Using Iterated Best-Response to Find Bayes-Nash Equilibria in Auctions

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Introduction

Bayes-Nash equilibria (BNE) have been derived analytically only for the simplest auction settings (Krishna 2002). Such settings include single-item first- and second-price auctions with continuous distributions of bidders' values[1]. Very little research has been devoted to auctions with discrete bids and values. We take some important first steps in this direction by computationally investigating when an iterated best-response procedure might lead to a BNE.

Standard auctions have a special structure that simplifies best-response computation. We present experimental results for single auction settings and symmetrically distributed bidders' values. Unlike most past research, both bids and values are discrete. Only first- and second-price auctions are considered, but the ultimate goal of this research is to design an iterative procedure capable of finding BNE in more complicated settings (e.g., multiple one-shot simultaneous auctions with valuations over bundles).

BNE in Standard Auctions

The strategies for n bidders $s^* = (s_1^*, \ldots, s_n^*)$ are a BNE if for each bidder i and for each of i's values v_i, $b_i = s_i^*(v_i)$ solves

$$\max_{b_i \in B_i} \sum_{v_{-i} \in V_{-i}} u_i(s_1^*(v_1), \ldots, b_i, \ldots, s_n^*(v_n); v) f_i(v_{-i}|v_i) \tag{1}$$

where for bidder i the following notation is used: b_i is a bid from the set B_i of possible bids, v_i is a value from the set V_i of possible values, s_i is a strategy, $u_i(s; v)$ is utility, and $f_i(v_{-i}|v_i)$ is the probability of the other bidders having values v_{-i} from the set V_{-i} of possible values (e.g., (Gibbons 1992)). We assume that bidders have independent and identically distributed values. This assumption means that the set of possible values (and bids) is the same for all bidders, $f_i(v_{-i}|v_i) = f(v_{-i})$, and $u_i(s_1^*(v_1), \ldots, b_i, \ldots, s_n^*(v_n); v) = u_i(s_1^*(v_1), \ldots, b_i, \ldots, s_n^*(v_n); v_i)$. One key observation is that the only piece of information relevant to a bidder's utility in standard auctions is the maximum bid of the other bidders. We will refer to the maximum bid of the other bidders as the price. Equation 1 simplifies to

$$\max_{b_i \in B_i} \sum_{v_{-i} \in V_{-i}} u_i(b_i, \max(v_{-i}); v_i) f(v_{-i}) \tag{2}$$

where $\max(v_{-i}) = \max_{j \neq i} s_j^*(v_j)$ is the highest bid submitted by one of the other bidders in equilibrium. The price distribution g can be derived from the distribution of the other bidders' values.

$$\forall p \in B_i \quad g(p) = \sum_{v_{-i} \in V_{-i} | \max(v_{-i}) = p} f(v_{-i})$$

The probability $g(p)$ that the price is p is the sum of the probabilities of all combinations of values of the other bidders that result in the maximum bid equal to p. Rewriting Equation 2 with the price distribution we get

$$\max_{b_i \in B_i} \sum_{p \in B_i} u_i(b_i, p; v_i) g(p) \tag{3}$$

The strategies $s^* = (s_1^*, \ldots, s_n^*)$ are a BNE if for each bidder i and for each of i's values v_i, $b_i = s_i^*(v_i)$ solves Equation 3. Note that $g(p)$ in Equation 3 is determined by the strategies of the other bidders. Equation 3 is more compact than Equation 2. Given g (we will show that g can be easily calculated), maximizing Equation 3 is easier than maximizing Equation 2. The summation in Equation 2 has V_i^{n-1} terms. The summation in Equation 3 has at most B_i terms. This difference becomes important when there are more than 2 bidders.

Iterated Best Response

We use an iterated myopic best-response to search for a BNE:

- initialize g
- repeat until s are a symmetric equilibrium
  - for every $v_i \in V_i$
    - bidder i finds the bid $s_i(v_i)$ that is the best-response to g.[2]

[1] Some literature refers to values as types or signals.

