A Fuzzy Description Logic

Umberto Straccia

I.E.I - C.N.R.
Via S. Maria, 46
I-56126 Pisa (PI) ITALY
straccia@iei.pi.cnr.it

Abstract

Description Logics (DLs, for short) allow reasoning about individuals and concepts, *i.e.* set of individuals with common properties. Typically, DLs are limited to dealing with crisp, well defined concepts. That is, concepts for which the problem whether an individual is an instance of it is a yes/no question. More often than not, the concepts encountered in the real world do not have a precisely defined criteria of membership: we may say that an individual is an instance of a concept only to a certain degree, depending on the individual's properties. Concepts of this kind are rather vague than precise. As fuzzy logic directly deals with the notion of vagueness and imprecision, it offers an appealing foundation for a generalisation of DLs to vague concepts.

In this paper we present a general fuzzy DL, which combines fuzzy logic with DLs. We define its syntax, semantics and present constraint propagation calculi for reasoning in it.

Introduction

Description Logics (DLs, for short) provide a logical reconstruction of the so-called frame-based knowledge representation languages[1]. *Concepts, roles* and *individuals* are the basic building blocks of these logics. Concepts are expressions which collect the properties, described by means of roles, of a set of individuals. From a first order point of view, concepts can be seen as unary predicates, whereas roles are interpreted as binary predicates. A *knowledge base* (KB) typically contains a set of *assertions*. An assertion states either that an individual a is an instance of a concept C (written $C(a)$), or that two individuals a and b are related by means of a role R (written $R(a, b)$). A basic inference task with knowledge bases is *entailment* and amounts to verify whether the individual a is an instance of the concept C w.r.t. the KB Σ (written $\Sigma \models C(a)$).

Typically, DLs are limited to dealing with crisp concepts. However, many useful concepts that are needed by an intelligent system do not have well defined boundaries. That is, often it happens that the concepts encountered in the real world do not have a precisely defined criteria of membership, *i.e.* they are *vague concepts* rather than precise concepts. For instance, `Tall` is such a concept: we may say that an individual `tom` is an instance of the concept `Tall` only to a certain degree $n \in [0, 1]$ depending on `tom`'s height.

Fuzzy logic directly deals with the notion of vagueness and imprecision using fuzzy predicates. Therefore, it offers an appealing foundation for a generalisation of DLs in order to dealing with such vague concepts.

The aim of this work is to present a general fuzzy DL, which combines fuzzy logic with DLs. In particular we will extend DLs by allowing expressions of the form $\langle C(a)\, n\rangle$ ($n \in [0, 1]$), *e.g.* $\langle \mathtt{Tall}(\mathtt{tom})\, .7\rangle$, with intended meaning "the membership degree of individual a being an instance of concept C is at least n".

Extending DLs with fuzzy features has already be done in the past. For instance, in (Yen 1991) the very limited DL $\mathcal{FL}^-$ (Brachman & Levesque 1984) has been extended with some fuzzy features. In particular, it allows the definition of fuzzy concepts and the only supported reasoning mechanism is determining subsumption[2]. Unfortunately, it does not allow reasoning in presence of assertions. Recently, (Meghini, Sebastiani, & Straccia 1997) proposed a fuzzy DL as a tool for modelling multimedia document retrieval[3]. But this work was rather at a preliminary stage and no reasoning algorithm was given.

We present a more general framework in the sense that it is based both on the DL $\mathcal{ALC}$, a significant and expressive representative of the various DLs, and on sound and complete constraint propagation calculi for reasoning in it. This allows us to adapt it easily to the different DLs presented in the literature. Moreover, we will show that the additional expressive power has no impact from a computational complexity point of view. This is important as the nice trade-off between computational complexity and expressive power of DLs contributes to their popularity.

Finally, note that most existing work in extending DLs for uncertainty management lie in the category of probabilistic extension like *e.g.* (Heinsohn 1994; Jäger 1994; Koller, Levy, & Pfeffer 1997) with some exceptions like (Hollunder 1994). Even though these

[1]See the DL Web Home Page: `http://dl.kr.org/dl`.

[2]Roughly, a concept D subsumes a concept C iff from a first order point of view, $\forall x.C(x) \to D(x)$ is logically valid.

[3]The idea to use DLs in the context of multimedia document retrieval has been proposed in (Gobel, Haul, & Bechhofer 1996) too.

probabilistic extensions enlarge the applicability of DLs they do not directly address the issue of reasoning about individuals and vague concepts. Moreover, reasoning in a probabilistic framework is generally a harder task, from a computational point of view, than the relative non probabilistic case (see *e.g.* (Roth 1996) for an overview) and thus, the computational problems have to be addressed carefully like in (Koller, Levy, & Pfeffer 1997).

In the following sections we first introduce crisp $\mathcal{ALC}$, then we extend it to the fuzzy case. Thereafter, we will present constraint propagation calculi for reasoning in it.

A quick look to $\mathcal{ALC}$

The specific DL we will extend with fuzzy capabilities is $\mathcal{ALC}$, a significant representative of the best-known and most important family of DLs, the $\mathcal{AL}$ family.

