Maximal Tractable Subclasses of Allen's Interval Algebra: Preliminary Report

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Abstract

This paper continues Nebel and Bürckert's investigation of Allen's interval algebra by presenting nine more maximal tractable subclasses of the algebra (provided that $P \neq NP$), in addition to their previously reported *ORD-Horn* subclass. Furthermore, twelve tractable subclasses are identified, whose maximality is not decided. Four of these can express the notion of *sequentiality* between intervals, which is not possible in the ORD-Horn algebra. The satisfiability algorithm, which is common for all the algebras, is shown to be linear.

Introduction

For specifying qualitative temporal information about relations between intervals, Allen's interval algebra (Allen 1983) is often considered a convenient tool. However, due to its expressiveness (the satisfiability problem is NP-complete (Vilain, Kautz, & van Beek 1989)), it is unlikely that there will be a polynomial-time algorithm for reasoning about the full algebra. Trying to overcome this, several tractable fragments of the algebra have been identified (e.g. (Nebel & Bürckert 1995; van Beek 1989; Golumbic & Shamir 1993)), of which the largest known is Nebel and Bürckert's *ORD-Horn* algebra (Nebel & Bürckert 1995). Furthermore, this algebra has been proved (Nebel & Bürckert 1995) to be the *unique maximal* algebra containing all the basic relations, comprising approximately 10 percent of the full algebra.

None of these algebras, however, are capable of expressing the notion of *sequentiality*, which is that of specifying that some intervals have to occur in sequence in time, without any overlap. This is required e.g. in some cases of reasoning about action (Sandewall 1994). The maximality result of the ORD-Horn algebra then implies that the requirement that the algebra contain all the basic relations has to be sacrificed. Golumbic and Shamir (Golumbic & Shamir 1993) come close to expressing sequentiality, but require that *any* two intervals are related.

In this paper, we exploit a simple graph algorithm, similar to that of van Beek (van Beek 1992), and show that we can construct 21 algebras for which this algorithm solves satisfiability in linear time, and furthermore, that four of these can express sequentiality, and nine of them are *maximal* tractable algebras (assuming $P \neq NP$, which we take for true in the rest of the paper).

The structure of the paper follows. First we present the necessary background material, about Allen's interval algebra, and some results on the ORD-Horn algebra. Then, the concepts of "acyclic" and "DAG-satisfying" relations are introduced, after which the main results of the new tractable algebras are presented. Finally, a discussion concludes the paper.

Allen's Interval Algebra

Allen's interval algebra (Allen 1983) is based on the notion of *relations between pairs of intervals.* An interval X is represented as an ordered pair (X^-, X^+) of real numbers with $X^- < X^+$, denoting the left and right endpoints of the interval, respectively, and relations between intervals are composed as disjunctions of *basic interval relations*, which are those in Table 1. Such disjunctions are represented as *sets* of basic relations, but using a notation such that e.g. the disjunction of the basic interval relations $\prec$, m and $\mathsf{f}^{\smile}$ is written $(\prec\ \mathsf{m}\ \mathsf{f}^{\smile})$. Thus, we have that $(\prec\ \mathsf{f}^{\smile}) \subseteq (\prec\ \mathsf{m}\ \mathsf{f}^{\smile})$.

The algebra is provided with the operations of *converse, intersection and composition* on interval relations, but we shall only need the converse operation. The converse operation takes an interval relation i to its converse $i^{\smile}$, obtained by inverting each basic relation in i, by exchanging X and Y in the endpoint relations of Table 1.

By the fact that there are thirteen basic relations, we get $2^{13} = 8192$ possible relations between intervals in the full algebra. We denote the set of all interval relations by $\mathcal{A}$. Subclasses of the full algebra are obtained by considering subsets of $\mathcal{A}$.

Although there are several computational problems associated with Allen's interval algebra, this paper focuses on the problem of *satisfiability* (*ISAT*) of a set of interval variables with relations between them, i.e. deciding whether there exists an assignment of intervals

Basic relation		Example	Endpoints
X before Y	$\prec$	xxx	$X^+ < Y^-$
Y after X	$\succ$	yyy	
X meets Y	m	xxxx	$X^+ = Y^-$
Y met-by X	m˘	yyyy	
X overlaps Y	o	xxxx	$X^- < Y^- < X^+$,
Y overl.-by X	o˘	yyyy	$X^+ < Y^+$
X during Y	d	xxx	$X^- > Y^-$,
Y includes X	d˘	yyyyyyy	$X^+ < Y^+$
X starts Y	s	xxx	$X^- = Y^-$,
Y st.-by X	s˘	yyyyyyy	$X^+ < Y^+$
X finishes Y	f	xxx	$X^+ = Y^+$,
Y fin.-by X	f˘	yyyyyyy	$X^- > Y^-$
X equals Y	$\equiv$	xxxx	$X^- = Y^-$,
		yyyy	$X^+ = Y^+$

