A Counterexample to Theorems of Cox and Fine

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Abstract

Cox's well-known theorem justifying the use of probability is shown not to hold in finite domains. The counterexample also suggests that Cox's assumptions are insufficient to prove the result even in infinite domains. The same counterexample is used to disprove a result of Fine on comparative conditional probability.

1 Introduction

One of the best-known and seemingly most compelling arguments in favor of the use of probability is given by Cox (1946). Suppose we have a function Bel that associates a real number with each pair (U, V) of subsets of a domain W such that $U \neq \emptyset$. We write $\mathrm{Bel}(V|U)$ rather than $\mathrm{Bel}(U, V)$, since we think of $\mathrm{Bel}(V|U)$ as the credibility or likelihood of V given U.[1] Cox further assumes that $\mathrm{Bel}(\overline{V}|U)$ is a function of $\mathrm{Bel}(V|U)$ (where $\overline{V}$ denotes the complement of V in W), that is, there is a function S such that

A1. $\mathrm{Bel}(\overline{V}|U) = S(\mathrm{Bel}(V|U))$ if $U \neq \emptyset$,

and that $\mathrm{Bel}(V \cap V'|U)$ is a function of $\mathrm{Bel}(V'|V \cap U)$ and $\mathrm{Bel}(V|U)$, that is, there is a function F such that

A2. $\mathrm{Bel}(V \cap V'|U) = F(\mathrm{Bel}(V'|V \cap U), \mathrm{Bel}(V|U))$ if $V \cap U \neq \emptyset$.

Notice that if Bel is a probability function, then we can take $S(x) = 1 - x$ and $F(x, y) = xy$. Cox makes much weaker assumptions: he assumes that F is twice differentiable, with a continuous second derivative, and that S is twice differentiable. Under these assumptions, he shows that Bel is isomorphic to a probability distribution in the sense that there is a continuous one-to-one onto function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $g \circ \mathrm{Bel}$ is a probability distribution on W, and

$$g(\mathrm{Bel}(V|U)) \times g(\mathrm{Bel}(U)) = g(\mathrm{Bel}(V \cap U)) \text{ if } U \neq \emptyset, \quad (1)$$

where $\mathrm{Bel}(U)$ is an abbreviation for $\mathrm{Bel}(U|W)$.

Not surprisingly, Cox's result has attracted a great deal of interest in the AI literature. For example

- Cheeseman (1988) has called it the "strongest argument for use of standard (Bayesian) probability theory".
- Horvitz, Heckerman, and Langlotz (1986) used it as a basis for comparison of probability and other nonprobabilistic approaches to reasoning about uncertainty.
- Heckerman (1988) uses it as a basis for providing an axiomatization for belief update.

The main contribution of this paper is to show (by means of an explicit counterexample), that Cox's result does not hold in finite domains, even under strong assumptions on S and F (stronger than those made by Cox and those made in all papers proving variants of Cox's results). Since finite domains are arguably those of most interest in AI applications, this suggests that arguments for using probability based on Cox's result—and other justifications similar in spirit—must be taken with a grain of salt, and their proofs carefully reviewed. Moreover, the counterexample suggests that Cox's assumptions are insufficient to prove the result even in infinite domains.

It is known that some assumptions regarding F and S must be made to prove Cox's result. Dubois and Prade (1990) give an example of a function Bel, defined on a finite domain, that is not isomorphic to a probability distribution. For this choice of Bel, we can take $F(x, y) = \min(x, y)$ and $S(x) = 1 - x$. Since min is not twice differentiable, Cox's assumptions block the Dubois-Prade example.

Aczél (1966, Section 7 (Theorem 1)) does not make any assumptions about F, but he does make two other assumptions, each of which block the Dubois-Prade example. The first is that the $\mathrm{Bel}(V|U)$ takes on every value in some range $[e, E]$, with $e < E$. In the Dubois-Prade example, the domain is finite, so this certainly cannot hold. The second is that if V and V' are disjoint, then there is a continuous function $G : \mathbb{R}^2 \rightarrow \mathbb{R}$, strictly increasing in each argument, such that

A3. $\mathrm{Bel}(V \cup V'|U) = G(\mathrm{Bel}(V|U), \mathrm{Bel}(V'|U))$.

Dubois and Prade point out that, in their example, there is no function G satisfying A3 (even if we drop the requirement that G be continuous and strictly increasing in each argument).[2] With these assumptions, he gives a proof much

[1] Cox writes $V|U$ rather than $\mathrm{Bel}(V|U)$, and takes U and V to be propositions in some language rather than events, i.e., subsets of a given set. This difference is minor—there are well-known mappings from propositions to events, and vice versa. I use events here since they are more standard in the probability literature.

[2] In fact, Aczél allows there to be a different function G_U for each set U on the right-hand side of the conditional. However, the

in the spirit of that of Cox to show that Bel is essentially a probability distribution.

Reichenbach (1949) earlier proved a result similar to Aczél's, under somewhat stronger assumptions. In particular, he assumed A3, with G being $+$.

Other variants of Cox's result have also been considered in the literature. For example, Heckerman (1988) and Horvitz, Heckerman, and Langlotz (1986) assume that F is continuous and strictly increasing in each argument and S is continuous and strictly decreasing. Since min is not strictly continuous in each argument, it fails this restriction too.[3] Aleliunas (1988) gives yet another collection of assumptions and claims that they suffice to guarantee that Bel is essentially a probability distribution.