We assume three alphabets of symbols, called *primitive concepts* (denoted by A), *primitive roles* (denoted by R) and *individuals* (denoted by a and b). The *concepts* (denoted by C and D) of the language $\mathcal{ALC}$ are formed out of primitive concepts according to the following syntax rules[4]:

$$
\begin{array}{lrl}
C, D \quad \longrightarrow & \top & \text{(top concept)} \\
& \bot & \text{(bottom concept)} \\
& A| & \text{(primitive concept)} \\
& C \sqcap D| & \text{(concept conjunction)} \\
& C \sqcup D| & \text{(concept disjunction)} \\
& \neg C| & \text{(concept negation)} \\
& \forall R.C| & \text{(universal quantification)} \\
& \exists R.C & \text{(existential quantification)}
\end{array}
$$

An *interpretation* $\mathcal{I}$ is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non empty set $\Delta^{\mathcal{I}}$ (called the *domain*) and of an *interpretation function* $\cdot^{\mathcal{I}}$ mapping different individuals into different elements of $\Delta^{\mathcal{I}}$, primitive concepts into subsets of $\Delta^{\mathcal{I}}$ and primitive roles into subsets of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The interpretation of complex concepts is defined in the usual way: $\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$, $\bot^{\mathcal{I}} = \emptyset$, $(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$, $(C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$, $(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$, $(\forall R.C)^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} : \forall d'. \, (d, d') \in R^{\mathcal{I}} \text{ implies } d' \in C^{\mathcal{I}}\}$, and $(\exists R.C)^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} : \exists d'. \, (d, d') \in R^{\mathcal{I}} \text{ and } d' \in C^{\mathcal{I}}\}$. For instance, the concept $\mathtt{Tall} \sqcap \mathtt{Student}$ denotes the set of tall students.

An *assertion* (denoted by α) is an expression of type $C(a)$ (a is an instance of C), or an expression of type $R(a, b)$ (a is related to b by means of R). For instance, $(\mathtt{Tall} \sqcap \mathtt{Student})(\mathtt{tom})$ asserts that `tom` is a tall student, whereas `Friend(tim,tom)` asserts that `tom` is a friend of `tim`.

The semantics of assertions is specified by saying that the assertion $C(a)$ (resp. $R(a, b)$) is *satisfied* by $\mathcal{I}$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$ (resp. $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$). A set Σ of assertions will be called a *knowledge base* (KB). An interpretation $\mathcal{I}$ *satisfies* (*is a model of*) a KB Σ iff $\mathcal{I}$ satisfies each element in Σ. A KB Σ *entails* an assertion α (written $\Sigma \models \alpha$) iff every model of Σ also satisfies α. For instance, if Σ is $\{(\mathtt{Tall} \sqcap \mathtt{Student})(\mathtt{tom}), \mathtt{Friend(tim,tom)}\}$ then $\Sigma \models (\exists \mathtt{Friend.Tall})(\mathtt{tim})$, *i.e.* `tim` has a tall friend. Notice that $\Sigma \models R(a, b)$ iff $R(a, b) \in \Sigma$.

Fuzzy $\mathcal{ALC}$

From a syntax point of view, in *fuzzy* $\mathcal{ALC}$ we are dealing with *fuzzy assertions* (denoted with γ), *i.e.* expressions of type $\langle \alpha \, n \rangle$, where α is an $\mathcal{ALC}$ assertion and $n \in [0, 1]$.

From a semantics point of view, we will follow Zadeh's semantics. According Zadeh's work about fuzzy sets (Zadeh 1965), a *fuzzy set* X with respect to a set S is characterized by a membership function $\mu_X : S \to [0, 1]$, assigning a X-membership degree, $\mu_X(s)$, to each element s in S. This membership degree gives us an estimation of the belonging of s to X. Typically, if $\mu_X(s) = 1$ then s definitely belongs to X, while $\mu_X(x) = .7$ means that s is "likely" to be an element of X. Moreover, according to Zadeh, the membership function has to satisfy three well-known restrictions. For all $s \in S$ and for all fuzzy sets X, Y with respect to S: $\mu_{X \cap Y}(s) = \min\{\mu_X(s), \mu_Y(s)\}$, $\mu_{X \cup Y}(s) = \max\{\mu_X(s), \mu_Y(s)\}$, and $\mu_{\overline{X}}(s) = 1 - \mu_X(s)$, where $\overline{X}$ is the complement of X in S, *i.e.* $S \setminus X$[5].

In fuzzy $\mathcal{ALC}$, a concept is interpreted as a fuzzy set. Therefore, concepts and roles become *imprecise* (or *vague*). According to this view, the intended meaning of *e.g.* $\langle C(a) \, n \rangle$ we will adopt is: "the membership degree of individual a being an instance of concept C is at least n". Similarly for roles. Hence, *e.g.* $\langle \mathtt{Tall(tom)} \, .7 \rangle$ means that the degree of `tom` being `Tall` is at least .7, *i.e.* `tom` is likely tall; $\langle \mathtt{Tall(tom)} \, 1 \rangle$ means that `tom` is tall, whereas $\langle \neg \mathtt{Tall(tom)} \, 1 \rangle$ means that `tom` is not tall.

A *fuzzy interpretation* is now a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is, as for the crisp $\mathcal{ALC}$ case, the *domain*, whereas $\cdot^{\mathcal{I}}$ is an *interpretation function* mapping (*i*) individuals as for the crisp case; (*ii*) $\mathcal{ALC}$ concepts into a membership degree function $\Delta^{\mathcal{I}} \to [0, 1]$, and (*iii*) $\mathcal{ALC}$ roles into a membership degree function $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0, 1]$. Therefore, if C is a concept then $C^{\mathcal{I}}$ will naturally be interpreted as the *membership degree function* of the fuzzy concept (set) C w.r.t. $\mathcal{I}$, *i.e.* if $d \in \Delta^{\mathcal{I}}$ is an object of the domain $\Delta^{\mathcal{I}}$ then $C^{\mathcal{I}}(d)$ gives us the degree of being the object d an element of the fuzzy concept C under the interpretation $\mathcal{I}$. Similarly for roles. Additionally, $\cdot^{\mathcal{I}}$ has to satisfy the following equations: for all $d \in \Delta^{\mathcal{I}}$

$$
\begin{array}{lcl}
\top^{\mathcal{I}}(d) & = & 1 \\
\bot^{\mathcal{I}}(d) & = & 0 \\
(C \sqcap D)^{\mathcal{I}}(d) & = & \min\{C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)\} \\
(C \sqcup D)^{\mathcal{I}}(d) & = & \max\{C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)\} \\
(\neg C)^{\mathcal{I}}(d) & = & 1 - C^{\mathcal{I}}(d) \\
(\forall R.C)^{\mathcal{I}}(d) & = & \min_{d' \in \Delta^{\mathcal{I}}}\{\max\{1 - R^{\mathcal{I}}(d, d'), C^{\mathcal{I}}(d')\}\} \\
(\exists R.C)^{\mathcal{I}}(d) & = & \max_{d' \in \Delta^{\mathcal{I}}}\{\min\{R^{\mathcal{I}}(d, d'), C^{\mathcal{I}}(d')\}\}.
\end{array}
$$

Just note that w.r.t. the $\forall$ connective, $(\forall R.C)^{\mathcal{I}}(d)$ is the result of viewing $\forall R.C$ as the first order formula

---

[4] Through this work we assume that every metavariable has an optional subscript.

