Using Elimination Methods to Compute Thermophysical Algebraic Invariants from Infrared Imagery

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Abstract

We describe a new approach for computing invariant features in infrared (IR) images. Our approach is unique in the field since it considers not just surface reflection and surface geometry in the specification of invariant features, but it also takes into account internal object composition and thermal state which affect images sensed in the non-visible spectrum. We first establish a non-linear energy balance equation using the principle of conservation of energy at the surface of the imaged object. We then derive features that depend only on material parameters of the object and the sensed radiosity. These features are independent of the scene conditions and the scene-to-scene transformation of the "driving conditions" such as ambient temperature, and wind speed. The algorithm for deriving the invariant features is based on the algebraic elimination of the transformation parameters from the non-linear relationships. The elimination approach is a general method based on the extended Dixon resultant. Results on real IR imagery are shown to illustrate the performance of the features derived in this manner when used for an object recognition system that deals with multiple classes of objects.

Introduction

A very popular and increasingly affordable sensor modality is thermal imaging - where non-visible radiation is sensed in the long-wave infrared (LWIR) spectrum of $8\mu m$ to $14\mu m$. The current generation of LWIR sensors produce images of contrast and resolution that compare favorably with broadcast television quality visible light imagery. However, the images are no longer functions of only surface reflectance. As the wavelength of the sensor transducer passband increases, emissive effects begin to emerge as the dominant mode of electromagnetic energy exitance from object surfaces. The (primarily) emitted radiosity of LWIR energy has a strong dependence on internal composition, properties, and state of the object such as specific heat, density, volume, heat generation rate of internal sources, etc. This dependence may be exploited by specifying image-derived invariants that vary only if these parameters of the physical properties vary.

Here, we describe the use of the principle of conservation of energy at the surface of the imaged object to specify a functional relationship between the object's thermophysical properties (e.g., thermal conductivity, thermal capacitance, emissivity, etc.), scene parameters (e.g., wind temperature, wind speed, solar insolation), and the sensed LWIR image gray level. We use this functional form to derive invariant features that remain constant despite changes in scene parameters/driving conditions. In this formulation the internal thermophysical properties play a role that is analogous to the role of parameters of the conics, lines and/or points that are used for specifying geometric invariants when analyzing visible wavelength imagery. Thus, in addition to the currently available techniques of formulating features that depend only on external shape and surface reflectance discontinuities, the phenomenology of LWIR image generation can be used to establish new features that "uncover" the composition and thermal state of the object, and which do not depend on surface reflectance characteristics.

A general approach is described that enables the specification of invariant features that are satisfactorily justified in a thermophysical sense. The energy balance equation is inherently a non-linear form. We choose the variable labeling such that a polynomial is formed whose variables are the unknowns of the image formation and the coefficients are the object parameters. The choice of labels for the variables determines the form of the transformations from scene to scene. Consideration of the variable inter-dependencies specifies the set of transformation to be a subgroup of the general linear group.

A method based on elimination techniques is used to specify the features. Elimination methods eliminate a subset of variables from a finite set of polynomial equations to give a smaller set of polynomials in the remaining variables while keeping the solution set the same. Invariants can be computed using these methods in three steps – (1) Set up the transformation equa-

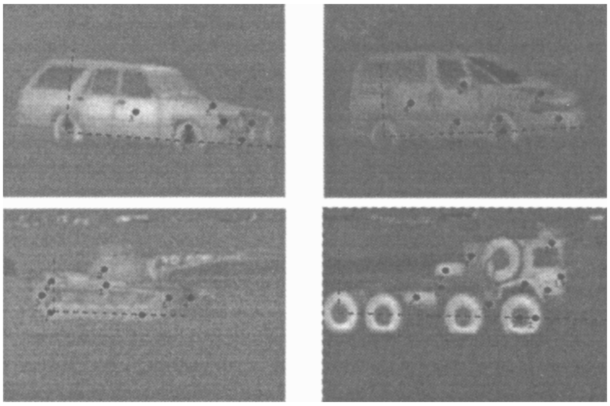

Figure 1: The vehicles used to test the object recognition approach, (from top left clockwise) car, van, truck 1, and tank. The axis superimposed on the image show the object centered reference frames. The numbered points indicate the object surfaces used to form the measurement matrices. These points are selected such that there are a variety of different materials and/or surface normals within the set.

