Splitting a Default Theory

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Abstract

This paper presents mathematical results that can sometimes be used to simplify the task of reasoning about a default theory, by "splitting it into parts." These so-called Splitting Theorems for default logic are related in spirit to "partial evaluation" in logic programming, in which results obtained from one part of a program are used to simplify the remainder of the program. In this paper we focus primarily on the statement and proof of the Splitting Theorems for default logic. We illustrate the usefulness of the results by applying them to an example default theory for commonsense reasoning about action.

Introduction

This paper introduces so-called Splitting Theorems for default logic, which can sometimes be used to simplify the task of reasoning about a default theory, by "splitting it into parts." These Splitting Theorems are related somewhat, in spirit, to "partial evaluation" in logic programming, in which results obtained from one part of a program are used to simplify the remainder of the program.[1] In fact, the Splitting Theorems for default logic closely resemble the Splitting Theorems for logic programming introduced in (Lifschitz & Turner 1994), despite complications due to the presence of arbitrary formulas in default theories.[2] Similar results for autoepistemic logic can be found in (Gelfond & Przymusińska 1992). Related, independently obtained, results for default logic, restricted to the finite case, appear in (Cholewinski 1995).

In this paper we focus on the statement and proof of the Splitting Theorems for default logic.[3] We illustrate the usefulness of the results by applying them to an example default theory for reasoning about action. An extended application of the Splitting Theorems to default theories for representing commonsense knowledge about actions will appear in (Turner 1997).

The paper is organized as follows. We present preliminary definitions. We then introduce the Splitting Set Theorem, followed by Splitting Sequence Theorem, which generalizes it. We exercise the Splitting Sequence Theorem on an example default theory for reasoning about action. We present an abridged proof of the Splitting Set Theorem, followed by a detailed proof of the Splitting Sequence Theorem.

Preliminary Definitions

Given a set U of atomic symbols (not including the special constants $\top$ and $\bot$), we denote by $\mathcal{L}(U)$ the language of propositional logic with exactly the atoms $U \cup \{\top, \bot\}$.[4] We say a set of formulas from $\mathcal{L}(U)$ is *logically closed* if it is closed under propositional logic. We write inference rules over $\mathcal{L}(U)$ as expressions of the form

$$\frac{\phi}{\psi}$$

where ϕ and ψ are formulas from $\mathcal{L}(U)$. When convenient, we identify a formula ϕ with the inference rule $\frac{\top}{\phi}$. We say a set Γ of formulas is *closed under* a set R of inference rules if for all $\frac{\phi}{\psi} \in R$, if $\phi \in \Gamma$ then $\psi \in \Gamma$. By $Cn_U(R)$ we denote the least logically closed set of formulas from $\mathcal{L}(U)$ that is closed under R.

A *default rule* over $\mathcal{L}(U)$ is an expression of the form

$$\frac{\alpha : \beta_1, \ldots, \beta_n}{\gamma} \tag{1}$$

where $\alpha, \beta_1, \ldots, \beta_n, \gamma$ are formulas from $\mathcal{L}(U)$ ($n \geq 0$). Let r be a default rule of form (1). We call α the *prerequisite* of r, and denote it by $pre(r)$. The formulas $\beta_1, \ldots, \beta_n$ are the *justifications* of r; we write $just(r)$ to denote the set $\{\beta_1, \ldots, \beta_n\}$. We call γ the *consequent* of r, and denote it by $cons(r)$. If $pre(r)$ is $\top$ we often

[1] See, for example, (Komorowski 1990).

[2] In (Lifschitz & Turner 1994) we presented without proof Splitting Theorems for logic programs with classical negation and disjunction, under the answer set semantics (Gelfond & Lifschitz 1991). The results for nondisjunctive logic programs follow from the Splitting Theorems for default logic. The definitions and proofs in this paper can be adapted to the more general case of disjunctive default logic (Gelfond *et al.* 1991), from which the Splitting Theorems for disjunctive logic programs would follow as well.

[3] We do not address the possible application of these ideas to the problem of automated default reasoning.

[4] Thus, $\mathcal{L}(\emptyset)$ consists of all formulas in which the only atoms are the constants $\top$ and $\bot$.

omit it, writing $\frac{:\beta_1,\ldots,\beta_n}{\gamma}$ instead. If $just(r) = \emptyset$, we identify r with the corresponding inference rule $\frac{\alpha}{\gamma}$.[5]

A *default theory* over $\mathcal{L}(U)$ is a set of default rules over $\mathcal{L}(U)$. Let D be a default theory over $\mathcal{L}(U)$, and E a set of formulas from $\mathcal{L}(U)$. The *reduct of D by E*, denoted by D^E, is the following set of inference rules.

$$\left\{ \frac{pre(r)}{cons(r)} : r \in D \text{ and for all } \beta \in just(r),\ \neg\beta \notin E \right\}$$

We say E is an *extension* of D if $E = Cn_U(D^E)$.

Default logic is due to Reiter (Reiter 1980). The (essentially equivalent) definitions given above follow (Gelfond *et al.* 1991). Although the definitions in this section are stated for the propositional case, they are taken, in the standard way, to apply in the first-order, quantifier-free case as well, by taking each non-ground expression to stand for all of its ground instances.