The first to observe potential problems with Cox's result is Paris (1994). As he puts it, "Cox's proof is not, perhaps, as rigorous as some pedants might prefer and when an attempt is made to fill in all the details some of the attractiveness of the original is lost." Paris provides a rigorous proof of the result, assuming that the range of Bel is contained in $[0, 1]$ and using assumptions similar to those of Horvitz, Heckerman, and Langlotz. In particular, he assumes that F is continuous and strictly increasing in $(0, 1]^2$ and that S is decreasing. However, he makes use of one additional assumption that, as he himself says, is not very appealing:

A4. For any $0 \leq \alpha, \beta, \gamma \leq 1$ and $\epsilon > 0$, there are sets U_1, U_2, U_3, and U_4 such that $U_3 \cap U_2 \cap U_1 \neq \emptyset$, and each of $|\mathrm{Bel}(U_4|U_3 \cap U_2 \cap U_1) - \alpha|$, $|\mathrm{Bel}(U_3|U_2 \cap U_1) - \beta|$, and $|\mathrm{Bel}(U_2|U_1) - \gamma|$ is less than ϵ.

Notice that this assumption forces the range of Bel to be dense in $[0, 1]$. This means that, in particular, the domain W on which Bel is defined cannot be finite.

Is this assumption really necessary? Paris suggests that Aczél needs something like it. (This issue is discussed in further detail below.) The counterexample of this paper gives further evidence. It shows that Cox's result fails in finite domains, even if we assume that the range of Bel is in $[0, 1]$, $S(x) = 1 - x$ (so that, in particular, S is twice differentiable and monotonically decreasing), $G(x, y) = x + y$, and F is infinitely differentiable and strictly increasing on $(0, 1]^2$. We can further assume that F is commutative, $F(0, x) = F(x, 0) = 0$, and that $F(x, 1) = F(1, x) = x$. The example emphasizes the point that the applicability of Cox's result is far narrower than was previously believed. It remains an open question as to whether there is an appropriate strengthening of the assumptions that does give us Cox's result in finite settings.

In fact, the example shows even more. In the course of his proof, Cox claims to show that F must be an associative function, that is, that $F(x, F(y, z)) = F(F(x, y), z)$. For the Bel of the counterexample, there can be no associative function F satisfying A2. It is this observation that is the key to showing that there is no probability distribution isomorphic to Bel.

What is going on here? Actually, Cox's proof just shows that $F(x, F(y, z)) = F(F(x, y), z)$ only for those triples (x, y, z) such that, for some sets U_1, U_2, U_3, and U_4, we have $x = \mathrm{Bel}(U_4|U_3 \cap U_2 \cap U_1)$, $y = \mathrm{Bel}(U_3|U_2 \cap U_1)$, and $z = \mathrm{Bel}(U_2|U_1)$. If the set of such triples (x, y, z) is dense in $[0, 1]^3$, then we conclude by continuity that F is associative. The content of A4 is precisely that the set of such triples is dense in $[0, 1]^3$. Of course, if W is finite, we cannot have density. As my counterexample shows, we do not in general have associativity in finite domains. Moreover, this lack of associativity can result in the failure of Cox's theorem.

A similar problem seems to exist in Aczél's proof (as already observed by Paris (1994)). While Aczél's proof does not involve showing that F is associative, it does involve showing that G is associative. Again, it is not hard to show that G is associative for appropriate triples, just as is the case for F. But it seems that Aczél also needs an assumption that guarantees that the appropriate set of triples is dense, and it is not clear that his assumptions do in fact guarantee this.[4] As shown in Section 2, the problem also arises in Reichenbach's proof.

This observation also shows that another well-known result in the literature is not completely correct. In his seminal book on probability and qualitative probability (1973), Fine considers a non-numeric notion of *comparative (conditional) probability*, which allows us to say "U given V is at least as probable as U' given V'", denoted $U|V \succeq U'|V'$. Conditions on $\succeq$ are given that are claimed to force the existence of (among other things) a function Bel such that $U|V \succeq U'|V'$ iff $\mathrm{Bel}(U|V) \geq \mathrm{Bel}(U'|V')$ and an associative function F satisfying A2. (This is Theorem 8 of Chapter II in (Fine 1973).) However, the Bel defined in my counterexample to Cox's theorem can be used to give a counterexample to this result as well.

The remainder of this paper is organized as follows. In the next section there is a more detailed discussion of the problem in Cox's proof. The counterexample to Cox's theorem is given in Section 3. The following section shows that it is also a counterexample to Fine's theorem. Section 5 concludes with some discussion.

2 The Problem With Cox's Proof

To understand the problems with Cox's proof, I actually consider Reichenbach's proof, which is similar in spirit Cox's proof (it is actually even close to Aczél's proof), but uses some additional assumptions, which makes it easier to explain in detail. Aczél, Cox, and Reichenbach all make critical use of functional equations in their proof, and they make the same (seemingly unjustified) leap at corresponding points in their proofs.

Dubois-Prade example does not even satisfy this weaker condition.

[3] Actually, the restriction that F be strictly increasing in each argument is a little too strong. If $e = \mathrm{Bel}(\emptyset)$, then it can be shown that $F(e, x) = F(x, e) = e$ for all x, so that F is not strictly increasing if one of its arguments is e.

[4] I should stress that my counterexample is not a counterexample to Aczél's theorem, since he explicitly assumes that the range of Bel is infinite. However, it does point out potential problems with his proof, and certainly shows that his argument does not apply to finite domains.