tions relating the generic coefficients of the polynomial form before and after the action of the transformation subgroup, (2) Eliminate the transformation parameters from the transformation equations using any of the elimination methods, and finally, (3) Extract the invariants from the result of elimination from step 2.

Using Elimination Methods for Computing Invariants

Elimination methods are a general class of algorithms designed to eliminate a given set of variables from a finite system of polynomial equations. Some of the most general elimination methods are the Gröbner basis method, characteristic set method, and various resultant methods see (Kapur & Lakshman 1992) for a survey. Such methods find applications in many areas of science and engineering and can be used to solve systems of polynomial equations. They can also be used to automatically compute invariants of a given configuration (or quintic) under various transformation groups see (Kapur, Lakshman, & Saxena 1995).

An *absolute invariant* is a rational function of the configuration parameters whose value remains constant under the action of a transformation group on this configuration. As a consequence, absolute invariants are very useful (Mundy & Zisserman 1992) in recognizing objects from images and building model-based object recognition libraries. Let $\mathbf{p}$ and $\mathbf{q}$ be the object and image parameters. Each absolute invariant f/g generates a *separable invariant relation*, $h(\mathbf{p},\mathbf{q}) = f(\mathbf{p})g(\mathbf{q}) - f(\mathbf{q})g(\mathbf{p})$. In other words, if these separable invariant relations can somehow be derived, then it may be possible to extract absolute invariants (which generate them) from them.

The process of computing invariants using elimination methods can be organized in three phases as follows:

1. **Phase 1:** Set up the *transformation equations* relating the image parameters to the object via the transformation parameters.
2. **Phase 2:** Eliminate transformation parameters from the transformation equations to derive separable invariant relations.
3. **Phase 3:** Extract the *absolute invariants* which generate the separable invariant relations. This is known as the *separability problem.*

In phase 2, elimination methods such as Gröbner basis algorithms, and in certain cases see (Kapur, Lakshman, & Saxena 1995) resultant computations can be used to derive separable invariant relations.

Given a separable invariant relation $h(\mathbf{p},\mathbf{q})$, there exist many (*algebraically dependent*) invariants $\frac{c}{d},\ldots,\frac{f}{g},\ldots,\frac{k}{l}$, which generate them, ie.:

$$\begin{aligned} c(\mathbf{p})d(\mathbf{q}) - c(\mathbf{q})d(\mathbf{p}) &= h(\mathbf{p},\mathbf{q}),\\ &\vdots\\ f(\mathbf{p})g(\mathbf{q}) - f(\mathbf{q})g(\mathbf{p}) &= h(\mathbf{p},\mathbf{q}),\\ &\vdots\\ k(\mathbf{p})l(\mathbf{q}) - k(\mathbf{q})l(\mathbf{p}) &= h(\mathbf{p},\mathbf{q}). \end{aligned}$$

But for a given ordering on the object parameters, there is a **unique** invariant $\mathbf{I} = f/g$ such that the:

1. leading term of g is strictly larger than the leading term of f,
2. leading term of f has zero coefficient in g and
3. leading coefficient of g is 1 (ie. g is monic).