Splitting Sets

Let D be a default theory over $\mathcal{L}(U)$ such that, for every rule $r \in D$, $pre(r)$ is in conjunctive normal form. (Of course any default theory can be easily transformed into an equivalent default theory, over the same language, satisfying this condition.) For any rule $r \in D$, a formula ϕ is a *constituent of r* if at least one of the following conditions holds: (i) ϕ is a conjunct of $pre(r)$; (ii) $\phi \in just(r)$; (iii) $\phi = cons(r)$.

A *splitting set* for D is a subset A of U such that for every rule $r \in D$ the following two conditions hold.

- Every constituent of r belongs to $\mathcal{L}(A) \cup \mathcal{L}(U \setminus A)$.
- If $cons(r)$ does not belong to $\mathcal{L}(U \setminus A)$, then r is a default rule over $\mathcal{L}(A)$.

If A is a splitting set for D, we say that A *splits D*. The *base of D relative to A*, denoted by $b_A(D)$, is the default theory over $\mathcal{L}(A)$ that consists of all members of D that are default rules over $\mathcal{L}(A)$.

Let $U_3 = \{a, b, c, d\}$. Consider the following default theory D_3 over $\mathcal{L}(U_3)$.

$$\frac{:\neg b}{a} \quad \frac{:\neg a}{b} \quad \frac{a \vee b : a, b}{c \vee d} \quad \frac{a \wedge (c \vee d) : \neg d}{\neg d} \quad \frac{b \wedge (c \vee d) : \neg c}{\neg c}$$

Take $A_3 = \{a, b\}$. It's easy to verify that A_3 splits D_3, with

$$b_{A_3}(D_3) = \left\{ \frac{:\neg b}{a}, \frac{:\neg a}{b} \right\}.$$

Notice that default theory $b_{A_3}(D_3)$ over $\mathcal{L}(A_3)$ has two consistent extensions: $Cn_{A_3}(\{a\})$ and $Cn_{A_3}(\{b\})$.

Given a splitting set A for D, and a set X of formulas from $\mathcal{L}(A)$, the *partial evaluation of D by X with respect to A*, denoted by $e_A(D, X)$, is the default theory over $\mathcal{L}(U \setminus A)$ obtained from D in the following manner. For each rule $r \in D \setminus b_A(D)$ such that

- every conjunct of $pre(r)$ that belongs to $\mathcal{L}(A)$ also belongs to $Cn_A(X)$, and
- no member of $just(r)$ has its complement in $Cn_A(X)$

there is a rule $r' \in e_A(D, X)$ such that

- $pre(r')$ is obtained from $pre(r)$ by replacing each conjunct of $pre(r)$ that belongs to $\mathcal{L}(A)$ by $\top$, and
- $just(r') = just(r) \cap \mathcal{L}(U \setminus A)$, and
- $cons(r') = cons(r)$.

For example, it is easy to verify that

$$e_{A_3}(D_3, Cn_{A_3}(\{a\})) = \left\{ \frac{\top}{c \vee d}, \frac{\top \wedge (c \vee d) : \neg d}{\neg d} \right\}$$

and that

$$e_{A_3}(D_3, Cn_{A_3}(\{b\})) = \left\{ \frac{\top}{c \vee d}, \frac{\top \wedge (c \vee d) : \neg c}{\neg c} \right\}.$$

Let A be a splitting set for D. A *solution to D with respect to A* is a pair $\langle X, Y\rangle$ of sets of formulas satisfying the following two properties.

- X is a consistent extension of the default theory $b_A(D)$ over $\mathcal{L}(A)$.
- Y is a consistent extension of the default theory $e_A(D, X)$ over $\mathcal{L}(U \setminus A)$.

For example, given our previous observations, it is easy to verify that D_3 has two solutions with respect to A_3:

$$\langle\, Cn_{A_3}(\{a\}),\ Cn_{U_3 \setminus A_3}(\{c, \neg d\})\,\rangle$$

and

$$\langle\, Cn_{A_3}(\{b\}),\ Cn_{U_3 \setminus A_3}(\{\neg c, d\})\,\rangle.$$

Splitting Set Theorem. *Let A be a splitting set for a default theory D over $\mathcal{L}(U)$. A set E of formulas is a consistent extension of D iff $E = Cn_U(X \cup Y)$ for some solution $\langle X, Y\rangle$ to D with respect to A.*

Thus, for example, it follows from the Splitting Set Theorem that default theory D_3 has exactly two consistent extensions: $Cn_{U_3}(\{a, c, \neg d\})$ and $Cn_{U_3}(\{b, \neg c, d\})$.

Splitting Set Corollary. *Let A be a splitting set for a default theory D over $\mathcal{L}(U)$. If E is a consistent extension of D, then the pair*

$$\langle\, E \cap \mathcal{L}(A),\ E \cap \mathcal{L}(U \setminus A)\,\rangle$$

is a solution to D with respect to A.

Splitting Sequences

A (transfinite) *sequence* is a family whose index set is an initial segment of ordinals $\{\alpha : \alpha < \mu\}$. We say that a sequence $\langle A_\alpha\rangle_{\alpha<\mu}$ of sets is *monotone* if $A_\alpha \subseteq A_\beta$ whenever $\alpha < \beta$, and *continuous* if, for each limit ordinal $\alpha < \mu$, $A_\alpha = \bigcup_{\gamma<\alpha} A_\gamma$.

A *splitting sequence* for a default theory D over $\mathcal{L}(U)$ is a nonempty, monotone, continuous sequence $\langle A_\alpha\rangle_{\alpha<\mu}$ of splitting sets for D s.t. $\bigcup_{\alpha<\mu} A_\alpha = U$.