[5] Other membership functions have been proposed in the literature. The interested reader can consult *e.g.* (Dubois & Prade 1980; Kundu & Chen 1994).

$\forall y.R(x,y) \rightarrow C(y)$, where $F \rightarrow G$ is $\neg F \vee G$[6] and the universal quantifier $\forall$ is viewed as a conjunction over the elements of the domain. Similarly, for the $\exists$ connective $(\exists R.C)^{\mathcal{I}}(d)$ is the result of viewing $\exists R.C$ as $\exists y.R(x,y) \wedge C(y)$, where the existential quantifier $\exists$ is considered a disjunction over the elements of the domain (see *e.g.* (Lee 1972)).

It is easily verified that for all interpretations $\mathcal{I}$ and individuals $d \in \Delta^{\mathcal{I}}$, $(\neg(C \sqcap D))^{\mathcal{I}}(d) = (\neg C \sqcup \neg D)^{\mathcal{I}}(d)$ and $(\neg(\forall R.C))^{\mathcal{I}}(d) = (\exists R.\neg C)^{\mathcal{I}}(d)$.

An interpretation $\mathcal{I}$ *satisfies* a fuzzy assertion $\langle C(a)\, n\rangle$ (resp. $\langle R(a,b)\, n\rangle$) iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq n$ (resp. $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq n$). An interpretation $\mathcal{I}$ *satisfies* (is *a model of*) a set of fuzzy assertions Σ, *i.e.* a *fuzzy KB*, iff $\mathcal{I}$ satisfies each element of Σ. A fuzzy KB Σ *fuzzy entails* a fuzzy assertion γ (written $\Sigma \models\!\!\!\approx \gamma$) iff every model of Σ also satisfies γ. Given a fuzzy KB Σ and an assertion α, we define the *maximal degree of truth* of α with respect to Σ (written $Maxdeg(\Sigma, \alpha)$) to be $\max\{n > 0 : \Sigma \models\!\!\!\approx \langle \alpha\, n\rangle\}$ ($\max \emptyset = 0$). Notice that $\Sigma \models\!\!\!\approx \langle \alpha\, n\rangle$ iff $Maxdeg(\Sigma, \alpha) \geq n$.

Example 1 Suppose we have two images `i1` and `i2` regarding `tim`, `tom` and `joe`. `i1` and `i2` have been indexed as follows: $\Sigma_{\mathtt{i1}} = \{\langle \mathtt{About(i1,tim)}\, .9\rangle, \langle \mathtt{Tall(tim)}\, .8\rangle, \langle \mathtt{About(i1,tom)}\, .6\rangle, \langle \mathtt{Tall(tom)}\, .7\rangle\}$, $\Sigma_{\mathtt{i2}} = \{\langle \mathtt{About(i1,joe)}\, .6\rangle, \langle \mathtt{Tall(joe)}\, .9\rangle\}$. Moreover, let $\Sigma_{\mathtt{B}} = \{\langle \mathtt{Student(tim)}\, 1\rangle, \langle \mathtt{Student(tom)}\, 1\rangle, \langle \mathtt{Student(joe)}\, 1\rangle, \langle \mathtt{Image(i1)}\, 1\rangle, \langle \mathtt{Image(i2)}\, 1\rangle\}$. We define $\Sigma_1 = \Sigma_{\mathtt{i1}} \cup \Sigma_{\mathtt{B}}$ and $\Sigma_2 = \Sigma_{\mathtt{i2}} \cup \Sigma_{\mathtt{B}}$. Our intention is to retrieve all images in which there is a tall student. This can be formalised by means of the query concept $C = \mathtt{Image} \sqcap \exists \mathtt{About}.(\mathtt{Student} \sqcap \mathtt{Tall})$. It can easily verified that $Maxdeg(\Sigma_1, C(\mathtt{i1})) = .8$, whereas $Maxdeg(\Sigma_1, C(\mathtt{i2})) = .6$. Therefore, we will retrieve both images and rank `i1` before `i2`. ■

Some properties

The following properties are easily verified: for all concepts C, D and for all $n, m \in [0,1]$, $\{\langle C(a)\, n\rangle, \langle \neg C(a)\, m\rangle\}$ is satisfiable iff $n \leq 1 - m$ and

$$Maxdeg(\emptyset, (\neg C \sqcup C)(a)) = .5 \quad (1)$$

$$\{\langle C(a)\, m\rangle, \langle (\neg C \sqcup D)(a)\, n\rangle\} \models\!\!\!\approx \langle D(a)\, n\rangle, \quad \text{if } m > 1 - n \quad (2)$$

Relation (2) is a sort of *modus ponens over concepts*. Similarly for $\forall$, the semantics of the $\forall$ connective gives us a sort of *modus ponens over roles*: if $k = \min\{n, m\}$ then

$$\{\langle R(a,b)\, m\rangle, \langle (\forall R.C)(a)\, n\rangle\} \models\!\!\!\approx \langle C(b)\, n\rangle, \quad \text{if } m > 1 - n \quad (3)$$

$$\{\langle (\exists R.D)(a)\, m\rangle, \langle (\forall R.C)(a)\, n\rangle\} \models\!\!\!\approx \langle (\exists R.D \sqcap C)(a)\, k\rangle, \quad \text{if } m > 1 - n. \quad (4)$$

It is natural to ask whether there is a relation between $\models$ and $\models\!\!\!\approx$. As first, given a fuzzy KB Σ, let $\overline{\Sigma}$ be the (crisp) KB $\overline{\Sigma} = \{\alpha : \langle \alpha\, n\rangle \in \Sigma\}$. Since every "crisp" interpretation is a fuzzy interpretation, the following proposition is easily verified.

