Qualitative Reasoning under Uncertainty

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Abstract

In this paper, we focus our attention on the processing of the uncertainty encountered in the natural language. Firstly, we explore the uncertainty concept and then we suggest a new approach which enables a representation of the uncertainty by using linguistic values. The originality of our approach is that it allows to reason on the symbolic uncertainty interval ⟦*Certain, Totally uncertain*⟧. The uncertainty scale that we use here, presents some advantages over other scales in the representation and in the management of the uncertainty. The axiomatic of our approach is inspired by the Shannon theory of entropy and built on the substrate of a symbolic many-valued logic. So, the uncertainty management in the symbolic logic framework leads to generalizations of classical inferences rules.

Introduction

In the common sense reasoning, the uncertainty is usually expressed by using the linguistic expressions like *"very uncertain"*, *"totally uncertain"*, *"almost certain"*... The main feature of this uncertainty is its *qualitative* nature (Chachoua & Pacholczyk 1996). So, several approaches have been proposed for the processing of this uncertainty category. Among these, we can quote the qualitative possibility theory (Dubois 1986), the qualitative evidence theory (Parsons & Mamdani 1993) and some qualitative probabilities theories (Savage 1954; Gärdenfors 1975; Wellman 1988; Aleliunas 1988; Bacchus 1990; Pacholczyk 1992; Darwiche & Ginsberg 1992; Pearl & Goldszmidt 1996; Lehmann 1996). These approaches offer better formalism for uncertainty representation of the common-sense knowledge. Nevertheless, in the natural language, the human subject uses generally two forms to express his uncertainty (Kant 1966): (1) *explicitly*, as for example in the statement *"I am totally uncertain of my future"* where, the term *"totally uncertain"* expresses an *ignorance degree* and (2) *implicitly*, very often in the *belief* form as for example in the statement *"It is very probable that Patty is Canadian"*. The term *"very probable"* designates a *belief degree* (Kant 1966). However, all approaches quoted previously concern rather the uncertainty expressed under belief form.

In this paper we suggest a new approach [1] whose objective is to contribute mainly to the qualitative processing of uncertainty expressed in the *ignorance* form. But it can also process the belief form. For example we can translate the statement *"It is very probable that Patty is Canadian"* as *"It is almost certain (i.e. little ignorance) that Patty is Canadian"*, whereas, a statement expressed in the ignorance form can not always be translated under the belief form. This is the case of the statement *"I am totally uncertain of my future"*.

In this approach, the uncertainty is represented by using the linguistic values in the interval [*Certain, Totally uncertain*]. This graduation scale presents at least two advantages. The first one concerns the representation of the ignorance situation that the scales used in other qualitative approaches do not allow. The second advantage concerns the management of the uncertainty. Indeed, in the qualitative management of uncertain knowledge, one often leads to intervals of *belief, certainty...* Then the question is to choose one value from these intervals. In our approach, to palliate this problem, we can choose the greatest value of an interval. The choice corresponds to the principle of maximum entropy (Jaynes 1982).

In section 2, we explore the uncertainty concept and we show the difference between ignorance form and belief form of the uncertainty concept.

Besides, the uncertainty concept is *gradual*. So, to take account of this feature, our approach is built on the substrate of a many-valued logic suggested by Pacholczyk (Pacholczyk 1992). This logic will be presented in section 3.

In section 4, we discuss our method of uncertainty representation. This method consists of the definition in the logical language of a many-valued predicate called *Uncert*. This predicate satisfies a set of axioms which governs the uncertainty concept, that we present in section 5. Thanks to this, in section 6 we obtain some theorems and particularly new generalization rule of Modus Ponens. These properties offer a formal framework for the qualitative management of the uncer-

[1] An earlier version of this has been presented in (Chachoua & Pacholczyk 1996).

tainty. In section 7, we presents the mains differences between our theory and others qualitative approaches of uncertainty. Finally, in section 8, we presents an application example.

Uncertainty concept

Generally, in the universe of discourse, knowledge is considered to be *certain* if it is *either true* or *false*. Otherwise, it is considered as *uncertain*. Let us consider the following example.