[5] Allowing the empty set of justifications is mathematically convenient (Brewka 1991; Marek & Truszczyński 1993), although it seems Reiter (Reiter 1980) may have meant to prohibit it.

$$\neg Holds(Up(Left), S_0) \wedge \neg Holds(Up(Right), S_0) \wedge \neg Holds(Spilled, S_0)$$
$$Precedes(S_0, S_1) \wedge Occurs(Raise(Left), S_0) \wedge Occurs(Raise(Right), S_0)$$
$$Precedes(S_1, S_2) \wedge Occurs(Lower(Left), S_1)$$
$$\frac{Occurs(Raise(x), s) \wedge Precedes(s, s')}{Holds(Up(x), s')} \qquad \frac{Occurs(Lower(x), s) \wedge Precedes(s, s')}{\neg Holds(Up(x), s')}$$
$$\frac{Holds(Up(Left), s) \not\equiv Holds(Up(Right), s)}{Holds(Spilled, s)}$$
$$\frac{Holds(f, s) \wedge Precedes(s, s') : Holds(f, s')}{Holds(f, s')} \qquad \frac{\neg Holds(f, s) \wedge Precedes(s, s') : \neg Holds(f, s')}{\neg Holds(f, s')}$$

Figure 1: Default theory D_1

The notion of a solution with respect to a splitting set is extended to splitting sequences as follows. Let $A = \langle A_\alpha \rangle_{\alpha<\mu}$ be a splitting sequence for D. A *solution to D with respect to A* is a sequence $\langle E_\alpha \rangle_{\alpha<\mu}$ of sets of formulas that satisfies the following three conditions.

- E_0 is a consistent extension of the default theory $b_{A_0}(D)$ over $\mathcal{L}(A_0)$.
- For any α such that $\alpha + 1 < \mu$, $E_{\alpha+1}$ is a consistent extension of the default theory
$$e_{A_\alpha}\left(b_{A_{\alpha+1}}(D), \bigcup_{\gamma\leq\alpha} E_\gamma\right)$$
over $\mathcal{L}(A_{\alpha+1} \setminus A_\alpha)$.
- For any limit ordinal $\alpha < \mu$, $E_\alpha = Cn_\emptyset(\emptyset)$.

We generalize the Splitting Set Theorem as follows.

Splitting Sequence Theorem. *Let $A = \langle A_\alpha \rangle_{\alpha<\mu}$ be a splitting sequence for a default theory D over $\mathcal{L}(U)$. A set E of formulas is a consistent extension of D if and only if*
$$E = Cn_U\left(\bigcup_{\alpha<\mu} E_\alpha\right)$$
for some solution $\langle E_\alpha \rangle_{\alpha<\mu}$ to D with respect to A.

The proof of the Splitting Sequence Theorem, which appears in later in the paper, relies on the Splitting Set Theorem. We also have the following counterpart to the Splitting Set Corollary.

Splitting Sequence Corollary. *Let $A = \langle A_\alpha \rangle_{\alpha<\mu}$ be a splitting sequence for a default theory D over $\mathcal{L}(U)$. Let $\langle U_\alpha \rangle_{\alpha<\mu}$ be the sequence of pairwise disjoint subsets of U such that for all $\alpha < \mu$*
$$U_\alpha = A_\alpha \setminus \bigcup_{\gamma<\alpha} A_\gamma\,.$$
If E is a consistent extension of D, then the sequence $\langle\, E \cap \mathcal{L}(U_\alpha)\, \rangle_{\alpha<\mu}$ is a solution to D with respect to A.

Example: Splitting a Default Theory for Reasoning About Action

We will illustrate the use of the Splitting Sequence Theorem by applying it to a default theory that formalizes commonsense knowledge about actions. The action domain we formalize is based on the "Soup Bowl" example from (Baral & Gelfond 1993), which involves reasoning about the effects of concurrent actions.[6] The default logic formalization we consider relies, implicitly, on the notion of "static causal laws," recently investigated in (McCain & Turner 1995; Turner 1996).

Default Theory D_1

We will formalize the following action domain in default logic. There is a bowl of soup. There are actions that raise and lower its left and right sides. If one side of the bowl is up while the other is not up, the soup is spilled. In the initial situation S_0, both sides of the bowl are not up, and the soup is not spilled. We let S_1 be the situation that would result from raising both sides of the bowl simultaneously. Notice that, intuitively speaking, it follows that in situation S_1 both sides of the bowl are up and the soup is not spilled. We also consider an additional situation S_2 that would result from lowering the left side of the bowl, when in situation S_1. Intuitively speaking, we can conclude that in situation S_2 the left side of the bowl is not up, the right side is up, and the soup is spilled.

We will use the Splitting Sequence Theorem to demonstrate that the default theory D_1, shown in Figure 1, is a correct formalization this action domain.

More precisely, what we will show is that the literals $Holds(Up(Left), S_1)$, $Holds(Up(Right), S_1)$, and $\neg Holds(Spilled, S_1)$ are among the consequences of default theory D_1, which establishes that D_1 correctly describes the values of the fluents in situation S_1.