Proposition 1 *Let Σ be a fuzzy KB and let α be an assertion. For all $n > 0$, if $\Sigma \models\!\!\!\approx \langle \alpha\, n\rangle$ then $\overline{\Sigma} \models \alpha$.* ⊣

Proposition 1 states that *there cannot be fuzzy entailment without entailment.* For instance, w.r.t. Example 1 we have $\Sigma_1 \models\!\!\!\approx \langle C(\mathtt{i1})\, .8\rangle$ and $\overline{\Sigma_1} \models C(\mathtt{i1})$. Unfortunately, the converse of Proposition 1 is not true in the general case. For instance, $\{\langle C(a)\, .3\rangle, \langle (\neg C \sqcup D)(a)\, .6\rangle\} \not\models\!\!\!\approx \langle D(a)\, n\rangle$ for all $n > 0$, whereas $\{C(a), (\neg C \sqcup D)(a)\} \models D(a)$.

A simple result concerning the "converse" relation between $\models\!\!\!\approx$ and $\models$ is the following. Let Σ be a crisp KB: we define $\tilde{\Sigma} = \{\langle \alpha\, 1\rangle : \alpha \in \Sigma\}$.

Proposition 2 *If $\Sigma \models \alpha$ then $\tilde{\Sigma} \models\!\!\!\approx \langle \alpha\, 1\rangle$.* ⊣

A closer relationship holds whenever we consider normalized fuzzy KBs. We will say that a fuzzy assertion $\langle \alpha\, n\rangle$ is *normalized* if $n > .5$. A fuzzy KB is normalized if every fuzzy assertion in it is. If we consider normalized fuzzy KBs only, then from (Lee 1972) it follows that

Proposition 3 *If Σ is normalized then there is $n \geq .5$ such that $\Sigma \models\!\!\!\approx \langle \alpha\, n\rangle$ iff $\overline{\Sigma} \models \alpha$.* ⊣

For instance, $\{\langle C(a)\, .6\rangle, \langle (\neg C \sqcup D)(a)\, .7\rangle\} \models\!\!\!\approx \langle D(a)\, .7\rangle$ and and $\{C(a), (\neg C \sqcup D)(a)\} \models D(a)$ hold. The reason relies on the fact that for $m, n > .5$, the condition $m > 1 - n$ in (2) – (4) is always true.

Deciding fuzzy entailment

Deciding whether $\Sigma \models\!\!\!\approx \langle \alpha\, n\rangle$ requires a calculus. We will develop a calculus in the style of the constraint propagation method, as this method is usually proposed in the context of DLs (see, *e.g.* (Buchheit, Donini, & Schaerf 1993)). The calculus extends the propositional framework described in (Chen & Kundu 1996) to the DL case.

Consider a new alphabet of *variables*. An Interpretation is extended to variables by mapping these into elements of the interpretation domain. An *object* (written w) is either an individual or a variable. A *constraint* (written τ) is an expression of the form $w{:}C$ or $(w_1, w_2){:}R$, where w, w_1, w_2 are objects, C is an $\mathcal{ALC}$ concept and R is a role. A *fuzzy constraint* (written σ) is an expression having one of the following forms: $\langle \tau \geq n\rangle$, $\langle \tau > n\rangle$, $\langle \tau \leq n\rangle$, $\langle \tau < n\rangle$. An interpretation $\mathcal{I}$ *satisfies* a fuzzy constraint $\langle w{:}C\ rel\ n\rangle$ (resp. $\langle (w_1, w_2){:}R\ rel\ n\rangle$) ($rel \in \{\geq, >, \leq, <\}$) iff $C^{\mathcal{I}}(w^{\mathcal{I}})\ rel\ n$ (resp. $R^{\mathcal{I}}(w_1{}^{\mathcal{I}}, w_2{}^{\mathcal{I}})\ rel\ n$). $\mathcal{I}$ *satisfies* a set S of fuzzy constraints iff $\mathcal{I}$ satisfies every element of it. In the following we will reduce the fuzzy entailment problem to the unsatisfiability problem of a set of fuzzy constraints. Given a fuzzy KB Σ, let

$$S_\Sigma = \begin{array}{l} \{\langle a{:}C \geq n\rangle \mid \langle C(a)\, n\rangle \in \Sigma\} \cup \\ \{\langle (a,b){:}R \geq n\rangle \mid \langle R(a,b)\, n\rangle \in \Sigma\}. \end{array} \quad (5)$$

It follows then that[7]

[6] In the literature, several different definitions of the fuzzy implication connective $\rightarrow$ has been proposed. See *e.g.* (Kundu & Chen 1994) for a discussion.

[7] Notice that $Maxdeg(\Sigma, R(a,b)) = \max\{n : \langle R(a,b)\, n\rangle \in \Sigma\}$.

$$\Sigma \approx \langle C(a)\, n\rangle \text{ iff } S_\Sigma \cup \{\langle a{:}C < n\rangle\} \text{ not satisfiable.} \tag{6}$$

Our calculus, determining whether a set S of fuzzy constraints is satisfiable or not, is based on a set of constraint propagation rules transforming a set S of fuzzy constraints into "simpler" model preserving sets S_i until either all S_i contains a clash (indicating that from all the S_i no model of S can be build) or some S_i is completed and clash-free, that is, no rule can be further be applied to S_i and S_i contains no clash (indicating that from S_i a model of S can be build).