To extract the absolute invariant from separable invariant relations, the algorithm in (Kapur, Lakshman, & Saxena 1995) fixes an ordering on the object and image parameters, and targets this unique invariant as follows. Let $\mathbf{p}^{e_f}$ and $\mathbf{p}^{e_g}$ be the leading terms of $f(\mathbf{p})$ and $g(\mathbf{p})$ respectively, and c_f, the leading coefficient of $f(\mathbf{p})$. Then, using the above properties of this unique invariant, the separable invariant relation can be expressed as

$$\begin{aligned} h(\mathbf{p},\mathbf{q}) &= f(\mathbf{p})g(\mathbf{q}) - g(\mathbf{p})f(\mathbf{q})\\ &= f(\mathbf{p})\,(\mathbf{q}^{e_g} + \cdots) - g(\mathbf{p})\,(c_f\mathbf{q}^{e_f} + \cdots)\\ &= f(\mathbf{p})\,\mathbf{q}^{e_g} - c_f\,g(\mathbf{p})\,\mathbf{q}^{e_f} + \cdots. \end{aligned}$$

As is evident from the above expansion of the separable invariant relation as a polynomial in $\mathbf{q}$, the numerator $f(\mathbf{p})$ of the absolute invariant is the coefficient of the leading term $\mathbf{q}^{e_g}$. Once $f(\mathbf{p})$ is known, and the denominator $g(\mathbf{p})$ is the coefficient of the term $-c_f\,\mathbf{q}^{e_f}$ and can be easily read off from $h(\mathbf{p},\mathbf{q})$ once it has been sorted according to a predetermined ordering.

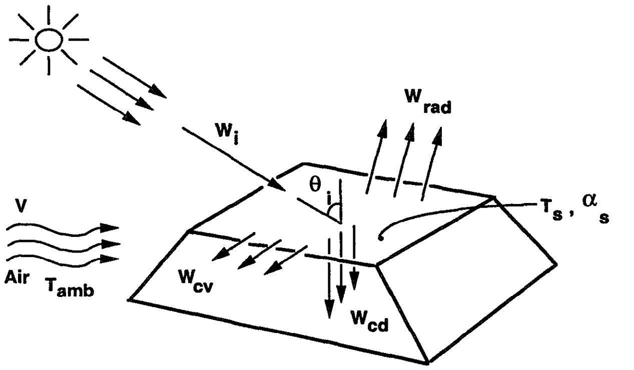

Figure 2: Energy exchange at the surface of the imaged object. Incident energy is primarily in the visible spectrum. Surfaces loses energy by convection to air, via radiation to the atmosphere, and via conduction to the interior of the object. The elemental volume at the surface also stores a portion of the absorbed energy.

A Thermophysical Approach to LWIR Image Analysis

At the surface of the imaged object (figure 2) energy absorbed by the surface equals the energy lost to the environment.

$$W_{abs} = W_{lost} \tag{1}$$

Energy absorbed by the surface is given by

$$W_{abs} = W_I \; cos\theta_I \; \alpha_s \; , \tag{2}$$

where, W_I is the incident solar irradiation on a horizontal surface, θ_i is the angle between the direction of irradiation and the surface normal, and α_s is the surface absorptivity which is related to the visual reflectance ρ_s by $\alpha_s = 1 - \rho_s$. Note that it is reasonable to use the visual reflectance to estimate the energy absorbed by the surface since approximately 90% of the energy in solar irradiation lies in the visible wavelengths (Incropera & DeWitt 1981).

The energy lost by the surface to the environment was given by

$$W_{lost} = W_{cv} + W_{rad} + W_{cnd} + W_{st} \tag{3}$$

The energy convected from the surface to the ambient air is given by $W_{cv} = h(T_s - T_{amb})$ where, T_{amb} is the ambient air temperature, T_s is the surface temperature of the imaged object, and h is the average convected heat transfer coefficient for the imaged surface, which depends on the wind speed, thermophysical properties of the air, and surface geometry (Incropera & DeWitt 1981). We note that surface temperature may be estimated from the thermal image based on an appropriate model of radiation energy exchange between the surface and the infrared camera.