[6] We will not attempt in this paper a general analysis of the problem of representing commonsense knowledge about concurrent actions. Nor will we attempt a general justification of the style of representation employed in the example default theory itself. Such analysis and justification is an important problem that is beyond the scope of this paper.

$$\frac{\neg Holds(Up(Left), S_0) \wedge \neg Holds(Up(Right), S_0) \wedge \neg Holds(Spilled, S_0) \quad Precedes(S_0, S_1) \wedge Occurs(Raise(Left), S_0) \wedge Occurs(Raise(Right), S_0) \quad Precedes(S_1, S_2) \wedge Occurs(Lower(Left), S_1) \quad Holds(Up(Left), S_0) \not\equiv Holds(Up(Right), S_0)}{Holds(Spilled, S_0)}$$

Figure 2: Default theory $b_{A_0}(D_1)$

$$Holds(Up(Left), S_1) \quad Holds(Up(Right), S_1)$$
$$\frac{Holds(Up(Left), S_1) \not\equiv Holds(Up(Right), S_1)}{Holds(Spilled, S_1)}$$
$$\frac{: \neg Holds(Up(Left), S_1)}{\neg Holds(Up(Left), S_1)} \quad \frac{: \neg Holds(Up(Right), S_1)}{\neg Holds(Up(Right), S_1)}$$
$$\frac{: \neg Holds(Spilled, S_1)}{\neg Holds(Spilled, S_1)}$$

Figure 3: Default theory $e_{A_0}(b_{A_1}(D_1), E_0)$

$$\neg Holds(Up(Left), S_2)$$
$$\frac{Holds(Up(Left), S_2) \not\equiv Holds(Up(Right), S_2)}{Holds(Spilled, S_2)}$$
$$\frac{: Holds(Up(Left), S_2)}{Holds(Up(Left), S_2)} \quad \frac{: Holds(Up(Right), S_2)}{Holds(Up(Right), S_2)}$$
$$\frac{: \neg Holds(Spilled, S_2)}{\neg Holds(Spilled, S_2)}$$

Figure 4: Default theory $e_{A_1}(b_{A_2}(D_1), E_0 \cup E_1)$

Similarly, we will also show that D_1 entails the literals $\neg Holds(Up(Left), S_2)$, $Holds(Up(Right), S_2)$, and $Holds(Spilled, S_2)$, which establishes that D_1 also correctly describes situation S_2.

Splitting Default Theory D_1

Before applying the Splitting Sequence Theorem, we must be more precise about the language of D_1.

Let U_1 consist of all ground atoms of the following many-sorted, first-order language $\mathcal{L}_1$. The sorts of $\mathcal{L}_1$ are *situation*, *action*, *fluent* and *side*. There are object constants S_0, S_1 and S_2 of sort *situation*, *Left* and *Right* of sort *side*, and *Spilled* of sort *fluent*. There is a unary function symbol *Up* of sort *side* $\rightarrow$ *fluent*, and unary function symbols *Raise* and *Lower* of sort *side* $\rightarrow$ *action*. There is a binary predicate symbol *Holds* of sort *fluent* $\times$ *situation*, a binary predicate symbol *Occurs* of sort *action* $\times$ *situation*, and a binary predicate symbol *Precedes* of sort *situation* $\times$ *situation*.

We take D_1 to be the (propositional) default theory over $\mathcal{L}(U_1)$ consisting of all ground instances in $\mathcal{L}_1$ of the rules shown in Figure 1.

Now we can specify a splitting sequence for D_1, which will allow us to break D_1 into simpler parts.

To begin, let A_0 consist of all ground instances in $\mathcal{L}_1$ of the following atoms:

$$Holds(f, S_0)\,,\; Occurs(a, s)\,,\; Precedes(s, s')\,.$$

Notice that A_0 splits D_1, with $b_{A_0}(D_1)$ the default theory over $\mathcal{L}(A_0)$ shown in Figure 2.

We obtain A_1 by adding to A_0 all ground instances in $\mathcal{L}_1$ of $Holds(f, S_1)$. Finally, let $A_2 = U_1$. One easily checks that $\langle A_0, A_1, A_2\rangle$ is a splitting sequence for D_1.

Let X_0 consist of the following literals:

$$\neg Holds(Up(Left), S_0)\,,\; \neg Holds(Up(Right), S_0)\,,\; \neg Holds(Spilled, S_0)\,,\; Precedes(S_0, S_1)\,,\; Occurs(Raise(Left), S_0)\,,\; Occurs(Raise(Right), S_0)\,,\; Precedes(S_1, S_2)\,,\; Occurs(Lower(Left), S_1)\,.$$

Take

$$E_0 = Cn_{A_0}(X_0)\,.$$

Notice that E_0 is the unique extension of $b_{A_0}(D_1)$.

The default theory $e_{A_0}(b_{A_1}(D_1), E_0)$ over $\mathcal{L}(A_1 \setminus A_0)$ is (essentially) as shown in Figure 3.

Thus, if we take X_1 to consist of the literals

$$Holds(Up(Left), S_1)\,,\; Holds(Up(Right), S_1)\,,\; \neg Holds(Spilled, S_1)\,,$$

and let

$$E_1 = Cn_{A_1 \setminus A_0}(X_1)$$

we find that E_1 is the unique extension of default theory $e_{A_0}(b_{A_1}(D_1), E_0)$.

Finally, observe that the default theory $e_{A_1}(b_{A_2}(D_1), E_0 \cup E_1)$ over $\mathcal{L}(A_2 \setminus A_1)$ is essentially as shown in Figure 4.