A set of fuzzy constraints S contains a *clash* iff it contains either $\langle w{:}\bot \geq n\rangle$ with $n > 0$, or $\langle w{:}\bot > n\rangle$, or $\langle w{:}\bot < 0\rangle$, or $\langle w{:}\top \leq n\rangle$ with $n < 1$, or $\langle w{:}\top < n\rangle$, or $\langle w{:}\top > 1\rangle$, or S contains a conjugated pair of fuzzy constraints. Each entry in the table below says us under which condition the row-column pair of fuzzy constraints is a *conjugated pair*.

	$\langle \tau < m\rangle$	$\langle \tau \leq m\rangle$
$\langle \tau \geq n\rangle$	$n \geq m$	$n > m$
$\langle \tau > n\rangle$	$n \geq m$	$n \geq m$

Given a fuzzy constraint σ, with σ^c we indicate a conjugate of σ (if there exists one). Just notice that a conjugate of a fuzzy constraint may be not unique, as there are could be infinitely many. For instance, both $\langle a{:}C < .6\rangle$ and $\langle a{:}C \leq .7\rangle$ are conjugates of $\langle a{:}C \geq .8\rangle$.

Concerning the rules, for each connective $\sqcap, \sqcup, \neg, \forall$ and $\exists$ there is a rule for each relation $rel \in \{\geq, >, \leq, <\}$, *i.e.* there are 20 rules. We will restrict our presentation to the set $rel \in \{\geq, \leq\}$. The rules for the case $rel \in \{>, <\}$ are quite similar. The rules can take the following two forms:

$$\Phi \rightarrow \Psi \text{ if } \Gamma \qquad \Phi \Rightarrow \Psi \text{ if } \Gamma \tag{7}$$

where Φ and Ψ are sequences of fuzzy constraints and Γ is a condition. Both rules fire only if the condition Γ holds and if the current set S of fuzzy constraints contains fuzzy constraints matching Φ. After execution, the first deletes the fuzzy constraints matching Φ from S, while the second keeps them. Both forms add the constraints from Ψ to S after firing. In order to prevent infinite application of the second type of rules, we assume that each instantiation of the rules is applied only once. The rules are the following:

$$\begin{array}{ll}
(\neg_\geq) & \langle w{:}\neg C \geq n\rangle \rightarrow \langle w{:}C \leq 1-n\rangle \\
(\neg_\leq) & \langle w{:}\neg C \leq n\rangle \rightarrow \langle w{:}C \geq 1-n\rangle \\
& \\
(\sqcap_\geq) & \langle w{:}C \sqcap D \geq n\rangle \rightarrow \langle w{:}C \geq n\rangle, \langle w{:}D \geq n\rangle \\
(\sqcup_\leq) & \langle w{:}C \sqcup D \leq n\rangle \rightarrow \langle w{:}C \leq n\rangle, \langle w{:}D \leq n\rangle \\
& \\
(\sqcup_\geq) & \langle w{:}C \sqcup D \geq n\rangle \rightarrow \langle w{:}C \geq n\rangle \mid \langle w{:}D \geq n\rangle \\
(\sqcap_\leq) & \langle w{:}C \sqcap D \leq n\rangle \rightarrow \langle w{:}C \leq n\rangle \mid \langle w{:}D \leq n\rangle \\
& \\
(\forall_\geq) & \langle w_1{:}\forall R.C \geq n\rangle, \sigma^c \Rightarrow \langle w_2{:}C \geq n\rangle \\
& \quad \text{if } \sigma \text{ is } \langle (w_1, w_2){:}R \leq 1-n\rangle \\
(\exists_\leq) & \langle w_1{:}\exists R.C \leq n\rangle, \sigma^c \Rightarrow \langle w_2{:}C \leq n\rangle \\
& \quad \text{if } \sigma \text{ is } \langle (w_1, w_2){:}R \leq n\rangle \\
(\exists_\geq) & \langle w{:}\exists R.C \geq n\rangle \rightarrow \langle (w,x){:}R \geq n\rangle, \langle x{:}C \geq n\rangle \\
& \quad \text{if } x \text{ new variable} \\
(\forall_\leq) & \langle w{:}\forall R.C \leq n\rangle \rightarrow \langle (w,x){:}R \geq 1-n\rangle, \langle x{:}C \leq n\rangle \\
& \quad \text{if } x \text{ new variable}
\end{array} \tag{8}$$

An instance of the $(\forall_\geq)$ rule is *e.g.*

$$\langle a{:}\forall R.C \geq .8\rangle, \langle (a,b){:}R \geq .7\rangle \Rightarrow \langle b{:}C \geq .8\rangle,$$

where σ is $\langle (a,b){:}R \leq .2\rangle$ and $\sigma^c = \langle (a,b){:}R \geq .7\rangle$ is a conjugate of σ.

A set of fuzzy constraints S is said to be *complete* if no rule is applicable to it. Any complete set of fuzzy constraints S_2 obtained from a set of fuzzy constraints S_1 by applying the above rules is called a *completion* of S_1. Due to the presence of the rules $\sqcup_\geq$, $\sqcup_>$, $\sqcap_\leq$ and $\sqcap_<$, more than one completion can be obtained. These rules are called *nondeterministic rules*. All other rules are called deterministic rules.

Example 2 Consider $\gamma = \langle (\exists R.D \sqcap C)(a)\, .6\rangle$ and $\Sigma = \{\langle (\exists R.D)(a)\, .7\rangle, \langle (\forall R.C)(a)\, .6\rangle\}$. We show that $\Sigma \approx \gamma$, confirming (4), by verifying that all completions of $S = S_\Sigma \cup \{\langle a{:}\exists R.D \sqcap C < .6\rangle\}$ contain a clash. In fact, we have the following two sequences.