[2] If multiple bids are best responses, we set $s_i(v_i)$ to the mixed strategy of submitting any of the best-response bids with equal probability.

- h is a frequency distribution of bids s_i
- H is the corresponding cumulative distribution
- $g(p) = H^{n-1}(p) - H^{n-1}(p-1)$

The strategies $s = (s_1, \ldots, s_n)$ are a symmetric BNE if s_i is a best response to the price distribution resulting from $n-1$ bidders playing s_i. We check this condition at the start of the loop.

In each iteration we calculate bidder i's best response s_i to the price distribution g. The best response is calculated for each value $v_i \in V_i$. At the end of an iteration, g is set to the price distribution resulting from $n-1$ bidders playing s_i. This price distribution is calculated using the cumulative distribution H of bids submitted by bidder i. The cumulative distribution of the maximum of $n-1$ bids distributed according to H is H^{n-1}. For integer prices p the probability density function $g(p)$ can be calculated from H: $g(p) = H^{n-1}(p) - H^{n-1}(p-1)$.

We assume that bidders' values are symmetrically distributed. Therefore all bidders have the same best-response strategy s_i and we only need to find the best-response of one bidder (bidder i).

The iterated best-response procedure is inspired by a procedure for finding self-confirming price distributions (Osepayshvili *et al.* 2005).

Experiments

We test the procedure in first- and second-price auctions. The number of bidders ranges between 2 and 11. Bidders have integer values uniformly distributed between 0 and k. The parameter k ranges from 2 to 50. Given a value v and a price distribution, an optimal bid is calculated by comparing profits from bidding each of the $v+1$ bids $0, 1, \ldots, v$. If the bid is equal to the price, the bidder wins the object with probability $\frac{1}{n}$. The bidder pays his bid in the first-price auction and the price in the second-price auction. We tried two ways of initializing the price distribution: to zero and to the value distribution.

Second-Price Auctions

The second-price auction has a known dominant strategy of bidding the true value. The iterative procedure converges to the dominant strategy after 2 iterations when the price distribution is initialized to zero and after 1 iteration when the price distribution is initialized to the value distribution.

First-Price Auctions

The procedure converges to an equilibrium in most of our experiments with varying number of bidders and initial price distributions for values of k below 9. It takes under 8 iterations (usually 1 iteration) to reach convergence when the price distribution is initialized to the value distribution. It takes under 27 iterations (5 iterations on average) to reach convergence when the price distribution is initialized to zero. Examples of equilibria are in Table 1.

Bidding $\frac{n-1}{n}$ of the value is a BNE of a first-price auction when bidders' values are identically distributed according to a continuous uniform distribution (Krishna 2002). The equilibrium strategies in Table 1 are similar to the equilibrium strategies for the continuous case. The continuous counterparts for these discrete strategies are to bid $\frac{10}{11}$ and $\frac{1}{2}$ of the value respectively.

k	# bidders	equilibrium strategy
8	11	0:0,1:0,2:1,3:2,4:3,5:4,6:5,7:6,8:7
7	2	0:0,1:0,2:1,3:1,4:2,5:2,6:3,7:3

Table 1: Equilibria in First-Price Auctions. 3:1, 4:2,... denotes the strategy of bidding 1 when the value is 3 and bidding 2 when the value is 4.

Related and Future Work

The procedure described here coincides with Cournot adjustment (e.g. (Fudenberg & Levine 1998)) when each bidder's set of values contains only one element.

(Reeves & Wellman 2004) describe a procedure for computing best-response strategies and finding Bayes-Nash equilibrium for two-player infinite games with types drawn from a piecewise-uniform distribution.

This paper is a preliminary study of discrete auctions. In future work we are going to search for iterative procedures with better convergence properties. Pure strategy BNE does not always exist. We are also interested in procedures capable of finding mixed Bayes-Nash equilibria.

The procedure discussed in this paper can be extended to multiple one-shot simultaneous auctions with valuations over bundles. In the multi-auction setting, the strategies of bidders are vectors of bids for all auctions and the probability distribution is over vectors of prices in all auctions.

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