Weak Representations of Interval Algebras

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Abstract

Ladkin and Maddux [*LaMa87*] showed how to interpret the calculus of time intervals defined by Allen [*All83*] in terms of representations of a particular relation algebra, and proved that this algebra has a unique countable representation up to isomorphism. In this paper, we consider the algebra A_n of n-intervals, which coincides with Allen's algebra for n=2, and prove that A_n has a unique countable representation up to isomorphism for all n≥ 1. We get this result, which implies that the first order theory of A_n is decidable, by introducing the notion of a weak representation of an interval algebra, and by giving a full classification of the connected weak representations of A_n. We also show how the topological properties of the set of atoms of A_n can be represented by a n-dimensional polytope.

1. Introduction

In [*All83*] James Allen introduced a calculus of time intervals conceived as ordered pairs of real numbers. He considered all possible relations between two intervals defined in this way and described the axioms governing the composition of two such relations. He showed that these axioms are summed up in a transitivity table with 144 entries.

In [*LiBe88*] Bestougeff and Ligozat introduce a geometrical object to describe the topological structure of the set of relations of Allen. By using the properties of symmetry of this structure, they improve on a result of [*Zhu87*] and show that the axioms of Allen can be described by a transitivity table with only 43 entries.

Ladkin and Maddux [*Lad87, LaMa87*] observed that the definitions given by Allen can be expressed as defining a particular relation algebra, in the sense of Tarski [*JoTa52*]. They showed that there is (up to isomorphism) a unique countable representation of this algebra. They also showed how to reformulate their results in terms of a first order theory, which is complete, countably categorical, and decidable.

In this paper we concentrate on the algebraic point of view. We show how the results of Allen, Ladkin and Maddux fit into a more general setting, where the objects considered are n-intervals (for n ≥ 1). For each positive integer n, there is a corresponding relation algebra A_n; in the special case where n=2, A_2 is the Allen algebra. A geometrical object $\mathbf{H}_{n,n}$, which is a n-dimensional polytope, describes the topological structure of the set of relations between n-intervals.

We then examine the general problem of describing the representations of A_n, for an arbitrary n. Slightly more generally, we first examine what we call *weak* representations of A_n, which are in fact the objects used in Artificial Intelligence. We show that basically the same result which is true for n=2 holds for any n. More precisely, extending the results cited in [*Lad87*], we define canonical functors between the class of weak representations of A_n and those of A_1. Applying the classification to the special case of representations yields the uniqueness of the countable representation of A_n.

Because this paper is mainly concerned with representations of interval algebras, we do not give here a complete characterization of the polytope $\mathbf{H}_{n,n}$. We show in another paper how the topological constructions can be applied to the non convex intervals introduced by Ladkin and Maddux [*Lad86*].

2. Algebras and representations

2.1. Binary relations

A binary relation R on a set U is by definition a subset R of U × U. If R is a binary relation, the transpose R^t of R is defined by $R^t = \{(x,y) \in U \times U \mid (y,x) \in R\}$. Particular binary relations in U are the empty relation Ø, the total relation U × U, and the identity relation $\Delta = \{(x,x) \mid x \in U\}$.

The composition of two binary relations R_1, R_2, noted $R_1 \circ R_2$, is defined by
$R_1 \circ R_2 = \{(x,y) \in U \times U \mid (\exists z \in U)\ (x,z) \in R_1 \text{ and } (z,y) \in R_2\}$.

2.2. Relation algebras

An algebra A= (A,+, 0, ., 1, ; , 1', $^{-1}$), where +, ., and ; are binary operations on A, $^{-1}$ is a unary operation on A, and 0, 1, and 1' are elements of A is called a relation algebra if the following conditions are satisfied :

- (A,+, 0, ., 1) is a Boolean algebra.
- (x ;y); z = x; (y; z) for any x,y,z ∈ A.
- 1' ; x = x = x; 1' for every x ∈ A.

- The formulas (x ; y) . z = 0, (x^{-1} ; z) . y = 0, and (z ; y^{-1}). x = 0 are equivalent for any x,y,z $\in$ A [*JoTa52*].

The prototypical example of a relation algebra is the set $P(U \times U)$ of binary relations in a set U, with its usual boolean structure, and where ; is composition, 1' the identity relation, and $^{-1}$ is transposition .