In the notation of this paper, Reichenbach (1949, pp. 65–67) assumes (1) that the range of $\mathrm{Bel}(\cdot|\cdot)$ is a subset of $[0, 1]$, (2) $\mathrm{Bel}(V|U) = 1$ if $U \subseteq V$, (3) that if V and V' are disjoint, then $\mathrm{Bel}(V \cup V'|U) = \mathrm{Bel}(V|U) + \mathrm{Bel}(V'|U)$ (thus, he assumes that A3 holds, with G being $+$), and (4) that A2 holds with a function F that is differentiable. (He remarks that the result holds even without assumption (4), although the proof is more complicated; Aczél in fact does not make an assumption like (4).)

Reichenbach's proof proceeds as follows: Replacing V' in A2 by $V_1 \cup V_2$, where V_1 and V_2 are disjoint, we get that

$$\mathrm{Bel}(V \cap (V_1 \cup V_2)|U) = F(\mathrm{Bel}(V_1 \cup V_2|V \cap U), \mathrm{Bel}(V|U)). \tag{2}$$

Using the fact that G is $+$, we immediately get

$$\mathrm{Bel}(V \cap (V_1 \cup V_2)|U) = \mathrm{Bel}(V \cap V_1|U) + \mathrm{Bel}(V \cap V_2|U) \tag{3}$$

and

$$\begin{aligned} &F(\mathrm{Bel}(V_1 \cup V_2|V \cap U), \mathrm{Bel}(V|U)) \\ &\quad = F(\mathrm{Bel}(V_1|V \cap U) + \mathrm{Bel}(V_2|V \cap U), \mathrm{Bel}(V|U)) \end{aligned} \tag{4}$$

Moreover, by A2, we also have, for $i = 1, 2$,

$$\mathrm{Bel}(V \cap V_i|U) = F(\mathrm{Bel}(V \cap V_i|V \cap U), \mathrm{Bel}(V|U)). \tag{5}$$

Putting together (2), (3), (4), and (5), we get that

$$\begin{aligned} &F(\mathrm{Bel}(V \cap V_1|V \cap U), \mathrm{Bel}(V|U)) + \\ &\quad F(\mathrm{Bel}(V \cap V_2|V \cap U), \mathrm{Bel}(V|U)) \\ &= F(\mathrm{Bel}(V \cap V_1|V \cap U) + \mathrm{Bel}(V \cap V_2|V \cap U, \mathrm{Bel}(V|U)). \end{aligned} \tag{6}$$

Taking $x = \mathrm{Bel}(V \cap V_1|V \cap U)$, $y = \mathrm{Bel}(V \cap V_2|V \cap U)$, and $z = \mathrm{Bel}(V|U)$ in (6), we get the functional equation

$$F(x, z) + F(y, z) = F(x + y, z). \tag{7}$$

Suppose we assume (as Reichenbach implicitly does) that this functional equation holds for all $(x, y, z) \in P = \{(x, y, z) \in [0, 1]^3 : x + y \leq 1\}$. The rest of the proof now follows easily. First, taking $x = 0$ in (7), it follows that

$$F(0, z) + F(y, z) = F(y, z),$$

from which we get that

$$F(0, z) = 0.$$

Next, fix z and let $g_z(x) = F(x, z)$. Since F is, by assumption, differentiable, from (7) we have that

$$g'_z(x) = \lim_{y \to 0}(F(x + y, z) - F(x, z)/y) = \lim_{y \to 0} F(y, z)/y.$$

It thus follows that $g'_z(x)$ is a constant, independent of x. Since the constant may depend on z, there is some function h such that $g'_z(x) = h(z)$. Using the fact that $F(0, z) = 0$, elementary calculus tells us that

$$g_z(x) = F(x, z) = h(z)x.$$

Using the assumption that for all U, V, we have $\mathrm{Bel}(V|U) = 1$ if $U \subseteq V$, we get that

$$\begin{aligned} &\mathrm{Bel}(V|U) = \mathrm{Bel}(V \cap V|U) \\ &= F(\mathrm{Bel}(V|V \cap U), \mathrm{Bel}(V|U)) = F(1, \mathrm{Bel}(V|U)). \end{aligned}$$

Thus, we have that

$$F(1, z) = h(z) = z.$$

We conclude that $F(x, z) = xz$.

Note, however, that this conclusion depends in a crucial way on the assumption that the functional equation (7) holds for all $(x, y, z) \in P$.[5] In fact, all that we can conclude from (6) is that it holds for all (x, y, z) such that there exist U, V, V_1, and V_2, with V_1 and V_2 disjoint, such that $x = \mathrm{Bel}(V \cap V_1|V \cap U)$, $y = \mathrm{Bel}(V \cap V_2|V \cap U)$, and $z = \mathrm{Bel}(V|U)$.

Let us say that a triple that satisfies this condition is *acceptable*. As I mentioned earlier, Aczél also assumes that $\mathrm{Bel}(V|U)$ takes on all values in $[e, E]$, where $e = \mathrm{Bel}(\emptyset|U)$ and $E = \mathrm{Bel}(U|U)$. (In Reichenbach's formulation, $e = 0$ and $E = 1$.) There are two ways to interpret this assumption. The weak interpretation is that for each $x \in [0, 1]$, there exist U, V such that $\mathrm{Bel}(V|U) = x$. The strong interpretation is that for each U and x, there exists V such that $\mathrm{Bel}(V|U) = x$. It is not clear which interpretation is intended by Aczél. Neither one obviously suffices to prove that every triple in P is acceptable, although it does seem plausible that it might follow from the second assumption.

In any case, both Aczél and Reichenbach (as well as Cox, in his analogous functional equation) see no need to check that Equation (7) holds throughout P. However, it turns out to be quite necessary to do this. Moreover, it is clear that if W is finite, there are only finitely tuples in P which are acceptable, and it is not the case that all of P is. As we shall see in the next section, this observation has serious consequences as far as all these proofs are concerned.