Table 1: The thirteen basic relations. The endpoint relations $X^- < X^+$ and $Y^- < Y^+$ that are valid for all relations have been omitted.

on the real line for the interval variables, such that all of the relations between the intervals hold. Such an assignment is said to be a *model* for the interval variables and relations. For $\mathcal{A}$, we have the following result.

Proposition 1 $ISAT(\mathcal{A})$ is NP-complete.
Proof: See Vilain et al (1989). □

The following auxiliary concept shall be needed.

Definition 2 ("satisfied as")
Let $\mathcal{I}$ be an instance of the satisfiability problem, M a model for $\mathcal{I}$ and $r \in \mathcal{A}$ a relation between two interval variables I_1 and I_2 in $\mathcal{I}$. Then r is said to be *satisfied as* r' for any relation $r' \in \mathcal{A}$, such that $I_1 r' I_2$ is satisfied in M. □

Example 3 Let I_1, I_2 be interval variables related by $I_1(\prec\ \succ)I_2$, and M a model where I_1 is interpreted as $[1,2]$ and I_2 as $[3,4]$. Then in M, $(\prec\ \succ)$ is satisfied as $(\prec)$, but also as $(\prec\ \succ)$. □

The ORD-Horn Subclass

Nebel and Bürckert (Nebel & Bürckert 1995) identify a subclass of the interval algebra, having the property that it is a maximal subclass containing all the basic interval relations, for which satisfiability can be solved using a polynomial-time algorithm, and is in fact the *unique* such maximal class[1]. This algebra, the *ORD-Horn* algebra, contains 868 relations, and thus covers slightly more than 10 percent of $\mathcal{A}$.

One of the main tools for analysing the ORD-Horn subclass is a *closure operation* on subclasses of the algebra, which preserves tractability.

[1]The uniqueness is proved under the assumption that the subclass shall contain the empty relation ( ) and the full relation ($\prec\ \succ$ m m˘ o o˘ d d˘ s s˘ f f˘ $\equiv$).

Definition 4 (Closure)
Let $S \subseteq \mathcal{A}$. Then we denote by $\overline{S}$ the *closure* of S under converse, intersection and composition, i.e. the least subalgebra containing S closed under the three operations. □

The key result for extrapolating tractability results is the following.

Proposition 5 Let $S \subseteq \mathcal{A}$. Then $ISAT(S)$ is polynomial iff $ISAT(\overline{S})$ is.
Proof: See Nebel and Bürckert (1995). □

The following stronger result follows from the proof of Proposition 5, although it is not stated explicitly in (Nebel & Bürckert 1995).

Corollary 6 Let $S \subseteq \mathcal{A}$. If $ISAT(S) \in O(f(n))$ for some $f(n) \in \Omega(n)$, then also $ISAT(\overline{S}) \in O(f(n))$.
Proof: In the proof of Proposition 5, the amount of time needed to transform a problem in $\overline{S}$ to a problem in S is linear in the size of the problem. □

Acyclic and DAG-satisfying Relations

This section introduces some auxiliary notions and results needed for defining the new algebras, and proving their properties.

Note that a satisfiability problem instance of the interval algebra can be represented by a directed graph, where the nodes are intervals variables, and the arcs are labelled by relations between intervals. Thus, in the rest of this paper, we let P be an arbitrary $ISAT(X)$ instance for $X \subseteq \mathcal{A}$, and $G = \langle V, E\rangle$ the labelled directed graph representing it.

Definition 7 (Acyclic relation)
A relation r is said to be an *acyclic relation* iff any cycle in any G with r on every arc is never satisfiable. □

Example 8 $\prec$ is an acyclic relation, and so is $(\prec$ m$)$. □

Corollary 9 Let r be an acyclic relation. Then every relation $r' \subseteq r$ is acyclic.
Proof: Since taking subsets of r constrains satisfiability further, the result follows. □

Corollary 10 Let r be an acyclic relation, and A such that $A \subseteq \{r' | r' \subseteq r\}$. Then, any cycle in G where every arc is labelled by some relation in A is unsatisfiable.
Proof: Same argument as in Corollary 9. □

Definition 11 (Maximal acyclic relation)
An acyclic relation r for which there is no acyclic relation $r' \supset r$, is said to be a *maximal acyclic relation*. □

In Proposition 13, we shall list all possible maximal acyclic relations.