2.3. Weak representations

A *representation* of a relation algebra A is a map Φ of A into a direct product of algebras of the form $P(U \times U)$, such that:

(a) Φ is one-to-one;

(b) Φ defines a homomorphism of boolean algebras;

(c) $\Phi(;) = \circ$;

(d) $\Phi(1') = \Delta$;

(e) $\Phi(^{-1}) = {}^{t}$.

Condition (c), in particular, means that for any α, β in A, we have:

(f) $\Phi(\alpha ; \beta) = \Phi(\alpha) \circ \Phi(\beta)$.

More generally, a *weak* representation is defined by dropping condition (a) and replacing condition (f) by the weaker condition :

(g) $\Phi(\alpha ; \beta) \supseteq \Phi(\alpha) \circ \Phi(\beta)$.

If A is a simple algebra, we shall say that a weak representation of A into $P(U \times U)$ is connected if $\Phi(1) = U \times U$.

3. Interval algebras

3. 1. n-intervals and (p,q)-positions

Let (U,<) be any totally ordered set. A n-interval is by definition an ordered n-uple (x_1,..,x_n) of points of U, that is such that $x_1<x_2<...<x_n$. A 1-interval is just a point in U. A 2-interval is an interval in the sense of Allen.

The consideration of n-intervals is motivated by several reasons:

- they are the natural entities for describing processes with a finite number of consecutive phases;
- they can be used to represent unions-of-convex intervals, as defined in Ladkin [*Lad86*];
- they appear in a natural way for the representation of temporal data in natural language, cf. [*BeLi85*, *BeLi89*].

We are primarily interested in the relative positions between two generalized intervals a and b. We define them abstractly as (p,q)-positions:

Definition Let p,q be two positive integers. A *(p,q)-position* π is a map

π:[1,...,p+q] -----> N^+ (N^+ is the set of strictly positive integers) subject to the two conditions:

(i) the image of π is an initial segment of N^+;

(ii) the restrictions of π to [1,...,p] and [p+1,...,p+q] are strictly increasing (hence injective) maps.

We denote by $\Pi_{p,q}$ the set of (p,q)-positions. A convenient way of representing a given (p,q)-position π is by its associated sequence ($\pi(1)$,...,$\pi(p+q)$).

Examples (1) Let p=q=1. Then a and b are two points in T. If a <b, we get the map associating 1 to 1, 2 to 2, which is represented by (1,2); if a=b, we get (1,1); if a > b, we get (2,1).

(2) Let p=q=2. Then we get the 13 elements considered by Allen. One is equality represented by (1,2,1,2). Six others are:

< = (1,2,3,4) (a strictly precedes b);
m = (1,2,2,3) (a meets b);
o = (1,3,2,4) (a overlaps b);
d = (2,3,1,4) (a during b);
e = (2,3,1,3) (a ends b);
s = (1,2,1,3) (a starts b).

Finally, we get six more relations by exchanging the roles of a and b.

(3) The element $\mathbf{1}'_{p,p}$ = (1,...,p,1,...,p) is called the unit position in $\Pi_{p,p}$.

Remark. It can be convenient in some cases to identify the initial segment [1,...,p+q] with the sequence of variables (x_1,...,x_p,y_1,...,y_q); hence π can be considered as mapping the set {x_1,...,x_p,y_1,...,y_q} into N^+.

More generally, for any finite sequence (p,...,s) of integers, we can define the notion of a (p,...,s)-position in a similar way. For example :

Definition A (p,r,q)-position σ is a map

σ:[1,...,p+r+q] -----> N^+ subject to the conditions:

(i) the image of σ is an initial segment of N^+;

(ii) consider the decomposition of [1,...,p+r+q] into three subsegments: initial of length p, middle of length r, terminal of length q; then the restrictions of σ to each of the subsegments is strictly increasing.

We denote by $\Pi_{p,r,q}$ the set of (p,r,q)-positions.

Clearly, we have canonical projections of $\Pi_{p,r,q}$ onto $\Pi_{p,r}$, $\Pi_{p,q}$, and $\Pi_{r,q}$.