The radiation energy loss is computed from $W_{rad} = \epsilon \, \sigma \, (T_s^4 - T_{amb}^4)$, where σ denotes the Stefan-Boltzmann constant. The energy conducted to the interior of the object is given by $W_{cnd} = -k \; dT/dx$, where k is the thermal conductivity of the material, and x is distance below the surface. Here, we assume that lateral energy conduction is insignificant compared to conduction along the direction normal to the surface. The increase in the stored, internal energy of an elemental volume at the surface is given by $W_{st} = C_T \frac{dT_s}{dt}$, where C_T denotes the lumped thermal capacitance of the object and is given by $C_T = DVc$, D is the density of the object, V is the volume, and c is the specific heat. In the following section we use the energy conservation model described above to derive invariant features using ideas in algebraic elimination theory.

Thermophysical Algebraic Invariants (TAI's)

The balance of energy expression,

$$W_{abs} = W_{rad} + W_{cv} + W_{st} + W_{cnd} \tag{4}$$

is the governing equation in our approach for computing invariant features. Each term in the above equation can be expanded, which results in equation 4 being expressed as a polynomial. The choice of labels for the variables determines both the form of the polynomial and transformation form. Since an absolute invariant feature value is not affected by transformations of the variables, the variables of the form are chosen to be the unknown parameters of the image formation. The coefficients are, then, the known/hypothesized object parameters and sensed measurements.

An Algebraic Invariance Formulation

The balance of energy expression, equation 4, may be written in the non-linear form

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 - a_5x_5 + a_6x_1x_6^4 + a_7x_4x_6 = 0. \tag{5}$$

where the variables and coefficients are labeled as

$$\begin{array}{ll} a_1 = \sigma T_s^4 & x_1 = \epsilon \\ a_2 = C_T & x_2 = \frac{dT_s}{dt} \\ a_3 = k & x_3 = \frac{dT_s}{dx} \\ a_4 = T\Delta & x_4 = \frac{h}{\Delta} \\ a_5 = -\cos\theta & x_5 = W_I\alpha_s \\ a_6 = -\sigma & x_6 = T_{amb} \\ a_7 = \Delta & \end{array} \tag{6}$$

Thus, the polynomial chosen to represent equation 4 is a quintic form in six variables.

Any pixel in a LWIR image of an object will yield a 7-D measurement vector, **a**. The image measurement (gray value) specifies a_1 and a_4. The values for a_2, a_3, and a_5 are known when the identity and pose of the object are hypothesized. The coefficient a_7, related to the convection term, h, is explained in greater detail in the discussion section. The driving conditions, x_i, $i = \{1 \ldots 6\}$ are the unknown scene parameters that change from scene to scene.

Consider two different LWIR images of a scene obtained under different scene conditions and from different viewpoints. Consider N points on the object that are visible in both views. Assume (for the nonce) that the object pose for each view, and point correspondence between the two views are available (or hypothesized). A point in each view yields a measurement vector $\mathbf{a}$. The ith component of the vector is denoted a_i, where $i = 1, \ldots, 7$ as defined by eqn (6). Let the collection of these vectors be denoted by $a_{i,k}$, $k = 1, \ldots, N$ for the first scene/image and $a'_{i,k}$, $k = 1, \ldots, N$ for the second scene. In the same vein, consider an associated set of driving condition vectors for the first scene. We express the collection as $x_{i,k}$ where $k = 1, \ldots, N$ and $i = 1, \ldots, 6$ as defined in eqn (6). Similarly, the driving condition vector from the second scene is denoted $x'_{i,k}$.

Thermophysical Transformation

Consider a set of $N \leq 6$ points imaged from the surface of an object. This creates a set of N vectors $x_{i,k}$, $k = 1 \ldots N, i = 1, \ldots, 6$ which define the driving conditions on the surface of the object in a scene at time t_n. This forms a variable matrix of dimension $6 \times N$, call it X. These points are transformed from their values at time t_n to their value at time t_{n+1}, $t_{n+1} > t_n$, by a GL transformation, M, $MX = X'$. The transformation matrix M is 6×6.