$$\begin{array}{lll}
(1) & \langle a{:}\exists R.D \geq .7\rangle & \text{Hypothesis:}S \\
(2) & \langle a{:}\forall R.C \geq .6\rangle & \\
(3) & \langle a{:}\exists R.D \sqcap C < .6\rangle & \\
\hline
(4) & \langle (a,x){:}R \geq .7\rangle, \langle x{:}D \geq .7\rangle & (\exists_\geq) : (1) \\
(5) & \langle x{:}C \geq .6\rangle & (\forall_\geq) : (2), (4) \\
(6) & \langle x{:}D \sqcap C < .6\rangle & (\exists_<) : (3), (4) \\
& \quad \Omega_1 \quad | \quad \Omega_2 &
\end{array}$$

where the two sequences Ω_1 and Ω_2 are respectively

$$\begin{array}{lll}
(7a) & \langle x{:}D < .6\rangle & (\sqcap_<) : (6) \\
(8a) & \text{clash} & (4), (7a)
\end{array}$$

and

$$\begin{array}{lll}
(7b) & \langle x{:}C < .6\rangle & (\sqcap_<) : (6) \\
(8b) & \text{clash} & (5), (7b).
\end{array} \quad \blacksquare$$

Soundness, completeness and complexity

It is easily verified that the above rules are sound, *i.e.* if S_1 is satisfiable then there is a satisfiable completion S_2 of S_1 and, thus, S_2 contains no clash. Vice-versa, completeness, *i.e.* if there is a completion S_2 of S_1 containing no clash then S_1 is satisfiable, can be shown by building an interpretation $\mathcal{I}$ from S_2 satisfying S_1. Roughly, given a clash-free completion S_2 of S_1 we consider $N_1[\tau] = \max\{n : \langle \tau \geq n\rangle \in S_2\}$, and $N_2[\tau] = \max\{n : \langle \tau > n\rangle \in S_2\}$. Since S_2 is clash-free, it follows that there is $\epsilon > 0$ such that the interpretation $\mathcal{I}$, (i) with domain $\Delta^\mathcal{I}$ being the set of objects appearing in S_2, (ii) $w^\mathcal{I} = w$ for all $w \in \Delta^\mathcal{I}$ and (iii) $\top^\mathcal{I}(w^\mathcal{I}) = 1$, $\bot^\mathcal{I}(w^\mathcal{I}) = 0$, $A^\mathcal{I}(w^\mathcal{I}) = \max\{N_1[w{:}A], N_2[w{:}A] + \epsilon\}$, $R^\mathcal{I}({w_1}^\mathcal{I}, {w_2}^\mathcal{I}) = \max\{N_1[(w_1,w_2){:}R], N_2[(w_1,w_2){:}R] + \epsilon\}$, satisfies both S_2 and S_1. It can be shown that

Proposition 4 *A set of fuzzy constraints S is satisfiable iff there exists a clash free completion of S.* $\dashv$

From a computational complexity point of view, it is easily verified that termination of the above algorithm is guaranteed. Moreover, from Proposition 2 and from PSPACE-completeness of the entailment problem in crisp $\mathcal{ALC}$ (Schmidt-Schauß & Smolka 1991), PSPACE-hardness of the fuzzy entailment problem follows. It can be verified that *trace rules* as in (Schmidt-Schauß & Smolka 1991) can be defined. Therefore,

Proposition 5 *Let Σ be a fuzzy KB and and let γ be a fuzzy assertion. Determining whether $\Sigma \approx \gamma$ is a PSPACE-complete problem.* $\dashv$

Computing the maximal degree of truth

The problem of determining $Maxdeg(\Sigma, \alpha)$ is important, as computing $Maxdeg(\Sigma, \alpha)$ is in fact the way to answer a query of type "to which degree is α (at least) true, given the (vague) facts in Σ ?". An easy algorithm can be given in terms of a sequence of fuzzy entailment tests. It is based on the observation that $Maxdeg(\Sigma, \alpha) \in \{0, .5, 1\} \cup N_\Sigma$, where $N_\Sigma = \{n : \langle A\, n\rangle \in \Sigma\}$. The algorithm is described below.

Algorithm $Max(\Sigma, \alpha)$
Let Σ be a set of $\mathcal{ALC}$ fuzzy assertions, let α be an assertion. Set $Min := 0$ and $Max := 2$.

1. Pick $n \in N_\Sigma \cup \{.5, 1\}$ such that $Min < n < Max$. If there is no such n, then set $Maxdeg(\Sigma, \alpha) := Min$ and exit.
2. If $\Sigma \approx \langle \alpha\, n\rangle$ then set $Min = n$ and go to Step 1, else set $Max = n$ and go to Step 1. ■

By a binary search on N_Σ the value of $Maxdeg(\Sigma, \alpha)$ can be determined in at most $\log |N_\Sigma + 1|$ fuzzy entailment tests. As checking fuzzy entailment is time consuming, this approach may be unfeasible.

In the extended version of this work, we present an alternative method for computing $Maxdeg(\Sigma, \alpha)$ performing the fuzzy entailment test only *once*. Essentially, the method extends the ideas described in (Straccia 1997) to our DL context.

Roughly, in order to determine $Maxdeg(\Sigma, C(a))$, we start with a set of constraints of the form $S = S_\Sigma \cup \{\langle a{:}C < \lambda\rangle\}$, where λ is a new variable symbol. Thereafter, we apply to S constraint propagation rules similar to those in (8) until each derived set S_i of constraints is completed. Finally, we are looking for the *maximal value* $n \in [0, 1]$ such that for each of the completions S_i, the constraint set $S_i[\lambda/n]$ (if not empty) contains a clash, where $S_i[\lambda/n]$ is the set obtained by replacing each occurrence of λ by n.

Concerning computational complexity, it can be shown that the problem of determining $Maxdeg(\Sigma, \alpha)$ inherits the result of determining (fuzzy) entailment, and thus, determining $Maxdeg(\Sigma, \alpha)$ is a PSPACE-complete problem.