Associated inequations

Let π be a (p,q)-position, where Im(π) = [1,...,k]. We can associate to it a set E_π (x,y) of inequations in the following way:

i) for each n such that $\pi^{-1}(n)$ contains two elements x_i and y_j, $E_\pi(x,y)$ contains the equation $x_i=y_j$;

ii) for each n, $1 \le n \le k$, let u_n be an element in $\pi^{-1}(n)$; $E_\pi(x,y)$ contains the inequations $u_1 < u_2 <...<u_k$.

Clearly E_π (x,y) is essentially uniquely defined: because of the equations in (i), different choices in (ii) do not really matter.

Operations on (p,q)-positions

The set of (p,q)-positions is naturally provided with a number of operations. We now examine the principal ones.

Transposition

If π is an element of $\Pi_{p,q}$, the transpose π^t of π is an element of $\Pi_{q,p}$, defined by:

$\pi^t(i) = \pi(p+i)$ for $1 \le i \le q$;

$\pi^t(i) = \pi(i-q)$ for $q+1 \leq i \leq p+q$.

In terms of pairs (a,b) of generalized intervals, transposition corresponds to exchanging the roles of a and b. Clearly, it is an involution, namely $(\pi^t)^t = \pi$ for any π.

Composition

Definition Let $\pi_1 \in \Pi_{p,r}$ and $\pi_2 \in \Pi_{r,q}$; then $\pi_1 \circ \pi_2 = \{pr_{p,q}(\sigma) | \sigma \in \Pi_{p,r,q}, pr_{p,r}(\sigma) = \pi_1, pr_{r,q}(\sigma) = \pi_2\}$. We say that $\pi_1 \circ \pi_2$ is the composition of π_1 and π_2.

Symmetries

If π is an element of $\Pi_{p,q}$ with $Im(\pi) = \{1,...,k\}$, we get an element π^h of $\Pi_{q,p}$ by setting

$$\pi^h(i) = (k+1) - \pi(p+q+1-i).$$

This corresponds to reversing the order on T, and associating to each n-interval $(t_1,...,t_n)$ for the initial order the n-interval $(t_n,...,t_1)$ for the new order.

The symmetry $v = h \circ t$ is an involution on $\Pi_{p,q}$, which commutes to transposition.

Proposition The following properties obtain, for any $\pi_1 \in \Pi_{p,r}$, $\pi_2 \in \Pi_{r,q}$ and $\pi_3 \in \Pi_{q,s}$:

i) $(\pi_1 \circ \pi_2) \circ \pi_3 = \pi_1 \circ (\pi_2 \circ \pi_3)$;
ii) $\pi_1 \circ 1'_{r,r} = \pi_1$ and $1'_{p,p} \circ \pi_1 = \pi_1$;
iii) $1'_{p,p} \in \pi_1 \circ \pi_1^t$ et $1'_{r,r} \in \pi_1^t \circ \pi_1$;
iv) $\pi \in (\pi_1 \circ \pi_2)$ implies $\pi_1 \in (\pi \circ \pi_2^t)$ and $\pi_2 \in (\pi_1^t \circ \pi)$;
v) $(\pi_1 \circ \pi_2)^t = \pi_2^t \circ \pi_1^t$;
vi) $(\pi_1 \circ \pi_2)^v = \pi_1^v \circ \pi_2^v$.

3.2. Constructing interval algebras

We can now use the preceding results to construct a family of relation algebras. Intuitively, $\mathbf{A}_S$ will be the algebra defining the calculus of n-intervals, for n in a fixed subset S of the integers.

Let S be a non empty subset of **N**. We define Π_S as the disjoint sum of all $\Pi_{p,q}$, where p and q belong to S. Let the product $\pi_1 ; \pi_2$ of two elements $\pi_1 \in \Pi_{p,q}$ and $\pi_2 \in \Pi_{p',q'}$ of Π_S be defined as $\pi_1 \circ \pi_2$ if q = p', as the empty set otherwise; let $\mathbf{1'}_S$ be the set of $\mathbf{1'}_{p,p}$, where p belongs to S; let finally transposition on Π_S be defined componentwise. Then we have:

Theorem The system $\Pi_S = (\Pi_S, ;, I_S, ^t)$ is a connected polygroupoid in the sense of Comer [*Com83*]; it is a polygroup if and only if S has a unique element.