Proposition 12 Let r be an acyclic relation, and A, A' sets such that $A \subseteq \{r' | r' \subseteq r\}$, and $A' = \{a \cup (\equiv) | a \in A\}$. Then, every cycle C labelled by relations in $A \cup A'$ is satisfiable iff it contains only relations from A', and, furthermore, all relations in the cycle have to be satisfied as $\equiv$.

Proof:

$\Rightarrow$) Suppose that a cycle C is satisfiable, and that it contains some relation from A. Induction on the number n of arcs in the cycle. For $n = 1$, we get a contradiction by the assumption. So, suppose for the induction that C contains $n+1$ arcs. Let M be a model for the relations in C. We cannot have that every relation in C is satisfied in M as some relation in A, by Corollary 10. Thus, some relation r' in C has to be satisfied as $\equiv$. But then we can collapse the two interval variables connected by r' to one interval variable, and we have a cycle of size n containing a relation from A. This contradicts the induction hyphothesis.

$\Leftarrow$) Suppose that a cycle C contains only relations in A'. Then C can be satisfied by choosing $\equiv$ on every arc, thus forcing the satisfying intervals to be identical. □

Next, we find all possible acyclic relations.

	m	m$^\smile$	$\prec$	$\succ$	o	o$^\smile$
l	+	−	+	−	+	−
r	+	−	+	−	+	−

	d	d$^\smile$	f	f$^\smile$	s	s$^\smile$
l	−	+	−	+	=	=
r	+	−	=	=	+	−

Table 2: The effect of relations on interval endpoints.

Proposition 13 The only maximal acyclic relations in $\mathcal{A}$ are (m $\prec$ o d$^\smile$ f$^\smile$ s), (m $\prec$ o d$^\smile$ f$^\smile$ s$^\smile$), (m $\prec$ o d f s), (m $\prec$ o d f$^\smile$ s), and their respective converses.

Proof: Obviously, a maximal acyclic relation cannot contain both a basic relation and its converse, and thus cannot contain $\equiv$. One consequence of this is that a maximal acyclic relation cannot contain more than six basic relations. So, if the above relations are shown to be acyclic, then they are also maximal.

Now, consider Table 2, which extracts from Table 1 how the basic relations (except for $\equiv$) relate the ending points of intervals. The table is to be read as follows. Suppose that the intervals i_1 and i_2 are related by some basic relation b, i.e. $i_1(b)i_2$, and consider the l row entry for b.

- If it is +, then the starting point of i_2 must be strictly *after* the starting point of i_1
- If it is − then the starting point of i_2 must be strictly *before* the starting point of i_1
- If it is =, then the starting points of i_1 and i_2 have to coincide.

Similarly, the r row states the same information for the ending points.

Now consider the l row. If we choose a relation r' to contain exactly the basic relations which have a + there, we know that r' will be an acyclic relation, because if in a cycle, the left ending points of the intervals have to increase at every arc, it cannot be satisfied. In addition to those basic relations in r', we can include in r' one basic relation b' which has a = in the l row, yielding the relation r'', since then, a cycle labelled by r'' on every arc has to be satisfied as b' on every arc (otherwise, we would get a contradiction, by strictly increasing starting point values). But since neither of s and s$^\smile$ has a = in their r row, this is impossible. This gives us two choices of acyclic relations, which are the two first ones listed.

Symmetrically, by inspecting the r row, we see that we get the next two relations listed. Finally, by taking the − entries instead of the + entries, we get the converse relations of the listed ones.

It remains to prove that these are the only maximal acyclic relations. So, suppose that some acyclic relation e is not a subset of (or equal to) any of the relations in the statement of the proposition. First, note that e cannot be a basic relation, since every basic acyclic relation is included in some of the listed relations. Thus, e has to contain at least two distinct basic relations b_1 and b_2. Without loss of generality (using Corollary 9), we have that $e = (b_1 b_2)$.