3 The Counterexample to Cox's Theorem

The goal of this section is to prove

Theorem 3.1: *There is a function Bel_0, a finite domain W, and functions S, F, and G satisfying A1, A2, and A3 respectively such that*

- *$Bel_0(V|U) \in [0, 1]$ for $U \neq \emptyset$,*
- *$S(x) = 1 - x$ (so that S is strictly decreasing and infinitely differentiable),*
- *$G(x, y) = x + y$ (so that G is strictly increasing in each argument and is infinitely differentiable),*
- *F is infinitely differentiable, nondecreasing in each argument in $[0, 1]^2$, and strictly increasing in each argument in $(0, 1]^2$. Moreover, F is commutative, $F(x, 0) = F(0, x) = 0$, and $F(x, 1) = F(1, x) = x$.*

However, there is no one-to-one onto function $g : [0, 1] \to [0, 1]$ satisfying (1).

Note that the hypotheses on Bel_0, S, G, and F are at least as strong as those made in all the other variants of Cox's result, while the assumptions on g are weaker than those made in the variants. For example, there is no requirement that g be continuous or increasing nor that $g \circ \mathrm{Bel}_0$ is a probability distribution (although Paris and Aczél both prove that,

[5] Actually, using the continuity of F, it suffices that the functional equation holds for a set of triples which is dense in P.

under their assumptions, g can be taken to satisfy all these requirements). This serves to make the counterexample quite a strong one.

Proof: Consider a domain W with 12 points: $w_1, ..., w_{12}$. We associate with each point $w \in W$ a weight $f(w)$, as follows.

$$\begin{array}{ll} f(w_1) = 3 & f(w_4) = 5 \times 10^4 \\ f(w_2) = 2 & f(w_5) = 6 \times 10^4 \\ f(w_3) = 6 & f(w_6) = 8 \times 10^4 \end{array}$$

$$\begin{array}{ll} f(w_7) = 3 \times 10^8 & f(w_{10}) = 3 \times 10^{18} \\ f(w_8) = 8 \times 10^8 & f(w_{11}) = 2 \times 10^{18} \\ f(w_9) = 8 \times 10^8 & f(w_{12}) = 14 \times 10^{18} \end{array}$$

For a subset U of W, we define $f(U) = \sum_{w \in U} f(w)$. Thus, we can define a probability distribution Pr on W by taking $\Pr(U) = f(U)/f(W)$.

Let f' be identical to f, except that $f'(w_{10}) = (3-\delta) \times 10^{18}$ and $f'(w_{11}) = (2+\delta) \times 10^{18}$, where δ is defined below. Again, we extend f' to subsets of W by defining $f'(U) = \sum_{w \in U} f'(w)$. Let $W' = \{w_{10}, w_{11}, w_{12}\}$. If $U \neq \emptyset$, define

$$\mathrm{Bel}_0(V|U) = \begin{cases} f'(V \cap U)/f(U) & \text{if } W' \subseteq U \\ f(V \cap U)/f(U) & \text{otherwise.} \end{cases}$$

Bel_0 is clearly very close to Pr. If $U \neq \emptyset$, then it is easy to see that $|\mathrm{Bel}_0(V|U) - \Pr(V|U)| = |f'(V \cap U) - f(V \cap U)|/f(U) \leq \delta$. We choose $\delta > 0$ so that

$$\text{if } \Pr(V|U) > \Pr(V'|U'), \text{ then } \mathrm{Bel}_0(V|U) > \mathrm{Bel}_0(V'|U'). \quad (8)$$

Since the range of Pr is finite, all sufficiently small δ satisfy (8).

The exact choice of weights above is not particularly important. One thing that is important though is the following collection of equalities:

$$\begin{array}{l} \Pr(w_1|\{w_1, w_2\}) = \Pr(w_{10}|\{w_{10}, w_{11}\}) = 3/5 \\ \Pr(\{w_1, w_2\}|\{w_1, w_2, w_3\}) = \Pr(w_4|\{w_4, w_5\}) = 5/11 \\ \Pr(\{w_4, w_5\}|\{w_4, w_5, w_6\}) = \\ \quad \Pr(\{w_7, w_8\}|\{w_7, w_8, w_9\}) = 11/19 \\ \Pr(w_4|\{w_4, w_5, w_6\}) = \\ \quad \Pr(\{w_{10}, w_{11}\}|\{w_{10}, w_{11}, w_{12}\}) = 5/19 \\ \Pr(w_1|\{w_1, w_2, w_3\}) = \Pr(w_7|\{w_7, w_8\}) = 3/11. \end{array} \quad (9)$$

It is easy to check that exactly the same equalities hold if we replace Pr by Bel_0.

Although, as is shown below, the function F satisfying A2 can be taken to be infinitely differentiable and increasing in each argument, the equalities in (9) suffice to guarantee that it cannot be taken to be associative, that is, we do not in general have

$$F(x, F(y, z)) = F(F(x, y), z).$$

Indeed, there is no associative function F satisfying A2, even if we drop the requirements that F be differentiable or increasing.