Applying to Π_S the standard construction which associates to a polygroupoid its complex algebra, and using the results of [*Com83*], we get simple relation algebras:

Theorem For any subset S of **N**, the complex algebra $\mathbf{A}_S$ of Π_S is a complete, simple, atomic relation algebra, with $0 \neq 1$. Moreover, $\mathbf{A}_S$ is integral if and only S has a unique element.

If $S = \{n\}$, we write $\mathbf{A}_n$ instead of $\mathbf{A}_{\{n\}}$.

In particular, $\mathbf{A}_1$ is the point algebra with 3 atoms. $\mathbf{A}_2$ is Allen's algebra. $\mathbf{A}_{\{1,2\}}$ is a simple algebra with 26 atoms implicitly considered by Vilain in [*Vil82*].

3.3. Associated polytopes

There is a canonical way of associating a labelled polytope $\mathbf{H}_{p,q}$ to the set of (p,q)-relations [*BeLi89*]. A complete description of the construction is given in [*Lig90b*]. Here we just consider the cases p=q=1,2,3.

The 1-dimensional case

Here there are three possible relations between two points: <, >, and equality δ. $\mathbf{H}_{1,1}$ is the graph in Fig. 1.

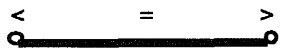

Figure 1: $\mathbf{H}_{1,1}$

It can be interpreted

-in "physical" terms: Suppose U is **R** (the reals); any point x in U defines three regions; two are 1-dimensional, corresponding resp. to y < x and y > x; they meet in a 0-dimensional one, corresponding to δ.

- in terms of permutations: The relative positions of two points x and y are of two kinds; the first kind comprizes general positions < and >, corresponding to two permutations of the list (x,y); the two permutations are joined by a permutation of the adjacent elements x and y, corresponding to collapsing x and y.

The 2-dimensional case

Here $\mathbf{H}_{2,2}$, is as represented in Fig. 2. It is a 2-dimensional polygon, with 6 vertices (0-faces), 6 arcs (1-vertices) and one 2-face. Here again, it has two interpretations:

-A physical interpretation: Suppose (x_1,x_2) and (y_1,y_2) are two intervals in **R**, with (x_1,x_2) entirely on the left of (y_1,y_2); then, the position is <; moving x to the right, we first get $x_2=y_1$; this is position m (meets); going further, we have $y_1<x_2<y_2$; this is position o (overlaps); then, depending on whether x is shorter, longer, or of the same length as y, we either get position s (x starts y), or position e^t (x is ended by y), or δ (x =y); in the first two cases, we then get d (x during y) or d^t (x contains y), respectively; then e or s^t, resp. Going still further, we then get o^t (x is overlapped by y), then m^t, then finally >.

- An interpretation in terms of permutations: Each general position, where all four points x_1,x_2,y_1,y_2, are distinct, corresponds to a permutation of the list (x_1,x_2,y_1,y_2); associate a vertex to each general position, and join two vertices if the two corresponding permutations are related by exchanging two adjacent points. In this manner, one gets the graph underlying $\mathbf{H}_{2,2}$.

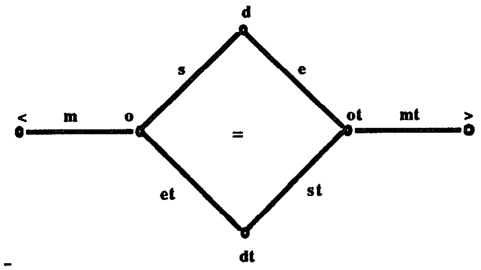

Figure 2: $\mathbf{H}_{2,2}$

The 3-dimensional case

The polyhedron associated to the relations between 3-intervals is represented in Fig. 3.

