T$, where $s \prec t$ iff $f(s) < f(t)$.

Definition (Lehmann 1989) A P-model or R-model $M = \langle S, \varphi, \prec\rangle$ *satisfies* a conditional assertion $\alpha \mid\sim \beta$ (written $\alpha \mid\sim_M \beta$) iff for any $\prec$-minimal s in $||\alpha||^M$, $s \in ||\beta||^M$. $\mid\sim_M$ is the *consequence relation defined by* M.

The following completeness results are also obtained.

Theorem 5 (Lehmann 1989) *$\mid\sim$ is a preferential consequence relation iff it is the consequence relation defined by some P-model. $\mid\sim$ is a rational consequence relation iff it is the consequence relation defined by some R-model.*

The logic defined by preferential consequence relations is denoted P in (Lehmann 1989; Kraus, Lehmann and Magidor 1990). We will denote the logic of rational relations by R. An apparently discouraging result presented in (Lehmann 1989) states that an assertion α follows in P from a set of assertions KB iff α follows in R from that set. This result, however, is due to the limited language in which reasoning is done. $KB \cup \{\alpha\}$ must contain only sentences of the form $\beta \mid\sim \gamma$. In particular, one cannot assert as a premise, nor derive as a consequence, propositions or boolean combinations of assertions such as $\neg(\alpha \mid\sim \beta)$ or $(\alpha \mid\sim \beta) \vee (\alpha \mid\sim \gamma)$. However, P and R can be extended in an obvious way to include this enriched language: we will allow as well-formed formulae any propositional formula, any conditional assertion formed from propositional formulae, and any boolean combination of these. In particular, only nested conditional assertions, of the form, say, $\alpha \mid\sim (\beta \mid\sim \gamma)$ are disqualified. Such well-formed formulae will be called *extended conditional assertions.* In order to capture reasonable inferences using this language of extended assertions, we must enhance the systems R and P to reason with propositions. P* and R* will denote the systems obtained by augmenting P and R with the axiom and rule schemata of CPL together with the axiom $(\neg A \mid\!\sim A) \supset A$[6]. The notions of satisfiability and validity in P-models will be adjusted as follows:

Definition Let $M = \langle S, \varphi, \prec \rangle$ be a P-model, and let $s \in S$. The truth of an extended conditional assertion α at s ($M \models_s \alpha$ means α is true at s) is defined inductively as follows:

1. $M \models_s \alpha$ iff $s \in \varphi(\alpha)$ for atomic sentence α.
2. $M \models_s \neg\alpha$ iff $M \not\models_s \alpha$.
3. $M \models_s \alpha \supset \beta$ iff $M \models_s \beta$ or $M \not\models_s \alpha$.
4. $M \models_s \alpha \mid\!\sim \beta$ iff $\alpha \mid\!\sim_M \beta$.

M satisfies α ($M \models \alpha$) iff $M \models_s \alpha$ for each $s \in S$. α is P*-valid ($\models_{P*} \alpha$) iff $M \models \alpha$ for each P-model M. α is R*-valid ($\models_{R*} \alpha$) iff $M \models \alpha$ for each R-model M.

It is not hard to see P* and R* correspond to the classes of P-models and R-models, respectively, using this extended notion of validity, and that these logics extend P and R in a very natural way. In fact, P* and R* are not much more interesting than P and R, except they will allow us to show a correspondence between the notions of consequence described in (Lehmann 1989; Kraus, Lehmann and Magidor 1990) and CLNs.

The language of $\mathbf{L}_C$ allows nested occurrences of the conditional connective, something which is forbidden in the language of extended assertions, so we will define $\mathbf{L}_C^-$ to be the conditional language without such nesting. For any CLN S, S– will denote S restricted to sentences of $\mathbf{L}_C^-$. To show the connection between CLNs and the notions of preferential and rational consequence, we will consider the logics P* and R* to be as before with the relation symbol $\mid\!\sim$ replaced by the connective $\Rightarrow$ in every sentence of the language of extended assertions.

Theorem 6 *Let* $\alpha \in \mathbf{L}_C^-$. $\models_{P*} \alpha$ *iff* $\models_{CT4-} \alpha$.

Theorem 7 *Let* $\alpha \in \mathbf{L}_C^-$. $\models_{R*} \alpha$ *iff* $\models_{CT4D-} \alpha$.

The "only if" half of these theorems is easy to show by demonstrating the validity of the inference rules of P* (R*) in CT4 (CT4D). The "if" portion is proven by showing any P*-satisfiable (R*-satisfiable) sentence is satisfiable in CT4 (CT4D). The interesting case is for conditional sentences and proceeds by constructing a CT4-model (CT4D-model) which satisfies the same extended assertions (or unnested conditionals) as a particular P-model (R-model).