In order to determine the form of the transformation we view the components of a driving condition vector in terms of the inter-dependencies of the parameters. By doing so, superfluous parameters are eliminated. The dependency of the value of a variable at the current instance on other variables at a previous instance is established by the physical phenomena that cause scene-to-scene change in the different parameter values. The dependencies are shown below (and explanations follow):

$$
\begin{array}{ll}
\text{variable} & \text{dependency} \\
x'_1 = \epsilon & x_1 \quad (\epsilon) \\
x'_2 = \frac{dT_s}{dt} & x_2, x_3, x_4, x_5 (\frac{dT_s}{dt}, \frac{dT_s}{dx}, \frac{h}{\Delta}, W_I \alpha_s) \\
x'_3 = \frac{dT_s}{dx} & x_2, x_3, x_4, x_5 (\frac{dT_s}{dt}, \frac{dT_s}{dx}, \frac{h}{\Delta}, W_I \alpha_s) \\
x'_4 = h & x_4 \quad (h) \\
x'_5 = W_I \alpha_s & x_5 \quad (W_I \alpha_s) \\
x'_6 = T_{amb} & x_6 \quad (T_{amb})
\end{array}
\tag{7}
$$

The change in emissivity is independent of the values of any of the variables. Hence, it is dependent only on itself. The second component, x_2, is the temporal derivative of the surface temperature. Its value at t_{n+1} will be affected by all of the parameters at t_n except emissivity and the ambient temperature. Physically, the temporal derivative is independent of the ambient temperature and the emissivity of the surface; however, it is dependent on (1) its previous value, (2) the spatial derivative of the temperature in the material, (3) the convection coefficient (the surface patches propensity to convect into the air), (4) incident solar irradiation and surface absorptivity. The spatial derivative, x_3, has the same dependencies that x_2 has. The remaining variables, x_4, x_5, and x_6 depend, physically, only on their own previous values.

The variable inter-dependencies determine the thermophysical transformation. Thus the transformation of the variables of equation 5 can be represented by a subgroup of the GL group of the form

$$
M = \left\{ \begin{array}{cccccc}
m_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & m_{22} & m_{23} & m_{24} & m_{25} & 0 \\
0 & m_{32} & m_{33} & m_{34} & m_{35} & 0 \\
0 & 0 & 0 & m_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & m_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & m_{66}
\end{array} \right\} . \tag{8}
$$

Consider four points to compose X. Further explanation of the thermophysical behavior of these points is included in the discussion section. Each of the four points has seven components. Thus, the transformation induced on the coefficients, a_i, gives 28 constraining equations. Since there are 12 parameters of the transformation, every additional constraining equation that is added to a set of 12 constraining equations gives rise to an invariant relationship. Thus, for a configuration of four points in the thermophysical space and a transformation consisting of 12 parameters, there are 28-12=16 invariant functions; however, a subset of these relations are physically trivial invariant relationships.

Given X, consisting of four copies of the equation 5, the elimination technique described in section 2 was applied to the algebraic configuration. This results in the following non-trivial invariants:

$$
I1 = \frac{\begin{vmatrix} a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{vmatrix}}{\begin{vmatrix} a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,2} & a_{4,3} & a_{4,4} \end{vmatrix}} \tag{9}
$$

$$
I2 = \frac{\begin{vmatrix} a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{5,1} & a_{5,2} & a_{5,3} \end{vmatrix}}{\begin{vmatrix} a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,2} & a_{3,3} & a_{3,4} \\ a_{5,2} & a_{5,3} & a_{5,4} \end{vmatrix}} \tag{10}
$$

where $a_{i,k}$ is the ith component of the kth point.

Employing TAI's for Object Recognition

The feature computation scheme formulated above is suitable for use in an object recognition system that employs a hypothesize-and-verify strategy. The scheme would consist of the following steps:

1. extract geometric features, e.g., lines and conics.