By the choice of the listed relations, b_1 and b_2 must have opposite signs either in their l or r rows (or both). Suppose that b_1 and b_2 do not have opposite signs in their l row, i.e. that either they have the same sign, or at least one of them has a =. If both of them have a =, they have to be s and s$^\smile$, which is impossible. If they have the same sign, which is not =, then they are included in one of the listed relations, by definition. If at least one of them, say b_1, has = there, i.e. b_1 is either s or s$^\smile$, we see that for any basic acyclic relation c, c and b_1 occur together in some of the listed relations (or their converses), and in particular, this holds when c is b_2. Thus b_1 and b_2 have to have opposite signs in the l row. Symmetrically, b_1 and b_2 must have opposite signs also in the r row.

Now, the only remaining choice of b_1 and b_2, for which the signs of the l and r rows do not coincide, is for the basic intervals d and d$^\smile$. But trivially, these cannot together be part of any acyclic relation, and thus b_1 and b_2 have to be chosen such without loss of generality, b_1 has + in both its l and r rows, and similarly for b_2, − in both its l and r rows. Obviously, also every choice when b_1 and b_2 are converses is impossible.

This leaves us with six relations to check: (m $\succ$), (m o$^\smile$) and ($\prec$ o$^\smile$) and their converses, and it is enough to check the first three ones due to symmetry. Now, it is easy to construct satisfiable cycles using relations containing either of these relations. □

Definition 14 (DAG-satisfying relation) A basic relation b is said to be *DAG-satisfying* iff any DAG (directed acyclic graph) labelled only by relations containing b is satisfiable. □

Now, we shall classify the DAG-satisfying relations, after an auxiliary definition.

Definition 15 (Minimal node)
Let G be a DAG. Then a node v in G is said to be *minimal* iff there are no arcs which end in v. □

Proposition 16 The basic relations $\prec$, d, o, f, s and $\equiv$, and their respective converses, are DAG-satisfying.
Proof: We show that any DAG labelled only by relations containing a fixed basic relation b, when b is one of the above relations, is satisfiable with some model M. Indeed, we prove the stronger result that we can choose the satisfying M such that

- when b is d or o, all intervals overlap at some open interval
- when b is f, every interval has the same right ending point
- when b is s, every interval has the same left ending point
- when b is $\equiv$, all intervals are identical.

The result for the converse relations follows by an analogous construction.

Induction on the number of nodes in the DAG G. The case when $n = 0$ is trivial. Suppose that G has $n + 1$ elements, and remove a minimal node g. By induction, the remaining graph G' is satisfiable by a model M satisfying the required condition for the relation b. We shall now construct a model M' of G, which agrees with M on every interval variable in G'. The satisfying interval, denoted s, for the remaining interval variable represented by the node g, is chosen as follows, depending on b and M. Note that M satisfies the above conditions.

- When b is $\prec$, choose s to be any interval strictly before every interval in M
- When b is d, choose s to be an interval which is within the common open interval of the intervals in M
- When b is o, choose s to have its left ending point to the left of every interval in M, and its right ending point to be in the middle of the common interval of the intervals in M
- When b is f, choose s to have the same right ending point as the intervals in M, and the left ending point to be in the middle of the interval in M which has the rightmost left ending point
- When b is s, choose s to have the same left ending point as the intervals in M, and the right ending point to be in the middle of the interval in M which has the leftmost right ending point
- When b is $\equiv$, choose s to be identical to the intervals in M.

Obviously, M' is a model of G satisfying the requirements. □

We may note that m is not DAG-satisfying: take interval variables I_1, I_2 and I_3 related by $I_1(\mathsf{m})I_2$, $I_2(\mathsf{m})I_3$ and $I_1(\mathsf{m})I_3$. This is a DAG which is not satisfiable.

Tractable Algebras

Now we define the class of algebras which is to be analysed.

Definition 17 (The subclasses $A(r,b)$)
Let b be a DAG-satisfying basic relation and r an acyclic relation. First define the subclasses $A_1(b)$ and $A_2(r,b)$ by $A_1(b) = \{r' \cup (b\ b^{\smile}) | r' \in \mathcal{A}\}$ and $A_2(r,b) = \{r' \cup (\equiv b) | r' \subseteq r\}$. Then define the subclass $A(r,b)$ by $A(r,b) = A_1(b) \cup A_2(r,b)$. □

Corollary 18 Let r be an acyclic relation, $r' \subseteq r$, and b be some DAG-satisfying basic relation. Then $A(r',b) \subseteq A(r,b)$.
Proof: By the construction of $A(r,b)$. □

Thus, by Corollary 18 and Corollary 9, it is sufficient to use maximal acyclic relations when constructing the algebras $A(r,b)$. We now state the algorithm which we shall show in Theorem 23 solves satisfiability for these algebras, after a short definition.