These theorems show that P* and R* are equivalent to the "flat" portions of CT4 and CT4D, respectively, and hence are equivalent to the "flat" portions of the modal systems S4 and S4.3[7]. This is somewhat surprising. That two independently motivated and developed conditional logics for default reasoning should turn out to be equivalent to standard modal systems is rather unexpected.

[6] CPL allows propositional reasoning and the new axiom captures the interaction between propositions and assertions.

[7] *A fortiori* P and R are equivalent to even smaller fragments of these logics, those which only allow "conditional assertions" of the form $\alpha \Rightarrow \beta$.

Concluding Remarks

The framework presented for conditional logics of normality seems very general and intuitively appealing. However, its generality and applicability is reinforced by the fact that logics within the literature, while independently motivated, turn out to be equivalent to the "unnested" fragments of logics developed in this framework.

Viewing CLNs as extensions of CT4 provides a number of conceptual and practical advantages from the standpoint of default reasoning research. This perspective suggests a wide variety of conditional logics, which may determine useful interpretations of "normality". The correspondence with standard modal systems provides a widely-studied, and well-developed and understood, semantics for such logics. Furthermore, this relationship allows the appropriation of a host of readymade results for these logics, results regarding axiomatizability, axiomatic bases, decision procedures and their complexity, and the like. For example, Lehmann (1989) showed that deciding whether $K \models_P \alpha \mid\!\sim \beta$ is a problem in co-NP when K is a finite set of assertions. Using the correspondence between R* and CT4D, and the fact that the problem of deciding S4.3-satisfiability is NP-complete (Ono and Nakimura 1980), we can state the following stronger result.

Corollary 2 *For a finite set of extended assertions* $K \cup \{\alpha\}$, *deciding whether* $K \models_{R*} \alpha$ *is in co-NP.*

As well, the validity problem for CT4D– is in co-NP and that of CT4D is co-NP-hard.

Regarding conditional logics as CLNs not only provides a uniform basis for comparison of such logics, but also extends the sort of reasoning that can be performed using conditional logics, as they typically appear in the literature. More specifically, conditional logics, including those of (Delgrande 1987; Delgrande 1988; Lehmann 1989; Kraus, Lehmann and Magidor 1990; Nute 1984), do not allow nested occurrences of the conditional connective in the language or do not provide an adequate semantic account of such sentences. CLNs, on the other hand, do allow such sentences, which are of some value. For example, the following sentences are theorems of CT4 and its extensions:

$$(A \wedge (A \Rightarrow B)) \Rightarrow B,$$

$$(A \Rightarrow C) \Rightarrow ((A \wedge B) \Rightarrow C).$$

The first sentence appears to embody a rough version of the probabilistic principle of *direct inference* (cf. (Bacchus 1988)) whereby the degree of belief associated with a sentence B, given that A holds, is equal to the conditional probability $P(B|A)$. Here we do not deal with degrees of belief or numerical probabilities, but rather with acceptance or rejection of facts, assuming "normality". So when A and $A \Rightarrow B$ hold, we are willing to conclude B (in normal circumstances).

The latter sentence is important when dealing with a "principle of irrelevance" (see (Delgrande 1988; Lehmann 1989; Pearl 1988)), which states that unless otherwise informed, assume that attributes are irrelevant or independent of one another. This principle allows one to conclude, for instance, that yellow birds normally fly, given that birds normally fly. This inference is problematic for most logics of normality (and probabilistic logics (Pearl 1988)) and requires the meta-inference of irrelevance. This theorem of CT4 can be seen as justifying this principle as being true in the normal state of affairs, and therefore "irrelevance" (or "independence" in probabilistic terms) is just another default inference. This idea has been used to develop an account of default reasoning with conditionals based on the notion of minimal or preferred models (Boutilier 1988). This system is similar in spirit to those of (Delgrande 1988; Lehmann 1989), except the "supported" sentences which characterize independence are derivable from the more standard assumption of normality, whereby one concludes $A \supset B$ from $A \Rightarrow B$. Thus, independence derives from theorems in the form of the second sentence above.

Several avenues for future study of CLNs remain open. One concerns weaker notions of normal implication. These may be investigated by studying logics weaker than CT4, or by allowing weaker definitions of the connective $\Rightarrow$. For instance, in CT4D, $A \Rightarrow B$ is equivalent to

$$\Box\neg A \vee \Diamond(A \wedge \Box(A \supset B))$$

which is weaker than its definition for CLNs in general. This weaker notion of normal implication may be interesting in subsystems of CT4D. In CT4, for example, this definition allows $A \Rightarrow B$ and $A \Rightarrow \neg B$ to be consistent with A, and has possibly useful interpretations. Also, the connection between the logic P and the notion of probabilistic entailment (Adams 1975; Pearl 1988) has been discussed by Lehmann (1989). These results show that probabilistic entailment is reducible to "flat" S4. This suggests a deeper connection, as yet unexplored, between probabilistic entailment and certain modal systems.

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