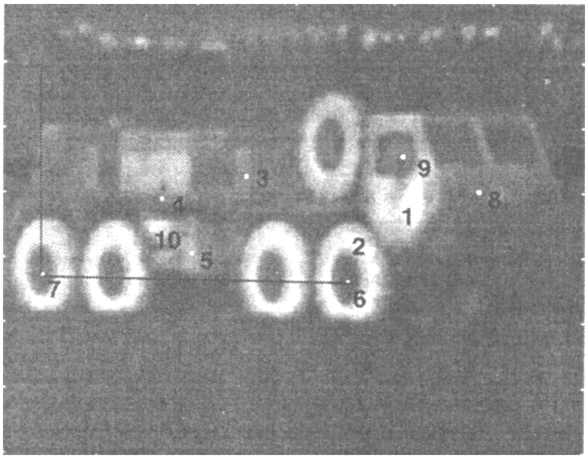

Figure 3: The truck 2 vehicle used in the recognition tests. The object centered coordinate axis is superimposed on the image. The numbered points correspond to the point sets given in table 1. These points are selected such that there are a variety of different materials and/or surface normals within the set.

2. for image region, r, hypothesize object class, k, and pose using, for example, geometric invariants as proposed by Forsyth, et al (Forsyth *et al.* 1991),
3. use the model of object k and project visible points labeled $i = 1, 2, \ldots$ onto image region r using scaled orthographic projection,
4. for point labeled i in the image region, assign thermophysical properties of point labeled i in the model of object k,
5. use the gray levels at each point and the assigned thermophysical properties, to compute the measurement vectors, $a_{i,k}$, and hence compute the feature I_1 or I_2, and finally,
6. compare feature $f^k(r)$ with model prototype F_k to verify the hypothesis.

Experimental Results

Object Recognition using TAIs

The method of computing thermophysical algebraic invariants discussed above was applied to real LWIR imagery acquired at different times of the day. Five types of vehicles were imaged: a van, two types of trucks, a military tank, and a car (figures 1). Several points were selected (as indicated in the figures) on the surfaces of different materials and/or orientation. The measurement vector given by eqn (6) was computed for each point, for each image/scene.

The features described in section 4 require four points. Given a model of an object that has some Q number of points defined, there is the possibility of forming q different features.

$$q = \binom{Q}{4}\binom{4}{3} \tag{11}$$

Point Set	Mean	STD	Quality
{2 3,4,5}	1.000	0.0026	0.0026
{2,3,4,9}	1.000	0.0061	0.0061
{2,3,4,8}	4.757	0.0352	0.0074
{2,3,4,7}	4.746	0.0280	0.0059
{8,9,3,10}	0.983	0.1951	0.1984
{8,9,10,6}	0.7361	0.1445	0.1963
{8,9,10,2}	0.0795	0.0146	0.1836
{5,6,7,8}	1.057	0.0443	0.0419

Table 1: Intra-class variation over time of the feature, I1, defined by equation 9 applied with the point sets given in column 1 for truck type 2. The features were evaluated at five time instances over two consecutive days, Day 1 - 11am, 12pm, 1pm, Day 2 - 9am, 10am. Column 2 is the mean of the feature over the five time instances and column 3 shows the feature stability in terms of standard deviation. Column 4 shows the quality factor defined as std divided by the mean. The points correspond to the points labeled in figure 3.

The first criterion for finding a useful feature is stable intra-class behavior. Nearly all of the point choices had low variation in intra-class tests; tests where the same object is viewed under different scene conditions. For example, a test was performed on the truck in figure 3. Table 1 shows the results for ten different features evaluated from truck 1. Although the performance of only ten features are shown, the performance is representative of the feature stability over all of the distinct point choices.

As mentioned in section 4, one must consider inter-class behavior as well as intra-class behavior for an object recognition application of the features. To investigate this we adopted the following procedure. Given an image of a vehicle, (1) assume the pose of the vehicle is known, then (2) use the front and rear wheels to establish an object centered reference frame. The center of the rear wheel is used as the origin, and center of the front wheel is used to specify the direction and scaling of the axes. The coordinates of the selected points are expressed in terms of this 2D object-centered frame. For example, when a van vehicle is hypothesized for an image actually obtained of a car or some unknown vehicle, the material properties of the van are used, but image measurements are obtained from the image of the car at locations given by transforming the coordinates of the van points (in the van-centered coordinate frame) to the image frame computed for the unknown vehicle.