Dealing with terminological axioms

We shortly show how to dealing with *terminological axioms*. In DLs, a general terminological axiom assumes the form $C \Rightarrow D$, where C and D are concepts. From a first-order point of view, $C \Rightarrow D$ is viewed as the formula $\forall x. C(x) \to D(x)$. For instance, $\texttt{Ferrari} \Rightarrow \texttt{SportCar} \sqcap \exists \texttt{Ownedby}.\texttt{CarFanatic}$ states that a Ferrari is a sport car which is owned by a car fanatic. When we switch to the fuzzy case, the simple form of *fuzzy terminological axiom* we allow is $\langle C \Rightarrow D\, n\rangle$, where $n \in [0, n]$. The semantics is given coherently to the above first order view of $C \Rightarrow D$: an interpretation $\mathcal{I}$ *satisfies* $\langle C \Rightarrow D\, n\rangle$ iff $\min_{d \in \Delta^\mathcal{I}}\{(\neg C \sqcup D)^\mathcal{I}(d)\} \geq n$. As for the $\forall$ connective, $F \to G$ is viewed as $\neg F \vee G$. It is easily verified that $\{\langle C(a)\, m\rangle, \langle C \Rightarrow D\, n\rangle\} \approx \langle D(a)\, n\rangle$ if $m > 1 - n$, which is similar to (2).

We will say that D *subsumes* C with degree n w.r.t. Σ (written $\Sigma \approx \langle C \Rightarrow D\, n\rangle$) iff all models of Σ are models of $\langle C \Rightarrow D\, n\rangle$. $Maxdeg(\Sigma, C \Rightarrow D)$ is the maximal degree n such that $\Sigma \approx \langle C \Rightarrow D\, n\rangle$. For instance, if Σ is $\{\langle A \Rightarrow C\, .6\rangle, \langle B \Rightarrow D\, .7\rangle\}$, then it can be verified that $Maxdeg(\Sigma, A \sqcap B \Rightarrow C \sqcap D) = .6$. Notice that $Maxdeg(\emptyset, C \Rightarrow C) = .5$, according to (1).

Example 3 Consider Example 1. Suppose we add $\{\langle \texttt{Student} \sqcap (\texttt{Male} \sqcup \texttt{Tall}) \Rightarrow \texttt{TallStudent}\, .7\rangle$, $\langle \texttt{Male}(\texttt{tim})\, 1\rangle$, $\langle \texttt{Male}(\texttt{tom})\, 1\rangle$, $\langle \texttt{Male}(\texttt{joe})\, 1\rangle\}$ to the background KB Σ_B. Suppose the query concept C is $\texttt{Image} \sqcap \exists \texttt{About}.\ \texttt{TallStudent}$. It can be verified that $Maxdeg(\Sigma_1, C(\texttt{i1})) = .7$, whereas $Maxdeg(\Sigma_1, C(\texttt{i2})) = .6$. ■

From a calculus point of view, we make the following assumptions: (i) $\langle C \Rightarrow D\, n\rangle$ is considered a constraint too; and (ii) given Σ, then S_Σ is defined as usual except that additionally we add $\langle C \Rightarrow D\, n\rangle$ to S_Σ for each $\langle C \Rightarrow D\, n\rangle \in \Sigma$. Just note that, w.r.t. subsumption, we have that $\Sigma \approx \langle C \Rightarrow D\, n\rangle$ iff $S_\Sigma \cup \{\langle (\neg C \sqcup D)(a) < n\rangle\}$ not satisfiable, where a is a new individual. Moreover, we will make the following *restrictions*: (i) the fuzzy terminological axioms in a fuzzy KB Σ have to be of the form $\langle A \Rightarrow C\, n\rangle$ (if A then C) or $\langle A{:=}\, C\, n\rangle$ (A iff C), where A is a primitive concept; and (ii) *we do not allow cycles*. Here, $\langle A{:=}\, C\, n\rangle$ is a macro for $\langle A \Rightarrow C\, n\rangle$ and $\langle C \Rightarrow A\, n\rangle$. This restriction guarantees us soundness, completeness and termination of the deduction process[8].

The rules are the following:

$$
\begin{array}{lll}
(\Rightarrow_R) & \langle A \Rightarrow C \geq n\rangle, \sigma^c \Rightarrow \langle w{:}C \geq n\rangle & \\
& \quad \text{if } \sigma = \langle w{:}A \leq 1 - n\rangle & \\
(\Rightarrow_L) & \langle C \Rightarrow A \geq n\rangle, \sigma^c \Rightarrow \langle w{:}C \leq 1 - n\rangle & (9)\\
& \quad \text{if } \sigma = \langle w{:}A \geq n\rangle &
\end{array}
$$

Example 4 Let $\Sigma = \{\langle A \Rightarrow C\, .6\rangle, \langle B \Rightarrow D\, .7\rangle\}$ and consider $\delta = A \sqcap B \Rightarrow C \sqcap D$. It is easily verified that $Maxdeg(\Sigma, \delta) = .6$. We show that $\Sigma \approx \langle \delta\, .6\rangle$ by verifying that all completions of $S = S_\Sigma \cup \{\langle a{:}\neg(A \sqcap B) \sqcup (C \sqcap D) < .6\rangle\}$ contain a clash. By applying rules (8) and (9), we have the following two sequences.

[8]Unfortunately, the technique used in (Buchheit, Donini, & Schaerf 1993) in order to reason in presence of axioms of the form $C \Rightarrow D$ is not directly applicable in the fuzzy case. This remains an open problem yet.

(1)	$\langle A \Rightarrow C\, .6\rangle$	Hypothesis:S
(2)	$\langle B \Rightarrow D\, .7\rangle$	
(3)	$\langle a{:}\neg(A \sqcap B) \sqcup (C \sqcap D) < .6\rangle$	
(4)	$\langle a{:}\neg(A \sqcap B) < .6\rangle, \langle a{:}(C \sqcap D) < .6\rangle$	$(\sqcup_<) : (3)$
(5)	$\langle a{:}A \sqcap B > .4\rangle$	$(\neg_<) : (4)$
(6)	$\langle a{:}A > .4\rangle, \langle a{:}B > .4\rangle$	$(\sqcap_>) : (5)$
(7)	$\langle a{:}C \geq .6\rangle$	$(\Rightarrow_R) : (1), (6)$
(8)	$\langle a{:}D \geq .7\rangle$	$(\Rightarrow_R) : (2), (6)$
	$\Omega_1 \quad \vert \quad \Omega_2$	

where the two sequences Ω_1 and Ω_2 (which determine the two c-completions S_1 and S_2, respectively) are