Definition 19 (Strong component)
A subgraph C of a graph G is said to be a *strong component* of G iff it is maximal such that for any nodes a, b in C, there is always a path in G from a to b. □

Algorithm 20 (*ISAT*$(A(r,b))$)
Let G' the graph obtained from G by removing arcs which are not labelled by some relation in $A_2(r,b)$.

Find all strong components C in G'
for every arc e in G whose relation
  does not contain $\equiv$
  if e connects two nodes in some C **then**
    Reject
  endif
endfor
Accept

□

In fact, this algorithm is very similar to that of van Beek (van Beek 1992), improved and used by Gerevini et al (1993), but here used on intervals instead of points.

We now state a simple result which holds for directed graphs in general.

Proposition 21 Let G be irreflexive[2] with an acyclic subgraph D. Then those arcs of G which are not in D can be reoriented so that the resulting graph is acyclic.
Proof: Induction over the number n of nodes in G that are not in D. For $n = 0$, the result is trivial. So, suppose that there are $n+1$ nodes in G that are not in

[2] A graph is said to be irreflexive if it has no arcs from a node v to the node v.

D, and remove an arbitrary node v of these, resulting in the graph G'. By induction, the arcs of G' can be reoriented to form a DAG G''. Now add the node v to G'' obtaining G''', and reorient any arcs between G'' and v (in either direction) towards v. Since the graph is irreflexive, no cycles are added by this operation, so G''' is acyclic. □

We now specialise this result.

Corollary 22 Let G be irreflexive with an acyclic subgraph D, b a DAG-satisfying basic relation, and let the arcs of D be labelled by relations containing b, and the arcs not in D be labelled by relations containing both b and $b^{\smile}$. Then G is satisfiable.

Proof: Reorient the arcs of G like in Proposition 21, yielding a DAG G'. In this construction, whenever an arc is reoriented, also invert the relation on that arc, so that G' is satisfiable iff G is. By the construction, only arcs containing *both* b and $b^{\smile}$ have been reoriented, so every arc in the DAG G' contains b and, thus, since b is DAG-satisfying, G' is satisfiable, and consequently, also G is satisfiable. □

Theorem 23 Algorithm 20 correctly solves satisfiability for $A(r, b)$.

Proof: Suppose the algorithm finds a strong component of G' (which then may contain only relations in $A_2(r, b)$), in which two of the nodes are connected by an arc e, labelled by a relation r', which does not contain $\equiv$. Then there exists a cycle C, in which the relation of every arc contains $\equiv$, such that e connects two nodes in that C (e is not included in the cycle itself). By the fact that the set $A_2(r, b)$ is a subcase of A' in Proposition 12, C can be satisfied only by choosing the relation $\equiv$ on every arc in C, and since r' does not admit the relation $\equiv$, C is unsatisfiable.

Otherwise, every such strong component can be collapsed to one interval, removing all arcs which would start and end in the collapsed interval, retaining the same condition for satisfiability, using the same argument as above. After the collapsing, the subgraph obtained by considering only arcs labelled by relations in $A_2(r, b)$ will be acyclic. Since by construction every relation in $A_2(r, b)$ contains the relation b, and the remaining arcs are labelled by relations containing both b and $b^{\smile}$, the graph is satisfiable by Corollary 22 (note that the graph will be irreflexive, since every node is contained in some strong component). □

Theorem 24 Algorithm 20 runs in linear time in the size of G (which is $|V| + |E|$).

Proof: Strong components can be found in linear time (see e.g. (Baase 1988)), and the remaining test can also be done in linear time. □

Corollary 25 Satisfiability of $\overline{A(r, b)}$ is solvable in linear time.

Proof: From Theorem 24 and Corollary 6. □

Using Proposition 13 we can construct the $A(r, b)$'s to get twenty $\overline{A(r, b)}$'s, by choosing r to be one of the maximal acyclic relations above, and choosing b to be an element in the chosen r except for m or $\mathsf{m}^{\smile}$. The reason why we get only twenty combinations is that the closure of an algebra is closed under the converse operation. Note that this exhausts the choices of parameters in Algorithm 20 except for the degenerate case when every relation contains $\equiv$, by Corollary 18 and Proposition 13. That case is covered in Definition 28.

Proposition 26 Each of the $\overline{A(r, b)}$ algebras contains 2178 elements, and each contains exacly three basic relations, namely $\equiv$, b and $b^{\smile}$. Furthermore, all of these twenty algebras are distinct.