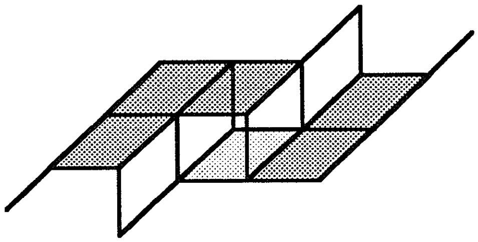

Figure 3: $\mathbf{H}_{3,3}$

The general case

For any pair (p,q) of integers, $\mathbf{H}_{p,q}$, is a polytope of dimension d(p,q)=inf(p,q), which is a connected union of k-cubes, for $k \leq d(p,q)$. In particular, $\mathbf{H}_{n,n}$ is n dimensional. It contains one n-cube, corresponding to equality, and (n-1) (n-1)-cubes (corresponding to collapsing n-1 points of x with (n-1) points of y), etc. It has two canonical symmetries: one corresponds to transposition; the other one to "reversing the time axis". For example, in the case of $\mathbf{H}_{2,2}$, this last symmetry corresponds to the vertical symmetry in Fig. 2.

From general results (cf. [*BeLi89*], [*Lig90a*]), the total number of relations (ie. the total number of faces of $\mathbf{H}_{n,n}$) is h(n,n), where more generally h(p,q) is defined by:

(ii) $h(p,q) = \Sigma_{m \geq 0} (p+q-m)! \, / \, m!(p-m)! \, (q-m)!$.

Remarks

By construction, $\mathbf{A}_n$ has a canonical symmetry associated with "reversing the time axis".

Moreover, because of their interpretation as sets of faces in $\mathbf{H}_{n,n}$, the elements of $\mathbf{A}_n$ can be considered as elements in the Euclidian n-space; so they inherit a topology and a dimension; a consequence of the physical interpretation is that all entries in the transitivity table defining the operation of composition have to be connected elements. More can be shown about them: they are in fact intervals in a suitable distributive lattice.

For A_2, the transitivity table shown in Fig. 4, together with the action of symmetry and transposition, characterizes composition [*LiBe88*].

	<	m	o	et	s	d	dt	e	st	ot	mt	>
<	<	<	<	<	<	< m o s d	<	< m o s d	<	< m o s d	< m o s d	any
m		<	<	<	m	o s d	<	o s d	m	o s d	e = et	
o			< m o	< m o	o	o s d	< m o et dt	o s d	o et dt	o ot d dt = s e st et		
s				< m o	s	d	< m o et dt	d	s = st			
et					o	o s d	dt	e = et				
d						d	any					
dt						o ot d dt = s e st et						

Figure 4: Transitivity table

4. Weak representations of interval algebras

In the remaining part of the paper, we consider the connected weak representations of the integral algebras $\mathbf{A}_n$.

4.1. The 1-dimensional case

The general setup is already apparent in the one-dimensional case. Consider a connected weak representation Φ of A_1: $\quad \Phi: A_1 \text{----}> \mathcal{P}(U \times U)$.

Let R be $\Phi(<)$, $\Delta = \Phi(\delta)$ the diagonal in $U \times U$. Then:

i) R, R^t and Δ are mutually disjoint, their union is $U \times U$;

ii) $R \circ R \subseteq R$;

By (i) and (ii), R is a strict total order.

If Φ is in fact a representation, we also have

v) $\quad R \circ R \supseteq R$;

vi) $\quad R \circ R^t \supseteq \Delta \cup R \cup R^t$;

vii) $R^t \circ R \supseteq \Delta \cup R \cup R^t$.

By (v), R is dense; by (vi) (resp. (vii)) it is unbounded on the right (resp. left).

Conversely, given a strict total order relation R on U, we get a weak representation; if R is dense and unbounded, it is in fact a representation.

A consequence of this fact and the countable categoricity of dense, unbounded total orders is that there exists a unique contable representation of A_1 up to isomorphism.

Example Let $U = \mathbf{R}$, and $R = \{(u,v) \mid u < v\}$. Then we get a representation, which deserves to be called Allen's 1-dimensional representation. The graph $\mathbf{H}_{1,1}$ is dually associated to the set of three regions in the plane $\mathbf{R}^2$ representing R, R^t and Δ.

4.2. The 2-dimensional case

A weak representation of A_2 is defined by a set U, together with six binary relations R, M, O, D, E, S on it, satisfying the following conditions:

(i) the thirteen relations Δ, together with R,M,O,D,E,S and their transposes are a partition of $U \times U$, ie. they are mutually disjoint and cover $U \times U$;

(ii) the composition of two relations is given by the transitivity table in Fig. 4 together with the identities (4.1) and (4.2).