Table 2 shows inter-class and intra-class variation when truck 1 is hypothesized. The data are gathered and images obtained at nine times during the daylight hours over a period of two days. The results show good inter-class separation and reasonable intra-class stability. Note that in the cases of wrong hypotheses, the feature values tend to be either indetermined or

Hypothesis: Data From:	Truck 1 Van	Truck 1 Car	Truck 1 Truck 2	Truck 1 Tank
11 am	4.62	1.00	-0.693	0.882
12 pm	1.00	1.00	15.74	-1.00
1 pm	1.00	NaN	1.00	2.846
2 pm	1.00	1.00	2.20	-1.00
3 pm	7.50	-Inf	1.00	1.00
4 pm	1.00	19.0	13.67	1.00
5 pm	2.95	51.0	1.71	4.20
9 am	1.00	1.20	3.00	-1.00
10 am	4.00	1.10	6.33	2.20

Table 2: Mistaken hypothesis feature values shows inter-class variation for feature A-1, consisting of point set $\{1,2,3,7\}$. The model for truck one is hypothesized. The feature value is formed using the model of truck 1 and the data from the respective other vehicles. When this feature is applied to the correctly hypothesized data of truck 1 it has a mean value of 0.0159 and a standard deviation of 0.0022. Thus feature, A-1, shows good separability when compared to the incorrect hypothesis feature value listed in the table.

unitary. This is a result of using the object centered coordinate system where the mistaken points fall on similar material types when dissimilar material types were expected.

Discussion

The approach described above is promising in that it makes available features that are (1) invariant to scene conditions, (2) able to separate different classes of objects, and (3) based on physics based models of the many phenomena that affect LWIR image generation.

Two aspects of the approach require further explanation. First, the factor, a_7, from equation 6 was used in this formulation to expand the number of degrees of freedom in the algebraic expression of the balance of energy equation. Although it is not interpreted directly as a physical parameter, it allows for the creation of a proper form and has no effect on the physical model. The motivation for including a_7 is that it is desirable to label as unknown variables both the convection parameter, h, and the ambient temperature, T_{amb}. These factors appear together in one of the terms of the balance of energy equation. With both factors labeled as variables, the coefficient can then only be unity, $a_7 = 1$. The resulting labeling produces a form that loses important degrees of freedom in the formation of invariant relations. Including $a_7 = \Delta$, implies that there is a scale of the temperature measurement, T_s, in the term $a_4 = T_s\Delta$. The transformation, M, of the variables induces a transformation on the coefficients. For the coefficient in question the induced transformation can be written $a_4' = m_{44}a_4$. Since the features found in section 4 are invariant to transformations of the form 8 it is invariant to an additional scale as in the action of the Δ parameter. Thus the term does not affect the relation of the physical model to the invariant feature. In addition, because a_7 does not appear in the feature there is no need to physically interpret its value.

Next, we consider the thermophysical justification of the transformation defined in the equation

$$X' = MX, \tag{12}$$

where X is a 6×4 collection of thermophysical variable vectors as defined in 6 at a time instance, t_n, and X' is the collection at a later time/scene t_{n+1}. The transformation M is defined in (8). The physical implication of such a transformation is that the four points in the thermophysical configuration are acted upon in the "same manner" by the environment. This is a reasonable assumption for the classes of objects under consideration. Note that if different types of surfaces are chosen (or points on surfaces with different surface orientations) the measurement vectors will, in general, be linearly independent. In other words, it is easy to select points such that the collection of measurement vectors span R^6. Then the existence of a non-singular transformation of the form of, M, for any pair of scenes and for a subset of four such points is guaranteed. Physically, the effect of the convection coefficient, solar irradiation and ambient temperature is consistent for the set of surface points. This fact taken with the fact that the emissivity can be considered relatively constant over time implies that it is reasonable to assume that equation (12) has physical justification.

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