Example 1 Let A, B, C, D, E, and F be six towns:
1. *All* the inhabitants of the town A always say the truth,
2. *All* the inhabitants of the town B are liars,
3. *A minority* of inhabitants of the town C always says the truth,
4. *A majority* of inhabitants of the town D always says the truth,
5. *Half* of inhabitants of town F are liars.

Supposing we don't have any knowledge about the town E. Let us designate by $H(X)$ an ordinary inhabitant from the town X ($X \in \{A, B, C, D, E, F\}$) and let us assume that we will be interested by the truth value of the sentence: "*H(X) is a liar*".

It is clear that the sentence "$H(A)$ *is a liar*" is *false* and the sentence "$H(B)$ *is a liar*" is *true*. So these two sentences are **certain**. In probability terms, we have Prob(*"H(A) is a liar"*)=0 and Prob(*"H(B) is a liar"*)= 1.

Nevertheless, the determination of the truth value of some sentences like "$H(X)$ *is a liar*" with $X \in \{C, D, E, F\}$ is impossible with the available knowledge. Thus, these sentences are **uncertain**. However, note that the ignorance is maximal about inhabitants of the towns E and F. Nevertheless the probability (belief) of "$H(E)$ *is a liar*" is not *estimable*. Indeed, we can not attribute Prob($H(E)$ *is a liar*) = Prob($H(E)$ *is not a liar*) $\approx \frac{1}{2}$ as in the town F, because in this town there is no ignorance about quantity of liars as in the town E. Note again that, intuitively, the ignorance about the inhabitants of the towns C and D are approximately equal, but their probabilities can be very different.

It results that one of the main features of the uncertainty concept is the ignorance of the truth values. According to Shannon's entropy theory (Shannon & Weaver 1949), the uncertainty concept refers also to the information deficiency. Indeed, Shannon has shown that a measure of *information* can also be used for measuring *ignorance*. It follows that *the ignorance* degree expresses the degree of the information deficiency to determine the truth value. However, *the belief degree* refers rather to the information available.

In the natural language, to evaluate uncertainty, the human subject refers to a set of adverbial expressions like *almost certain, very uncertain*... This set allows him to build a subjective scale of uncertainty degrees. In our approach we reproduce the same method. So, in the following section we introduce the algebraic structures on which our method will be constructed.

Algebraic structures

The algebraic structures and the many-valued logic that we present here have been already presented by Pacholczyk in (Pacholczyk 1992; 1994).

Chains of De Morgan

Let $M \geq 2$ be an integer. Let us designate by $\mathcal{W}$ the interval $[1, M]$ completely ordered by the relation "$\leq$", and by "n" the application such that $n(\beta) = M + 1 - \beta$. In these conditions $\{\mathcal{W}, \wedge, \vee, n\}$ is a lattice of De Morgan with: $\alpha \wedge \beta = min(\alpha, \beta)$ and $\alpha \vee \beta = max(\alpha, \beta)$. Let $\mathcal{L}_M = \{\tau_1, \tau_2, ..., \tau_M\}$ be a set where: $\alpha \leq \beta - \tau_\alpha \leq \tau_\beta$. Thus, $\{\mathcal{L}_M, \leq\}$ is a chain in which the least element is τ_1 and the greatest element is τ_M.
We define in $\mathcal{L}_M$, two operators "$\wedge$" and "$\vee$" and a decreasing involution "$\sim$" by the relation: $\tau_\alpha \wedge \tau_\beta - \tau_{min(\alpha,\beta)}$, $\tau_\alpha \vee \tau_\beta = \tau_{max(\alpha,\beta)}$, and $\sim \tau_\alpha = \tau_{n(\alpha)}$.
Likewise, operators v_γ ($\gamma \in \mathcal{W}$) are defined as $\mathcal{L}_M$ in the following way:
If $a = b$ then $v_a\tau_b = \tau_M$ else $v_a\tau_b = \tau_1$.