Let X_2 consist of the literals

$$\neg Holds(Up(Left), S_2)\,,\; Holds(Up(Right), S_2)\,,\; Holds(Spilled, S_2)\,.$$

Take

$$E_2 = Cn_{A_2 \setminus A_1}(X_2)\,.$$

It is not difficult to verify that E_2 is the unique extension of $e_{A_1}(b_{A_2}(D_1), E_0 \cup E_1)$.

We have shown that $\langle E_0, E_1, E_2\rangle$ is a solution to default theory D_1 with respect to $\langle A_0, A_1, A_2\rangle$. In fact it is the unique solution. It follows by the Splitting Sequence Theorem that $Cn_{U_1}(E_0 \cup E_1 \cup E_2)$ is the unique consistent extension of D_1.

This shows that the literals $Holds(Up(Left), S_1)$, $Holds(Up(Right), S_1)$, and $\neg Holds(Spilled, S_1)$ are among the consequences of D_1, which, as we discussed previously, is intuitively correct.

We have similarly established that the literals $\neg Holds(Up(Left), S_2)$, $Holds(Up(Right), S_2)$, and $Holds(Spilled, S_2)$ are among the consequences of D_1, which is again the intuitively correct result.

Proof of the Splitting Set Theorem

Due to space constraints, we present an abridged proof of the Splitting Set Theorem, omitting several intermediate lemmas, and proofs of the remaining lemmas.

Lemma 1 *Let U, U' be disjoint sets of atoms. Let R be a set of inference rules over $\mathcal{L}(U)$, and R' a set of inference rules over $\mathcal{L}(U')$. Let $X = Cn_{U \cup U'}(R \cup R')$.*

- *If X is consistent, then $X \cap \mathcal{L}(U) = Cn_U(R)$.*
- *$X = Cn_{U \cup U'}(Cn_U(R) \cup R')$.*

In proving the Splitting Set Theorem it is convenient to introduce a set of alternative definitions, differing very slightly from those used in stating the theorem. (We nonetheless prefer the original definitions, because they are more convenient in applications of the Splitting Theorems.)

Let D be a default theory over $\mathcal{L}(U)$ split by A. We define the following.

- $t^*_A(D) = \{ r \in D \ : \ cons(r) \in \mathcal{L}(U \setminus A) \}$
- $b^*_A(D) = D \setminus t^*_A(D)$

The advantage of these alternative definitions is captured in the following key lemma, which fails to hold for their counterparts $b_A(D)$ and $D \setminus b_A(D)$.

Lemma 2 *Let D be a default theory over $\mathcal{L}(U)$ with splitting set A. For any set X of formulas from $\mathcal{L}(U)$, $b^*_A(D)^X = b^*_A(D^X)$ and $t^*_A(D)^X = t^*_A(D^X)$.*

The proof will also make use of the following additional alternative definitions.

Given a set X of formulas from $\mathcal{L}(A)$, let $e^*_A(D, X)$ be the default theory over $\mathcal{L}(U \setminus A)$ obtained from D in the following manner. For each rule $r \in t^*_A(D)$ s.t.

- every conjunct of $pre(r)$ that belongs to $\mathcal{L}(A)$ also belongs to $Cn_A(X)$, and
- no member of $just(r)$ has its complement in $Cn_A(X)$

there is a rule $r' \in e^*_A(D, X)$ such that

- $pre(r')$ is obtained from $pre(r)$ by replacing each conjunct of $pre(r)$ that belongs to $\mathcal{L}(A)$ by $\top$, and
- $just(r') = just(r) \cap \mathcal{L}(U \setminus A)$, and
- $cons(r') = cons(r)$.

Notice that e^*_A differs from e_A only in starting with the rules in $t^*_A(D)$ instead of the rules in $D \setminus b_A(D)$.

Finally, let $s^*_A(D)$ be the set of all pairs $\langle X, Y \rangle$ s.t.

- X is a consistent extension of $b^*_A(D)$, and
- Y is a consistent extension of $e^*_A(D, X)$.

The following lemma shows that these alternative definitions are indeed suitable for our purpose.

Lemma 3 *If a default theory D over $\mathcal{L}(U)$ is split by A, then $s^*_A(D)$ is precisely the set of solutions to D with respect to A.*

Now we present the two main lemmas, and then use them to prove the Splitting Set Theorem.

Lemma 4 *Let D be a default theory over $\mathcal{L}(U)$ with splitting set A. Let E be a consistent set of formulas from $\mathcal{L}(U)$. $E = Cn_U(D^E)$ if and only if*

- *$E \cap \mathcal{L}(A) = Cn_A[b^*_A(D)^{E \cap \mathcal{L}(A)}]$ and*
- *$E = Cn_U[(E \cap \mathcal{L}(A)) \cup t^*_A(D)^E]$.*

Lemma 5 *Let D be a default theory over $\mathcal{L}(U)$ split by A. Let E be a logically closed set of formulas from $\mathcal{L}(U)$, with $X = E \cap \mathcal{L}(A)$ and $Y = E \cap \mathcal{L}(U \setminus A)$. We have $Cn_U[X \cup e^*_A(D, X)^Y] = Cn_U[X \cup t^*_A(D)^E]$.*

Proof of Splitting Set Theorem. Given a default theory D over $\mathcal{L}(U)$ with splitting set A, we know by Lemma 3 that $s^*_A(D)$ is precisely the set of solutions to D with respect to A. We will show that E is a consistent extension of D if and only if $E = Cn_U(X \cup Y)$ for some $\langle X, Y \rangle \in s^*_A(D)$.