Lemma 3.2: *For Bel_0 as defined above, there is no associative function F satisfying A2.*

Proof: Suppose there were such a function F. From (9), we must have that

$$\begin{array}{ll} & F(5/11, 11/19) \\ = & F(\mathrm{Bel}_0(w_4|\{w_4, w_5\}), \mathrm{Bel}_0(\{w_4, w_5\}|\{w_4, w_5, w_6\})) \\ = & \mathrm{Bel}_0(w_4|\{w_4, w_5, w_6\}) = 5/19 \end{array}$$

and that

$$\begin{array}{ll} & F(3/5, 5/11) \\ = & F(\mathrm{Bel}_0(w_1|\{w_1, w_2\}), \mathrm{Bel}_0(\{w_1, w_2\}|\{w_1, w_2, w_3\})) \\ = & \mathrm{Bel}_0(w_1|\{w_1, w_2, w_3\}) = 3/11. \end{array}$$

It follows that

$$F(3/5, F(5/11, 11/19)) = F(3/5, 5/19)$$

and that

$$F(F(3/5, 5/11), 11/19) = F(3/11, 11/19).$$

Thus, if F were associative, we would have

$$F(3/5, 5/19) = F(3/11, 11/19).$$

On the other hand, from (9) again, we see that

$$\begin{array}{ll} & F(3/5, 5/19) \\ = & F(\mathrm{Bel}_0(w_{10}|\{w_{10}, w_{11}\}), \mathrm{Bel}_0(\{w_{10}, w_{11}\}|\{w_{10}, w_{11}, w_{12}\})) \\ = & \mathrm{Bel}_0(w_{10}|\{w_{10}, w_{11}, w_{12}\}) = (3-\delta)/19, \end{array}$$

while

$$\begin{array}{ll} & F(3/11, 11/19) \\ = & F(\mathrm{Bel}_0(w_7|\{w_7, w_8\}), \mathrm{Bel}_0(\{w_7, w_8\}|\{w_7, w_8, w_9\})) \\ = & \mathrm{Bel}_0(w_7|\{w_7, w_8, w_9\}) = 3/19. \end{array}$$

It follows that F cannot be associative. ∎

The next lemma shows that Bel_0 cannot be isomorphic to a probability function.

Lemma 3.3: *For Bel_0 as defined above, there is no one-to-one onto function $g : [0,1] \to [0,1]$ satisfying (1).*

Proof: Suppose there were such a function g. First note that $g(\mathrm{Bel}_0(U)) \neq 0$ if $U \neq \emptyset$. For if $g(\mathrm{Bel}_0(U)) = 0$, then it follows from (1) that for all $V \subseteq U$, we have

$$g(\mathrm{Bel}_0(V)) = g(\mathrm{Bel}_0(V|U)) \times g(\mathrm{Bel}_0(U)) = g(\mathrm{Bel}_0(V|U)) \times 0 = 0.$$

Thus, $g(\mathrm{Bel}_0(V)) = g(\mathrm{Bel}_0(U))$ for all subsets V of U. Since the definition of Bel_0 guarantees that $\mathrm{Bel}_0(V) \neq \mathrm{Bel}_0(U)$ if V is a strict subset of U, this contradicts the assumption that g is one-to-one. Thus, $g(\mathrm{Bel}_0(U)) \neq 0$ if $U \neq \emptyset$. It now follows from (1) that if $U \neq \emptyset$, then

$$g(\mathrm{Bel}_0(V|U)) = g(\mathrm{Bel}_0(V \cap U))/g(\mathrm{Bel}_0(U)). \quad (10)$$

Now define $F(x, y) = g^{-1}(g(x) \times g(y))$. Notice that, by applying the observation above repeatedly, if $V \cap U \neq \emptyset$, we get

$$\begin{array}{ll} & F(\mathrm{Bel}_0(V'|V \cap U), \mathrm{Bel}_0(V|U)) \\ = & g^{-1}((g(\mathrm{Bel}_0(V'|V \cap U)) \times g(\mathrm{Bel}_0(V|U)) \\ = & g^{-1}(g(\mathrm{Bel}_0(V' \cap V \cap U))/g(\mathrm{Bel}_0(U))) \\ = & g^{-1}(g(\mathrm{Bel}_0(V' \cap V|U))) \\ = & \mathrm{Bel}_0(V' \cap V|U). \end{array}$$

Thus, F satisfies A2. Moreover, notice that F is associative, since

$$\begin{aligned} F(F(x,y),z) &= g^{-1}(g(g^{-1}(g(x)\times g(y)))\times g(z)) \\ &= g^{-1}(g(x)\times g(y)\times g(z)) \\ &= g^{-1}(g(x)\times g(g^{-1}(g(y)\times g(z)))) \\ &= F(x,F(y,z)). \end{aligned}$$

But this contradicts Lemma 3.2. ∎

Despite the fact that Bel_0 is not isomorphic to a probability function, functions S, F, and G can be defined that satisfy A1, A2, and A3, respectively, and all the other requirements stated in Theorem 3.1. The argument for S and G is easy; all the work goes into proving that an appropriate F exists.

Lemma 3.4 : *There exists an infinitely differentiable, strictly decreasing function $S : [0,1] \to [0,1]$ such that $Bel_0(\overline{V}|U) = S(Bel_0(V|U))$ for all sets $U, V \subseteq W$ with $U \neq \emptyset$. In fact, we can take $S(x) = 1 - x$.*

Proof: This is immediate from the observation that $\mathrm{Bel}_0(\overline{V}|U) = 1 - \mathrm{Bel}_0(V|U)$ for $U, V \subseteq W$. ∎

Lemma 3.5: *There exists an infinitely differentiable function $G : [0,1]^2 \to [0,1]$, increasing in each argument, such that if $U, V, V' \subseteq W$, $V \cap V' = \emptyset$, and $U \neq \emptyset$, then $Bel_0(V \cup V'|U) = G(Bel_0(V|U), Bel_0(V', U))$. In fact, we can take $G(x,y) = x + y$.*

Proof: This is immediate from the definition of Bel_0. ∎

Thus, all that remains is to show that an appropriate F exists. The key step is provided by the following lemma, which essentially shows that there is a well defined F that is increasing.