In [*BeLi89*] this data is called a connected system of intervals in the sense of Allen.

Example Let $U = \{i_1, i_2\}$, $O = (\{i_1, i_2\})$, $M = D = E = S = R = \emptyset$. This is a connected weak representation of A_2, as it is easy to verify. It corresponds to Fig. 5. It is not a representation.

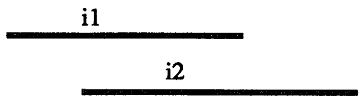

Figure 5

4.3. The general case

Consider the general case of A_n, with $n \geq 1$. Let Φ be a connected weak representation of A_n into $P(U \times U)$:

Φ: A_n----> $P(U \times U)$.

For each element π of $\Pi_{n,n}$, which can be considered as an atom of A_n, $\Phi(\pi)$ is a binary relation R_π on U. We have :

(i) (R_π), for $\pi \in \Pi_{n,n}$, is a partition of $U \times U$.

(ii) for any $\pi, \pi' \in \Pi_{n,n}$, $R_\pi \circ R_{\pi'} \subseteq R_{\pi;\pi'}$.

Recall the interpretation of the elements of $\Pi_{n,n}$ in terms of maps from the set $\{x_1,...,x_n,y_1,...,y_n\}$ into N^+. We consider the following elements in A_n, for $1 \leq i,j \leq n$:

$a_{i,j}$ is the sum of all π such that $\pi(x_i) = \pi(y_j)$.
$b_{i,j}$ is the sum of all π such that $\pi(x_i) < \pi(y_j)$.

Proposition

(i) $a_{i,i} \geq 1'_{n,n}$.
(ii) $a^t_{i,j} = a_{j,i}$.
(iii) $a_{i,j} ; a_{j,k} = a_{i,k}$.
(iv) $a_{i,j} ; b_{j,k} ; a_{k,l} = a_{i,l}$.
(v) $b_{i,j} ; b_{j,k} = b_{i,k}$.
(vi) $b_{i,j} . b^t_{j,i} = 0$.
(vii) $1 = b_{i,j} + b^t_{i,j} + a_{i,j}$.
(viii) if $i < j$, then $1'_{n,n} \in b_{i,j}$.

5. Classifying weak representations

5.1. From weak representations of A_1 to weak representations of A_n

Let (B,<) be a weak representation of A_1 *ie.* a strict total order. Let U be the set of n-intervals of B, which is non empty if B has more than n elements; we define a weak representation $G_n((B,<))$ in the following way:

For each atom π of A_n:

$R_\pi = \{ (x,y) \in U \times U \mid x,y \text{ satisfy } E_\pi(x,y)\}$.

Then associating R_π to π defines a connected weak representation of A_n.

It is easily shown that G_n in fact defines a functor from the category (in a suitable universe) of strict total orders to the category of connected weak representations of A_n.

We now show how to define a functor F_n in the opposite direction.

5.2. From weak representations of A_1 to weak representations of A_n

The construction

Let $\Phi = (U,(R_\pi))$ be a connected weak representation of A_n. Consider the disjoint sum $\mathbf{U} = U_1 \oplus ... \oplus U_n$ of n copies $U_1,..., U_n$ of U (indexed by $i = 1,...,n$).

a) Define on $\mathbf{U}$ the relation :

$u \equiv v$, where $u \in U_i$, $v \in U_j$ iff $(u,v) \in \Phi(a_{i,j})$.

Then, because of (i,ii,iii) of the proposition in 4.3 and the fact that Φ is a weak representation, $\equiv$ is an equivalence relation on $\mathbf{U}$. Let B be the quotient set $\mathbf{U} / \equiv$.

b) Define on $\mathbf{U}$ the relation:

$u < v$, where $u \in U_i$, $v \in U_j$ iff $(u,v) \in \Phi(b_{i,j})$.

Using the same proposition as before, we get by (iii) that < defines a binary relation on B; by (v,vi,vii), and (vii), this relation (still noted <) is transitive, irreflexive, and total. Hence $(B,<) = F_n(U,(R_\pi))$ is a strict total order.