($\Rightarrow$) Assume E is a consistent extension of D. Let $X = E \cap \mathcal{L}(A)$ and $Y = E \cap \mathcal{L}(U \setminus A)$. By Lemma 4, $X = Cn_A(b^*_A(D)^X)$ and $E = Cn_U(X \cup t^*_A(D)^E)$. By Lemma 5, $E = Cn_U(X \cup e^*_A(D, X)^Y)$. By Lemma 1, we can conclude that $Y = Cn_{U \setminus A}(e^*_A(D, X)^Y)$. So we have established that $\langle X, Y \rangle \in s^*_A(D)$. We can also conclude by Lemma 1 that $E = Cn_U[X \cup Cn_{U \setminus A}(e^*_A(D, X)^Y)]$. And since $Y = Cn_{U \setminus A}(e^*_A(D, X)^Y)$, we have $E = Cn_U(X \cup Y)$.

($\Leftarrow$) Assume $E = Cn_U(X \cup Y)$ for some $\langle X, Y \rangle \in s^*_A(D)$. Since $\langle X, Y \rangle \in s^*_A(D)$, we have $Y = Cn_{U \setminus A}(e^*_A(D, X)^Y)$. Hence $E = Cn_U[X \cup Cn_{U \setminus A}(e^*_A(D, X)^Y)]$. By Lemma 1, we can conclude that $E = Cn_U(X \cup e^*_A(D, X)^Y)$. Thus, by Lemma 5, $E = Cn_U(X \cup t^*_A(D)^E)$. By Lemma 1, $E \cap \mathcal{L}(A) = Cn_A(X)$, and since X is logically closed, $Cn_A(X) = X$. So $E \cap \mathcal{L}(A) = X$. Since $\langle X, Y \rangle \in s^*_A(D)$, we have $X = Cn_A(b^*_A(D)^X)$. We can conclude by Lemma 4 that $E = Cn_U(D^E)$. □

Proof of Splitting Set Corollary. Assume that E is a consistent extension of D. By the Splitting Set Theorem, there is a solution $\langle X, Y \rangle$ to D with respect to A such that $E = Cn_U(X \cup Y)$. Since $X \subseteq \mathcal{L}(A)$ and $Y \subseteq \mathcal{L}(U \setminus A)$, we can conclude by Lemma 1 that $E \cap \mathcal{L}(A) = Cn_A(X)$. And since X is logically closed, $Cn_A(X) = X$. So $E \cap \mathcal{L}(A) = X$. A symmetric argument shows that $E \cap \mathcal{L}(U \setminus A) = Y$. □

Proof of Splitting Sequence Theorem

Lemma 6 *Let D be a default theory over $\mathcal{L}(U)$ with splitting sequence $A = \langle A_\alpha \rangle_{\alpha < \mu}$. Let E be a set of formulas from $\mathcal{L}(U)$. Let $X = \langle X_\alpha \rangle_{\alpha < \mu}$ be a sequence of sets of formulas from $\mathcal{L}(U)$ such that*

- *$X_0 = E \cap \mathcal{L}(A_0)$,*
- *for all α s.t. $\alpha + 1 < \mu$, $X_{\alpha+1} = E \cap \mathcal{L}(A_{\alpha+1} \setminus A_\alpha)$,*
- *for any limit ordinal $\alpha < \mu$, $X_\alpha = Cn_\emptyset(\emptyset)$.*

If E is a consistent extension of D, then X is a solution to D with respect to A.

Proof. There are three things to check.

First, by the Splitting Set Corollary, we can conclude that $E \cap \mathcal{L}(A_0)$ is a consistent extension of $b_{A_0}(D)$.

Second, choose α such that $\alpha + 1 < \mu$. We must show that $X_{\alpha+1}$ is a consistent extension of

$$e_{A_\alpha}\left(b_{A_{\alpha+1}}(D), \bigcup_{\gamma \le \alpha} X_\gamma\right). \tag{2}$$

Let $\beta = \alpha + 1$. By the Splitting Set Corollary, $E \cap \mathcal{L}(A_\beta)$ is a consistent extension of $b_{A_\beta}(D)$. Let $D' = b_{A_\beta}(D)$ and let $E' = E \cap \mathcal{L}(A_\beta)$. By the Splitting Set Corollary, since A_α splits D', $E' \cap \mathcal{L}(A_\beta \setminus A_\alpha)$ is a consistent extension of $e_{A_\alpha}(D', E' \cap \mathcal{L}(A_\alpha))$. It is easy to verify that $X_{\alpha+1} = E' \cap \mathcal{L}(A_\beta \setminus A_\alpha)$. It is not difficult to verify also that $e_{A_\alpha}(D', E' \cap \mathcal{L}(A_\alpha))$ is the same as (2).

Third, for any limit ordinal $\alpha < \mu$, $X_\alpha = Cn_\emptyset(\emptyset)$. □

Lemma 7 *Let D be a default theory over $\mathcal{L}(U)$ with splitting sequence $A = \langle A_\alpha \rangle_{\alpha<\mu}$. Let $\langle E_\alpha \rangle_{\alpha<\mu}$ be a solution to D with respect to A. For all $\alpha < \mu$*

$$Cn_{A_\alpha}\left(\bigcup_{\gamma \le \alpha} E_\gamma\right)$$

is a consistent extension of $b_{A_\alpha}(D)$.