Lemma 3.6: *If $U_2 \cap U_1 \neq \emptyset$ and $V_2 \cap V_1 \neq \emptyset$, then*

(a) if $Bel_0(V_3|V_2 \cap V_1) \leq Bel_0(U_3|U_2 \cap U_1)$ and $Bel_0(V_2|V_1) \leq Bel_0(U_2|U_1)$, then $Bel_0(V_3 \cap V_2|V_1) \leq Bel_0(U_3 \cap U_2|U_1)$,

(b) if $Bel_0(V_3|V_2 \cap V_1) < Bel_0(U_3|U_2 \cap U_1)$, $Bel_0(V_2|V_1) \leq Bel_0(U_2|U_1)$, $Bel_0(U_3|U_2 \cap U_1) > 0$, and $Bel_0(U_2|U_1) > 0$, then $Bel_0(V_3 \cap V_2|V_1) < Bel_0(U_3 \cap U_2|U_1)$,

(c) if $Bel_0(V_3|V_2 \cap V_1) \leq Bel_0(U_3|U_2 \cap U_1)$, $Bel_0(V_2|V_1) < Bel_0(U_2|U_1)$, $Bel_0(U_3|U_2 \cap U_1) > 0$, and $Bel_0(U_2|U_1) > 0$, then $Bel_0(V_3 \cap V_2|V_1) < Bel_0(U_3 \cap U_2|U_1)$,

Proof: First observe that if $\mathrm{Bel}_0(V_3|V_2 \cap V_1) \leq \mathrm{Bel}_0(U_3|U_2 \cap U_1)$ and $\mathrm{Bel}_0(V_2|V_1) \leq \mathrm{Bel}_0(U_2|U_1)$, then from (8), it follows that $\Pr(V_3|V_2 \cap V_1) \leq \Pr(U_3|U_2 \cap U_1)$ and $\Pr(V_2|V_1) \leq \Pr(U_2|U_1)$. If we have either $\Pr(V_3|V_2 \cap V_1) < \Pr(U_3|U_2 \cap U_1)$ or $\Pr(V_2|V_1) < \Pr(U_2|U_1)$, then we have either $\Pr(V_3 \cap V_2|V_1) < \Pr(U_3 \cap U_2|U_1)$ or $\Pr(U_3|U_2 \cap U_1) = 0$ or $\Pr(U_2|U_1) = 0$. It follows that either $\mathrm{Bel}_0(V_3 \cap V_2|V_1) < \mathrm{Bel}_0(U_3 \cap U_2|U_1)$ (this uses (8) again) or that $\mathrm{Bel}_0(V_3 \cap V_2|V_1) = \mathrm{Bel}_0(U_3 \cap U_2|U_1) = 0$. In either case, the lemma holds.

Thus, it remains to deal with the case that $\Pr(V_3|V_2 \cap V_1) = \Pr(U_3|U_2 \cap U_1)$ and $\Pr(V_2|V_1) = \Pr(U_2|U_1)$, and hence $\Pr(V_3 \cap V_2|V_1) = \Pr(U_3 \cap U_2|U_1)$. The details of this analysis are left to the full paper. ∎

Lemma 3.7: *There exists a function $F : [0,1]^2 \to [0,1]$ satisfying all the assumptions of the theorem.*

Proof: Define a partial function F' on $[0,1]^2$ whose domain consists of all pairs (x,y) such that for some subsets U, V, V' of W, we have $x = \mathrm{Bel}_0(V'|V \cap U)$ and $y = \mathrm{Bel}_0(V|U)$. For such (x,y), we define $F'(x,y) = \mathrm{Bel}_0(V' \cap V|U)$. *A priori*, it is possible that there exist sets $U_1, U_2, U_3, V_1, V_2, V_3$ such that $x = \mathrm{Bel}_0(U_3|U_2 \cap U_1) = \mathrm{Bel}_0(V_3|V_2 \cap V_1)$ and $y = \mathrm{Bel}_0(U_2|U_1) = \mathrm{Bel}_0(V_2|V_1)$, yet $\mathrm{Bel}_0(U_3 \cap U_2|U_1) \neq \mathrm{Bel}_0(V_3 \cap V_2|V_1)$. If this were the case, then $F'(x,y)$ would not be well defined. However, Lemma 3.6 says that this cannot happen. Moreover, the lemma assures us that F' is increasing on its domain, and strictly increasing as long as one of its arguments is not 0. Notice that if $\mathrm{Bel}_0(V|U) = x \neq 0$ for some V, U, then $(0,x)$, $(x,1)$ and $(1,x)$ are in the domain of F', and $F'(x,1) = F'(1,x) = x$, while $F'(0,x) = 0$. It is easy to see that there are no pairs $(x,0)$ in the domain of F'. Finally, there are no pairs (x,y) and (y,x) that are both in the domain of F' unless one of x or y is 1.