Remark 1 In the context of the many-valued logic used here, v_α and τ_α are associated by the relation[2]:
"x *is* v_α A" $-$ x *is* v_α A $-$ "x *is* A" *is* τ_α $-$ *true*
where x and A designates respectively an object and a concept.

Example 2 For $M = 5$, one could introduce one possible set of linguistic degrees totally ordered like: $\mathcal{L}_5$ ={*not-at-all, little, enough, very, totally*}.

So, in these conditions, if the assertion "*Tom is very young*" is *true*, then it is equivalent to "*Tom is young*" is *very-true*.
In the following, the chain $\{\mathcal{L}_M, \leq, \wedge, \vee, \sim\}$ will be used as the support of the representation of truth degrees.

Interpretation and satisfaction of formulae

The formal system of many-valued predicate logic used here can be found in (Pacholczyk 1992; 1994).

Definition 1 We call an interpretation structure $\mathcal{A}$ of the language $\mathcal{L}$, the pair $< \mathcal{D}, \mathcal{R}_n >$ for $n \in N$, where $\mathcal{D}$ is a non-empty set called domain of $\mathcal{A}$ and $\mathcal{R}_n$ a multi-set[3] in $\mathcal{A}$.

Definition 2 Let $\mathcal{V} = \{z_1, z_2, .., z_n, ..\}$ be the infinite countable set of individual variables of the formal system. We call a ***valuation of variables***, a sequence denoted $x =< x_0, ..., x_n, ... >$ where $\forall i, x_i \in \mathcal{D}$. So, if x is a valuation, then $x(n/a) =< x_0, ..., x_{n-1}, a, x_{n+1}, ... >$.

[2]These equivalences can be viewed as generalizations of Tarski criteria (Pacholczyk 1992; 1994).

[3]The notion of multi-set was introduced by M. De Glas in 1988 (see (Akdag, De-Glas, & Pacholczyk 1992)). In this theory $x \in_\alpha \mathcal{R}_n \iff x$ belongs to degree τ_α to the multi-set $\mathcal{R}_n$. Note that this multi-set theory is an axiomatics approach to the fuzzy sets theory of Zadeh. In this theory $x \in_\alpha A$ in the formal representation of $\mu_A(x) = \alpha$.

Definition 3 Let Φ be a formula of $\mathcal{SF}$[4] and x a valuation. The relation "*x satisfies Φ to a degree v_α in $\mathcal{A}$*" denoted as $\mathcal{A} \models^x_\alpha \Phi$ is defined recursively as follows:

1. $\mathcal{A} \models^x_\alpha \mathcal{P}_n(z_{i_1}, ..., z_{i_k}) - < z_{i_1}, ..., z_{i_k} > \in_\alpha \mathcal{R}_n$
2. $\mathcal{A} \models^x_\alpha \neg\Phi - \mathcal{A} \models^x_\beta \Phi$, with $\tau_\beta = \sim \tau_\alpha$
3. $\mathcal{A} \models^x_M V_\alpha \Phi - \mathcal{A} \models^x_\alpha \Phi$
4. $\mathcal{A} \models^x_\alpha \Phi \cup \Psi - \mathcal{A} \models^x_\beta \Phi$ and $\mathcal{A} \models^x_\gamma \Psi \mid \tau_\gamma \vee \tau_\beta = \tau_\alpha$
5. $\mathcal{A} \models^x_\alpha \Phi \cap \Psi - \mathcal{A} \models^x_\beta \Phi$ and $\mathcal{A} \models^x_\gamma \Psi \mid \tau_\gamma \wedge \tau_\beta = \tau_\alpha$
6. $\mathcal{A} \models^x_\alpha \Phi \supset \Psi - \mathcal{A} \models^x_\beta \Phi$ and $\mathcal{A} \models^x_\alpha \Psi | \tau_\beta \rightarrow \tau_\gamma = \tau_\alpha$
7. $\mathcal{A} \models^x_\alpha \exists z_n \Phi - \tau_\alpha = \text{Max}\ \{\tau_\gamma \mid \mathcal{A} \models^{x(n/a)}_\gamma \Phi, a \in \mathcal{D}\}$
8. $\mathcal{A} \models^x_\alpha \forall z_n \Phi - \tau_\alpha = \text{Min}\ \{\tau_\gamma \mid \mathcal{A} \models^{x(n/a)}_\gamma \Phi, a \in \mathcal{D}\}$

Definition 4 Let ϕ be a formula. We say that *ϕ is v_α -true* in $\mathcal{A}$ if and only if we have a valuation x such that *x v_α -satisfies* ϕ in $\mathcal{A}$.