Proof. For all $\alpha < \mu$, let

$$X_\alpha = Cn_{A_\alpha}\left(\bigcup_{\gamma \le \alpha} E_\gamma\right).$$

Proof is by induction on α. Assume that for all $\gamma < \alpha$, X_γ is a consistent extension of $b_{A_\gamma}(D)$. We'll show that X_α is a consistent extension of $b_{A_\alpha}(D)$. There are two cases to consider.

Case 1: α is not a limit ordinal. Choose γ such that $\gamma + 1 = \alpha$. By the inductive hypothesis, X_γ is a consistent extension of $b_{A_\gamma}(D)$. We also know that E_α is a consistent extension of $e_{A_\gamma}(b_{A_\alpha}(D), \bigcup_{\beta \le \gamma} E_\beta)$. Let $D' = b_{A_\alpha}(D)$. It is clear that $b_{A_\gamma}(D) = b_{A_\gamma}(D')$. It is not difficult to verify that $e_{A_\gamma}(b_{A_\alpha}(D), \bigcup_{\beta \le \gamma} E_\beta)$ is the same as $e_{A_\gamma}(D', X_\gamma)$. So we've shown that X_γ is a consistent extension of $b_{A_\gamma}(D')$ and that E_α is a consistent extension of $e_{A_\gamma}(D', X_\gamma)$. By the Splitting Set Theorem, it follows that $Cn_{A_\alpha}(X_\gamma \cup E_\alpha)$ is a consistent extension of D'. And since it's easy to check that $Cn_{A_\alpha}(X_\gamma \cup E_\alpha) = X_\alpha$, we're done with the first case.

Case 2: α is a limit ordinal. First we show that X_α is closed under $b_{A_\alpha}(D)^{X_\alpha}$. So suppose the contrary. Thus there is an $r \in b_{A_\alpha}(D)^{X_\alpha}$ such that $pre(r) \in X_\alpha$ and $cons(r) \notin X_\alpha$. Since A is continuous and α is a limit ordinal, we know there must be a $\gamma < \alpha$ such that $r \in b_{A_\gamma}(D)^{X_\alpha}$. Since $b_{A_\gamma}(D)$ is a default theory over $\mathcal{L}(A_\gamma)$, we have $b_{A_\gamma}(D)^{X_\alpha} = b_{A_\gamma}(D)^{X_\gamma}$. So $r \in b_{A_\gamma}(D)^{X_\gamma}$. Furthermore, it follows that $pre(r) \in X_\gamma$ and $cons(r) \notin X_\gamma$. This shows that X_γ is not closed under $b_{A_\gamma}(D)^{X_\gamma}$, which contradicts the fact that, by the inductive hypothesis, X_γ is a consistent extension of $b_{A_\gamma}(D)$. So we have shown that X_α is closed under $b_{A_\alpha}(D)^{X_\alpha}$.

Now, let $E = Cn_{A_\alpha}(b_{A_\alpha}(D)^{X_\alpha})$. We will show that $E = X_\alpha$, from which it follows that X_α is a consistent extension of $b_{A_\alpha}(D)$. Since X_α is logically closed and closed under $b_{A_\alpha}(D)^{X_\alpha}$, we know that $E \subseteq X_\alpha$. Suppose $E \neq X_\alpha$, and consider any formula $\phi \in X_\alpha \setminus E$. Since A is continuous and α is a limit ordinal, there must be a $\gamma < \alpha$ such that ϕ is from $\mathcal{L}(A_\gamma)$ and therefore $\phi \in X_\gamma$. Thus, X_γ is a proper superset of $E \cap \mathcal{L}(A_\gamma)$. By the inductive hypothesis, we know that X_γ is a consistent extension of $b_{A_\gamma}(D)$. Thus, $X_\gamma = Cn_{A_\gamma}(b_{A_\gamma}(D)^{X_\gamma})$. And since $b_{A_\gamma}(D)^{X_\gamma} = b_{A_\gamma}(D)^{X_\alpha}$, we have $X_\gamma = Cn_{A_\gamma}(b_{A_\gamma}(D)^{X_\alpha})$. Since $E = Cn_{A_\alpha}(b_{A_\alpha}(D)^{X_\alpha})$ and $b_{A_\gamma}(D)^{X_\alpha} \subseteq b_{A_\alpha}(D)^{X_\alpha}$, we know that E is closed under $b_{A_\gamma}(D)^{X_\alpha}$. Moreover, since $b_{A_\gamma}(D)^{X_\alpha}$ is a default theory over $\mathcal{L}(A_\gamma)$, $E \cap \mathcal{L}(A_\gamma)$ is closed under $b_{A_\gamma}(D)^{X_\alpha}$. But X_γ is the least logically closed set closed under $b_{A_\gamma}(D)^{X_\alpha}$, so $X_\gamma \subseteq E \cap \mathcal{L}(A_\gamma)$, which contradicts the fact that X_γ is a proper superset of $E \cap \mathcal{L}(A_\gamma)$. We can conclude that $E = X_\alpha$, which completes the second case. □

Let D be a default theory over $\mathcal{L}(U)$ with splitting sequence $A = \langle A_\alpha \rangle_{\alpha<\mu}$. The *standard extension* of A is the sequence $B = \langle B_\alpha \rangle_{\alpha<\mu+1}$ such that

- for all $\alpha < \mu$, $B_\alpha = A_\alpha$, and
- $B_\mu = U$.