Proof: By generating the algebras using the utility **aclose** (Nebel & Bürckert 1993). □

We have four algebras $A(r, \prec)$, all containing the relations $(\equiv)$, $(\prec)$, $(\prec \equiv)$, $(\succ)$, $(\succ \equiv)$ and $(\prec \succ)$, expressing the notion of *sequentiality*, which is useful for solving reasoning problems under the assumption that actions always occur in sequence (Sandewall 1994). Note that the ORD-Horn algebra does *not* contain the relation $(\prec \succ)$, and thus cannot express sequentiality.

Proposition 27 The eight algebras $\overline{A(r, b)}$ which have $b \in \{\mathsf{f}, \mathsf{s}\}$ are maximal tractable algebras.

Proof: By running the utility **atry** (Nebel & Bürckert 1993), which generates minimal extensions of subclasses by adding a relation and computing the closure of that class. For these algebras, no nontrivial extensions were found (i.e. every extension results in $\mathcal{A}$), and since $ISAT(\mathcal{A})$ is NP-complete by Proposition 1, the result follows by Proposition 5. □

For the remaining algebras, we do not have a proof of maximality.

Finally, we cover the degenerate case when every relation contains $\equiv$.

Definition 28 (The algebra $A_{\equiv}$)
Define the algebra $A_{\equiv}$ to contain every relation that contains $\equiv$, and the empty relation ( ). It is easy to see that $A_{\equiv}$ contains 4097 elements. □

For this case, Algorithm 20 collapses to the following trivial algorithm.

Algorithm 29 (Satisfiability in $A_{\equiv}$)

```
if some arc is labelled by ( ) then
    Reject
else
    Accept
endif
```

□

Proposition 30 Algorithm 29 correctly solves satisfiability in $A_{\equiv}$ in linear time. Furthermore, it is a maximal tractable subclass of $\mathcal{A}$.

Proof: Correctness and complexity results are trivial. The maximality follows by running the utility **atry** (Nebel & Bürckert 1993), which generates no nontrivial extensions of the algebra. □

The algebra $A_{\equiv}$ certainly raises doubts about whether the *size* of a subalgebra can be used to judge its usefulness, since its expressivity is obviously too weak to be of any use.

Discussion

Nebel and Bürckert (Nebel & Bürckert 1995) argue that the ORD-Horn algebra is an improvement in quantitative terms over previous approaches, since it covers more than 10 percent of the full algebra. Certainly this is a valid argument only because the ORD-Horn algebra *includes* the previous algebras; otherwise we have a counterexample in the $A_{\equiv}$ algebra, which is much larger than the ORD-Horn algebra, but is clearly of no use. We may mention that the 21 algebras of this paper covers about 92 percent of $\mathcal{A}$, and that there are only two relations in the ORD-Horn algebra which are not elements of any of the algebras: (m) and $(\mathsf{m}^{\smile})$. From a cognitive perspective, the exclusion of these relations is not a serious restriction, as Freksa (Freksa 1992) notes, since they are not likely to occur in any context reasoning about e.g. perception of the physical world.

It is also argued by Nebel and Bürckert (Nebel & Bürckert 1995) that a useful algebra should contain all the basic relations, since otherwise, complete knowledge cannot be specified. However, since the unique maximality of the ORD-Horn class shows that there exists no tractable subalgebra which contains both all the basic relations and the relations expressing *sequentiality* (notably the $(\prec\,\succ)$ relation), this argument fails. Furthermore, four of our algebras can indeed express this sequentiality requirement, which underlies many systems (see e.g. (Sandewall 1994)).

Clearly, the question of maximality of the algebras needs to be settled. Also, as a long-term goal, it would be useful to classify *all* maximal tractable subalgebras of the full algebra, since then an application specifying a set of intervals could search for the best algebra to use, or otherwise report that no such algebra exists. Since there are 2^{8192} subsets of the full algebra, the task is clearly nontrivial, even using computer-supported proof methods.

Conclusions

We have identified 21 new tractable fragments of Allen's interval algebra, of which nine have been proved maximal tractable. Further, we have presented a linear time algorithm for deciding satisfiability of these. In addition, all the algebras are considerably larger (in quantity) than the ORD-Horn subalgebra, but thus cannot contain all the basic relations. Also, four of the algebras can express the relations $(\equiv)$, $(\prec)$, $(\prec\,\equiv)$, $(\succ)$, $(\succ\,\equiv)$ and $(\prec\,\succ)$ (in addition to the "nonrelation"), which is necessary and sufficient for expressing the notion of sequentiality.

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