Moreover, the canonical injection of U into each U_i defines a map β_i of U into B. By (viii), the sequence $(\beta_1(u),...,\beta_n(u))$ is a n-interval. Hence we have a canonical map from U into the set of n-intervals on B. In fact, this map defines a morphism of weak representations

$\eta_n : (U,(R_\pi))$ ---->$(G_n \circ F_n)$ $(U,(R_\pi))$.

In the opposite direction, it is easily seen that, starting with a total order (B,<) with at least n elements, applying G_n, then F_n, we get a canonical isomorphism of total orders:

ε_n : (B,<) ----> $(F_n \circ G_n)$ (B,<).

Theorem The situation $(F_n, G_n, \eta_n, \varepsilon_n)$ is an adjunction between categories. The functor F_n is left-adjoint to F_n .

In particular, the canonical map η_n is a closure operation. We can define:

Definition A connected weak representation of A_n is *closed* if and only if the canonical map η_n is an isomorphism. The closure of $(U,(R_\pi))$ is $(G_n \circ F_n)$ $(U,(R_\pi))$.

Intuitively, a closed weak representation is one which contains all the n-intervals it implicitly defines. For example, the weak representation of Fig. 5 is not closed, since it implicitly defines four boundaries, hence six intervals. If n=1, every weak representation is closed.

In the general case, a connected weak representation is canonically embedded into its closure by η_n.

Using general results about adjunctions, we get from the preceding theorem:

Corollary The pair of functors (F_n ,G_n) defines an equivalence of categories between the categories of *closed* connected weak representations of $\mathbf{A_n}$ and the category of strict total orders with at least n elements.

Hence, we can give a full classification of the connected weak representations of $\mathbf{A_n}$; in summary:
- the fact for weak representations of having isomorphic closures define classes of equivalence;
- each class contains (up to isomorphism) a closed representative;
- closed representatives are characterized by their underlying point sets, which are strict total orders.

In the special case where n=2, this classification was obtained in [*Lig86*].

5. 3. Representations of $\mathbf{A_n}$

Representations are special cases of weak representations. Moreover:

Proposition If Φ is a representation of $\mathbf{A_n}$, then it is a *closed* connected weak representation of $\mathbf{A_n}$.

This is proved as follows: consider n elements $u_1,..., u_n$, in U such that $\beta_{\sigma(1)}(u_1),...,\beta_{\sigma(n)}(u_n)$ is a n-interval in B, for some map σ of [1,...,n] into itself. Using the fact that Φ is a representation, we can find w_1 in U such that $\beta_1(w_1) = \beta_{\sigma(1)}(u_1)$ and $\beta_2(w_1) =\beta_{\sigma(2)}(u_2)$; hence we can replace u_1 and u_2 by w_1 and get the same n-interval ; after (n-1) steps, we get $w=w_{n-1}$ in U such that $\beta_1(w),...,\beta_n(w)$ is the n-interval we started with.

By the preceding results, F_n and G_n define an equivalence of categories between the representations of $\mathbf{A_n}$ and those of $\mathbf{A_1}$, that is, dense, unbounded linear orders. That is, by Cantor's theorem:

Theorem There is a unique countable representation of $\mathbf{A_n}$, up to isomorphism.

Since $G_n((\mathbf{Q},<))$ is such a representation, any other one is isomorphic to it.

This implies that the first order theory associated to $\mathbf{A_n}$ is countably categorical. Since it is finitely axiomatisable, because $\mathbf{A_n}$ is finite, it is in fact decidable:

Corollary The first order theory of $\mathbf{A_n}$ is decidable.

This result was also obtained independently by Ladkin and McKenzie.

6. Summary

We have generalized the calculus of time intervals defined by Allen to a calculus of n-intervals. We have shown how this generalization can be expressed in terms of relation algebras $\mathbf{A_S}$, whose atoms have a natural topological structure representable by a polytope $\mathbf{H_{p,q}}$. We introduce the notion of weak representation of an interval algebra, which are the objects of interest in Artificial Intelligence, and give a full classification of the connected weak representations of $\mathbf{A_n}$. We deduce from these results the fact that $\mathbf{A_n}$ has a unique countable representation, and that its first order theory is decidable.

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