The domain of F' is finite. It is straightforward to extend F' to a commutative, infinitely differentiable, and increasing function F defined on all of $[0,1]^2$, which is strictly increasing on $(0,1]^2$, and satisfies $F(x,1) = F(1,x) = x$ and $F(x,0) = F(0,x) = 0$. (Note that to make F commutative, we first define it on pairs (x,y) such that $x \geq y$, and then if $x < y$, we define $F((x,y) = F(y,x)$. Since F' is commutative on its domain of definition, this approach does not run into problems.) Clearly F satisfies A2, since (by construction) F' does, and A2 puts constraints only on the domain of F'. ∎

Theorem 3.1 now follows from Lemmas 3.3, 3.4, 3.5, and 3.7. ∎

4 The Counterexample to Fine's Theorem

Fine is interested in what he calls *comparative conditional probability*. Thus, rather than associating a real number with each "conditional object" $V|U$, he puts an ordering $\succeq$ on such objects. As usual, $V|U \succ V'|U'$ is taken to be an abbreviation for $V|U \succeq V'|U'$ and not$(V'|U' \succeq V|U)$.

Fine is interested in when such an ordering is induced by a real-valued belief function with reasonable properties. He says that a real-valued function P on such objects *agrees with* $\succeq$ if $P(V|U) \geq P(V'|U')$ iff $V|U \succeq V'|U'$. Fine then considers a number of axioms that $\succeq$ might satisfy. For our purposes, the most relevant are the ones Fine denotes QCC1, QCC2, QCC5, and QCC7.

QCC1 just says that $\succeq$ is a linear order:

QCC1. $V|U \succeq V'|U'$ or $V'|U' \succeq V|U$.

QCC2 says that $\succeq$ is transitive:

QCC2. If $V_1|U_1 \succeq V_2|U_2$ and $V_2|U_2 \succeq V_3|U_3$, then $V_1|U_1 \succeq V_3|U_3$.

QCC5 is a technical condition involving notions of order topology. The relevant definitions are omitted here (see (Fine 1973) for details), since QCC5, as Fine observes, holds vacuously in finite domains (the only ones of interest here).

QCC5. The set $\{V|U\}$ has a countable basis in the order topology induced by $\succ$.

Finally, QCC7 essentially says that $\succeq$ is increasing, in the sense of Lemma 3.6.

QCC7.

(a) If $V_3|V_2 \cap V_1 \succeq U_3|U_2 \cap U_1$ and $V_2|V_1 \succeq U_2|U_1$ then $V_3 \cap V_2|V_1 \succeq U_3 \cap U_2|U_1$.

(b) If $V_3|V_2 \cap V_1 \succeq U_2|U_1$ and $V_2|V_1 \succeq U_3|U_2 \cap U_1$ then $V_3 \cap V_2|V_1 \succeq U_3 \cap U_2|U_1$.

(c) If $V_3|V_2 \cap V_1 \succ U_3|U_2 \cap U_1$, $V_2|V_1 \succeq U_2|U_1$, and $V_2|V_1 \succ \emptyset|W$, then $V_3 \cap V_2|V_1 \succ U_3 \cap U_2|U_1$.

Fine then claims the following theorem:

Fine's Theorem: (Fine 1973, Chapter II, Theorem 8) *If $\succeq$ satisfies QCC1, QCC2, QCC5, then there exists some agreeing function P. There exists a function F of two variables such that*

1. $P(V \cap V'|U) = F(P(V'|V \cap U), P(V|U))$,[6]
2. $F(x, y) = F(y, x)$,
3. $F(x, y)$ *is increasing in x for $y > P(\emptyset|W)$,*
4. $F(x, F(y, z)) = F(F(x, y), z)$,
5. $F(P(W|U), y) = y$,
6. $F(P(\emptyset|U), y) = P(\emptyset|U)$.

iff $\succeq$ also satisfies QCC7.

The only relevant clauses for our purposes are Clause (1), which is just A2, and Clause (4), which says that F is associative. As Lemma 3.2 shows, there is no associative function satisfying A2 for Bel_0. As I now show, this means that Fine's theorem does not quite hold either.

Before doing so, let me briefly touch on a subtle issue regarding the domain of $\succeq$. In the counterexample of the previous section, $\mathrm{Bel}_0(V|U)$ is defined as long as $U \neq \emptyset$. Fine does not assume that the $\succeq$ relation is necessarily defined on all objects $V|U$ such that $U, V \subseteq W$ and $U \neq \emptyset$. He assumes that there is an algebra $\mathcal{F}$ of subsets of W (that is, a set of subsets closed under finite intersections and complementation) and a subset $\mathcal{F}'$ of $\mathcal{F}$ closed under finite intersections and not containing the empty set such that $\succeq$ is defined on conditional objects $V|U$ such that $V \in \mathcal{F}$ and $U \in \mathcal{F}'$. Since $\mathcal{F}'$ is closed under intersection and does not contain the empty set, $\mathcal{F}'$ cannot contain disjoint sets. If W is finite, then the only way a collection $\mathcal{F}'$ can meet Fine's restriction is if there is some nonempty set U_0 such that all elements in $\mathcal{F}'$ contain U_0. This restriction is clearly too strong to the extent that comparative conditional probability is intended to generalize probability. If Pr is a probability function, then it certainly makes sense to compare $\Pr(V|U)$ and $\Pr(V'|U')$ even if U and U' are disjoint sets. Fine [private communication, 1995] suggested that it might be better to constrain QCC7 so that we do not condition on events U that are equivalent to $\emptyset$ (where U is equivalent to $\emptyset$ if $\emptyset \succeq U$ and $U \succeq \emptyset$). Since the only event equivalent to $\emptyset$ in the counterexample of the previous section is $\emptyset$ itself, this means that the counterexample can be used without change. This is what is done in the proof below. In the full paper, I indicate how to modify the counterexample so that it satisfies Fine's original restrictions.

[6] Fine assumes that $P(V \cap V'|U) = F(P(V|U), P(V'|V \cap U))$. I have reordered the arguments here for consistency with Cox's theorem.