$(9a)$	$\langle a{:}C < .6\rangle$	$(\sqcap_<) : (4)$
$(10a)$	clash	$(9a), (7)$

and

$(9b)$	$\langle a{:}D < .6\rangle$	$(\sqcap_<) : (4)$
$(10b)$	clash	$(9b), (8)$

respectively. ■

A further extension (which we roughly address here without working it out formally) is to allow terminological axioms in which the membership function is specified explicitly. These axioms are of the form $A =_\mu \mu_A(f_1, \ldots, f_n)$, where A is a primitive concept, f_i are features (*i.e.* functional roles) and μ_A is a fuzzy membership function defined on a *concrete domain* (called *universe of discourse* in (Yen 1991)) such that μ_A depends on the features f_i (see (Baader & Hanschke 1991) for the formal aspects about concrete domains). For instance, the concept `Tall` could be defined as $\mathtt{Tall} =_\mu \mu_{\mathtt{Tall}}(\mathtt{height})$, where $\mu_{\mathtt{Tall}}$ is defined as a lambda abstraction on reals such that it relies on the `height` of an individual: *e.g.* $\lambda x. \min\{(x/200)^2, 1\}$. Therefore, if Σ contains both $\mathtt{height(tom)} = 180$ and the above axiom, then we may infer that $\langle \mathtt{Tall(tom)}\, .81\rangle$, where $.81 = \mu_{\mathtt{Tall}}(\mathtt{height(tom)}) = (\lambda x. \min\{(x/200)^2, 1\})(\mathtt{height(tom)}) = \min\{(180/200)^2, 1\}$.

Conclusions and Future Work

We have presented a fuzzy DL which enables us to reason in presence of imprecise concepts. In particular, syntax, semantics and sound and complete algorithms for reasoning in it has been presented. The complexity results shows that the additional expressive power has no impact from a computational complexity point of view. This work can be used as a basis both for extending existing DL based systems and for further research. In particular, the case of considering general terminological axioms (including cycles) and role-forming rules should be worked out. Another interesting point is to understand the impact of fuzziness on the computational complexity: is it always true that the upper bound of the complexity is the same as in the crisp case?

Acknowledgements

This work is funded by the European Community ESPRIT project FERMI 8134. Thanks go to C. Meghini, F. Sebastiani and the reviewers for their suggestions and comments.

References

Baader, F., and Hanschke, P. 1991. A schema for integrating concrete domains into concept languages. In *Proc. of the 12th Int. Joint Conf. on Artificial Intelligence (IJCAI-91)*, 452–457.

Brachman, R. J., and Levesque, H. J. 1984. The tractability of subsumption in frame-based description languages. In *Proceedings of AAAI-84, 4th Nat. Conf. on Artificial Intelligence*, 34–37.

Buchheit, M.; Donini, F. M.; and Schaerf, A. 1993. Decidable reasoning in terminological knowledge representation systems. In *Proc. of the 13th Int. Joint Conf. on Artificial Intelligence (IJCAI-93)*, 704–709.

Chen, J., and Kundu, S. 1996. A sound and complete fuzzy logic system using Zadeh's implication operator. In *Proc. of the 9th Int. Sym. on Methodologies for Intelligent Systems (ISMIS-96)*, LNAI 1079, 233–242.

Dubois, D., and Prade, H. 1980. *Fuzzy Sets and Systems.* New York, NJ: Academic Press.

Gobel, C. A.; Haul, C.; and Bechhofer, S. 1996. Describing and classifying multimedia using the description logic GRAIL. In *Proceedings of the SPIE Conference on Storage and Retrieval for Still Images and Video Databases IV (SPIE-96)*, 132–143.

Heinsohn, J. 1994. Probabilistic description logics. In *Proceedings of the 10th Conference on Uncertainty in Artificila Intelligence*, 311–318.

Hollunder, B. 1994. An alternative proof method for possibilistic logic and its application to terminological logics. In *10th Annual Conference on Uncertainty in Artificial Intelligence.*

Jäger, M. 1994. Probabilistic reasoning in terminological logics. In *Proceedings of KR-94, 5th International Conference on Principles of Knowledge Representation and Reasoning*, 305–316.

Koller, D.; Levy, A.; and Pfeffer, A. 1997. P-CLASSIC: A tractable probabilistic description logic. In *Proc. of the 14th Nat. Conf. on Artificial Intelligence (AAAI-97)*, 390–397.

Kundu, S., and Chen, J. 1994. Fuzzy logic or Lukasiewicz logic: A clarification. In *Proc. of the 8th Int. Sym. on Methodologies for Intelligent Systems (ISMIS-94)*, LNAI 869, 56–64.

Lee, R. C. T. 1972. Fuzzy logic and the resolution principle. *Journal of the ACM* 19(1):109–119.

Meghini, C.; Sebastiani, F.; and Straccia, U. 1997. Reasoning about the form and content for multimedia objects (extended abstract). In *Proceedings of AAAI 1997 Spring Symposium on Intelligent Integration and Use of Text, Image, Video and Audio*, 89–94.

Roth, D. 1996. On the hardness of approximate reasoning. *Artificial Intelligence* 82:273–302.

Schmidt-Schauß, M., and Smolka, G. 1991. Attributive concept descriptions with complements. *Artificial Intelligence* 48:1–26.

Straccia, U. 1997. A four-valued fuzzy propositional logic. In *Proc. of the 15th Int. Joint Conf. on Artificial Intelligence (IJCAI-97)*, 128–133.

Yen, J. 1991. Generalizing term subsumption languages to fuzzy logic. In *Proc. of the 12th Int. Joint Conf. on Artificial Intelligence (IJCAI-91)*, 472–477.

Zadeh, L. A. 1965. Fuzzy sets. *Information and Control* 8(3):338–353.