Notice that the standard extension of A is itself a splitting sequence for D.

Lemma 8 *Let D be a default theory over $\mathcal{L}(U)$ with splitting sequence $A = \langle A_\alpha \rangle_{\alpha<\mu}$. Let $B = \langle B_\alpha \rangle_{\alpha<\mu+1}$ be the standard extension of A. Let $X = \langle X_\alpha \rangle_{\alpha<\mu}$ be a sequence of sets of formulas from $\mathcal{L}(U)$. Let $Y = \langle Y_\alpha \rangle_{\alpha<\mu+1}$ be defined as follows.*

- *For all $\alpha < \mu$, $Y_\alpha = X_\alpha$.*
- *$Y_\mu = Cn_\emptyset(\emptyset)$.*

If X is a solution to D with respect to A, then Y is a solution to D with respect to B.

Proof. First, it's clear that Y_0 is a consistent extension of $b_{B_0}(D)$, since $Y_0 = X_0$, $b_{B_0}(D) = b_{A_0}(D)$, and X_0 is a consistent extension of $b_{A_0}(D)$. Similarly, it's clear that for any α such that $\alpha + 1 < \mu$, $Y_{\alpha+1}$ is a consistent extension of

$$e_{B_\alpha}\left(b_{B_{\alpha+1}}(D), \bigcup_{\gamma \le \alpha} Y_\gamma\right).$$

We also know that for any limit ordinal $\alpha < \mu$, $Y_\alpha = Cn_\emptyset(\emptyset)$. It remains to show that we handle μ correctly. There are two cases to consider.

Case 1: μ is a limit ordinal. In this case we must show that $Y_\mu = Cn_\emptyset(\emptyset)$, which it does.

Case 2: μ is not a limit ordinal. In this case, choose α such that $\alpha + 1 = \mu$. We must show that Y_μ is a consistent extension of the default theory

$$e_{B_\alpha}\left(b_{B_\mu}(D), \bigcup_{\gamma\leq\alpha} Y_\gamma\right) \tag{3}$$

over $\mathcal{L}(B_\mu \setminus B_\alpha)$. Since A is a splitting sequence for a default theory over $\mathcal{L}(U)$, we know that $\bigcup_{\gamma<\mu} A_\gamma = U$. Moreover, since A is monotone and μ is not a limit ordinal, it follows that $A_\alpha = U$. And since $B_\alpha = A_\alpha$, we know that $b_{B_\alpha}(D) = D$. It follows that default theory (3) is empty. It also follows that $B_\mu \setminus B_\alpha = \emptyset$, so the language of (3) is $\mathcal{L}(\emptyset)$. Since $Y_\mu = Cn_\emptyset(\emptyset)$, we have shown that Y_μ is a consistent extension of (3). □

Proof of Splitting Sequence Theorem. ($\Rightarrow$) Assume that E is a consistent extension of D. By Lemma 6, there is a solution $\langle E_\alpha\rangle_{\alpha<\mu}$ to D with respect to $\langle A_\alpha\rangle_{\alpha<\mu}$ for which it is not difficult to verify that

$$E = Cn_U\left(\bigcup_{\alpha<\mu} E_\alpha\right).$$

($\Leftarrow$) Assume that $X = \langle X_\alpha\rangle_{\alpha<\mu}$ is a solution to D with respect to $\langle A_\alpha\rangle_{\alpha<\mu}$. Let

$$E = Cn_U\left(\bigcup_{\alpha<\mu} X_\alpha\right).$$

Let $B = \langle B_\alpha\rangle_{\alpha<\mu+1}$ be the standard extension of $\langle A_\alpha\rangle_{\alpha<\mu}$. By Lemma 8, we know there is a solution $\langle Y_\alpha\rangle_{\alpha<\mu+1}$ to D with respect to B such that

$$E = Cn_U\left(\bigcup_{\alpha<\mu+1} Y_\alpha\right).$$

Moreover, we know there is an $\alpha < \mu + 1$ such that $B_\alpha = U$. Thus $b_{B_\alpha}(D) = D$ and

$$E = Cn_{B_\alpha}\left(\bigcup_{\gamma\leq\alpha} Y_\gamma\right).$$

It follows by Lemma 7 that E is a consistent extension of D. □

Proof of Splitting Sequence Corollary. Assume that E is a consistent extension of D. By the Splitting Sequence Theorem, there is a solution $\langle X_\alpha\rangle_{\alpha<\mu}$ to D with respect to A such that $E = Cn_U\left(\bigcup_{\alpha<\mu} X_\alpha\right)$. We will show that for all $\alpha < \mu$, $E \cap \mathcal{L}(U_\alpha) = X_\alpha$. Let $X = \bigcup_{\alpha<\mu} X_\alpha$. Consider any $\alpha < \mu$. We have $X_\alpha \subseteq \mathcal{L}(U_\alpha)$, $X \setminus X_\alpha \subseteq \mathcal{L}(U \setminus U_\alpha)$, and $E = Cn_U(X_\alpha \cup X \setminus X_\alpha)$. Thus, by Lemma 1 we can conclude that $E \cap \mathcal{L}(U_\alpha) = Cn_{U_\alpha}(X_\alpha)$. And since X_α is logically closed, we have $Cn_{U_\alpha}(X_\alpha) = X_\alpha$. □

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