Theorem 4.1: *There exists an ordering $\succeq$ satisfying QCC1, QCC2, QCC5, and QCC7, such that for every function P agreeing with $\succeq$, there is no associative function F of two variables such that $P(V \cap V')|U) = F(P(V'|V \cap U), P(V|U))$.*

Proof: Let W and Bel_0 be as in the counterexample in the previous section. Define $\succeq$ so that Bel_0 agrees with $\succeq$. Thus, $V|U \succeq V'|U'$ iff $\mathrm{Bel}_0(V|U) \geq \mathrm{Bel}_0(V'|U')$. Clearly $\succeq$ satisfies QCC1 and QCC2. As was mentioned earlier, since W is finite, $\succeq$ vacuously satisfies QCC5. Lemma 3.6 shows that $\succeq$ satisfies parts (a) and (c) of QCC7. To show that $\succeq$ also satisfies part (b) of QCC7, we must prove that if $\mathrm{Bel}_0(V_3|V_2 \cap V_1) \geq \mathrm{Bel}_0(U_2|U_1)$ and $\mathrm{Bel}_0(V_2|V_1) \geq \mathrm{Bel}_0(U_3|U_2 \cap U_1)$, then $\mathrm{Bel}_0(V_3 \cap V_2|V_1) \geq \mathrm{Bel}_0(U_3 \cap U_2|U_1)$. The proof of this is almost identical to that of Lemma 3.6; we simply exchange the roles of $\Pr(V_2|V_1)$ and $\Pr(V_3|V_2 \cap V_1)$ in that proof. I leave the details to the reader. Lemma 3.2 shows that there is no associative function F satisfying A2 for Bel_0. All that was used in the proof was the fact that Bel_0 satisfied the inequalities of (9). But these equalities must hold for any function agreeing with $\succeq$. Thus, exactly the same proof shows that if P is any function agreeing with $\succeq$, then there is no associative function F satisfying $P(V \cap V'|U) = F(P(V'|V \cap U), P(V|U))$. ∎

5 Discussion

Let me summarize the status of various results in the light of the counterexample of this paper:

- Cox's theorem as originally stated does not hold in finite domains. Moreover, even in infinite domains, the counterexample and the discussion in Section 2 suggest that more assumptions are required for its correctness. In particular, the claim in his proof that F is associative does not follow.
- Although the counterexample given here is not a counterexample to Aczél's theorem, his assumptions do not seem strong enough to guarantee that the function G is associative, as he claims it is.
- The variants of Cox's theorem stated by Heckerman (1988), Horvitz, Heckerman, and Langlotz (1986), and Aleliunas (1988) all succumb to the counterexample.
- The claim that the function F must be associative in Fine's theorem is incorrect. Fine has an analogous result (Fine 1973, Chapter II, Theorem 4) for unconditional comparative probability involving a function G as in Aczél's theorem. This function too is claimed to be associative, and again, this does not seem to follow (although my counterexample does not apply to that theorem).

Of course, the interesting question now is what it would take to recover Cox's theorem. Paris's assumption A4 suffices. As we have observed, A4 forces the domain of Bel to be infinite, as does the assumption that the range of Bel is all of [0, 1]. We can always extend a domain to an infinite—indeed, uncountable—domain by assuming that we have an infinite collection of independent fair coins, and that we can talk about outcomes of coin tosses as well as the original events in the domain. (This type of "extendibility" assumption is fairly standard; for example, it is made by Savage (1954) in quite a different context.) In such an extended domain, it seems reasonable to also assume that Bel varies uniformly between 0 (certain falsehood) and 1 (certain truth). If we also assume A4 (or something like it), we can then recover Cox's theorem. Notice, however, that this viewpoint disallows a notion of belief that takes on only finitely many or even countably many gradations.

Suppose we really are interested in a particular finite domain, and we do not want to extend it. What assumptions do we then need to get Cox's theorem? The counterexample given here could be circumvented by requiring that F be associative on all tuples (rather than just on the tuples (x, y, z) that arise as $x = \mathrm{Bel}_0(U_4|U_3 \cap U_2 \cap U_1)$, $y = \mathrm{Bel}_0(U_3|U_2 \cap U_1)$, and $z = \mathrm{Bel}_0(U_2|U_1)$). However, if we really are interested in a single domain, the motivation for making requirements on the behavior of F on belief values that do not arise is not so clear. Moreover, it is far from clear that assuming that F is associative suffices to prove the theorem. For example, Cox's proof makes use of various functional equations involving F and S, analogous to the equation (7) that appears in Section 2. These functional equations are easily seen to hold for certain tuples. However, as we saw in Section 2, the proof really requires that they hold for *all* tuples. Just assuming that F is associative does not appear to suffice to guarantee that the functional equations involving S hold for all tuples. Futher assumptions appear necessary.

One condition (suggested by Nir Friedman) that does seem to suffice (although I have not checked details) is that of assuming that essentially all beliefs are distinct. More precisely, we could assume

- if $\emptyset \subset U \subset V$, $\emptyset \subset U' \subset V'$, and $(U, V) \neq (U', V')$, then $\mathrm{Bel}(U|V) \neq \mathrm{Bel}(U'|V')$.

Even if this condition suffices, note that it precludes, for example, a uniform probability distribution, and thus again seems unduly restrictive.

So what does all this say regarding the use of probability? Not much. Although I have tried to argue here that Cox's justification of probability is not quite as strong as previously believed, and the assumptions underlying the variants of it need clarification, I am not trying to suggest that probability should be abandoned. There are many other justifications for its use.

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