Uncertainty representation

Let $\mathcal{L}$ be a first order language, $\mathcal{A}$ an interpretation structure of $\mathcal{L}$ and Ω a set of formulae of $\mathcal{SF}$ such that:

$$\Omega = \{\phi \in \mathcal{SF}, \mathcal{A} \models^x_M \phi \text{ or } \mathcal{A} \models^x_1 \phi\}$$

In other terms, for a valuation x, all formulae of Ω are either *satisfied*, or *not satisfied* in $\mathcal{A}$ (ie. true or false in $\mathcal{A}$). The uncertainty of a formula ϕ of Ω is treated thanks to a particular many-valued predicate called *Uncert* which has been added in the first order many-valued predicate logic.

Definition 5 *Uncert* is defined as follows:

- Uncert is a many-valued predicate such that: $\forall\varphi \in \Omega$ Uncert $(\varphi) \in \mathcal{SF}$.
- Given an interpretation $\mathcal{A}$, for all φ of Ω one has: $\mathcal{A} \models^{x(0/\phi)}_\gamma Uncert(V_\chi\varphi)$ $\iff Uncert(V_\chi\phi)$ is $v_\gamma - true$ in $\mathcal{A}$ $\iff$ "ϕ is $t_\chi - true$" is u_γ*-uncertain* in $\mathcal{A}$ with $\tau_\gamma \in [\tau_1, \tau_M]$ and $\tau_\chi \in \{\tau_1, \tau_M\}$.

Notations: In the following, we use respectively $\mathcal{A} \models_\gamma Uncert(V_\chi\phi)$ and $\mathcal{A} \models_\gamma Uncert(\phi)$ instead of $\mathcal{A} \models^{x(0/\phi)}_\gamma Uncert(V_\chi\varphi)$ and $\mathcal{A} \models_\gamma Uncert(V_M\phi)$.

Example 3 By choosing $M = 5$, one could introduce the following ordered set of linguistic degrees.
D_{t_5}:{*not-at-all-true, little-true, true-enough, very-true, totally-true*} which corresponds to $v_\alpha - true$.
D_{u_5}:{*not-at-all-uncertain, little-uncertain, uncertain-enough, very-uncertain, totally-uncertain*} which corresponds to $u_\alpha - uncertain$, with $\alpha = 1, 2, ..., 5$.

Besides, in the natural language, one normally uses "*certain*" and "*almost certain*" rather than "*not at all uncertain*" and "*little-uncertain*". So, with these terms we obtain the following uncertainty scale[5].

[*Certain, Almost-certain, Uncertain-enough, Very-uncertain, Totally-uncertain*]

So, if Uncert($V_M\phi$) is ***very true***, then "*ϕ is true*" is ***very uncertain***.

Besides, to manage rationally the uncertainty, it is necessary that the predicate "*Uncert*" satisfy an axioms set which governs this concept.

Axioms of Uncert

According to the sense that we have given in the previous section, a formula is *certain* if its *truth value*[6] is known. From this, we derive the first axioms:

Axiom 1 $\forall\phi \in \Omega, \mathcal{A} \models_M \phi \Longrightarrow \mathcal{A} \models_1 \mathbf{Uncert}(V_M\phi)$.

Axiom 2 $\forall\phi \in \Omega, \mathcal{A} \models_1 \phi \Longrightarrow \mathcal{A} \models_1 \mathbf{Uncert}(V_1\phi)$.

Besides, two equivalent formulae have the same truth value. By this fact, their uncertainty is the same. So, the third axiom is:

Axiom 3 $\forall\phi \in \Omega, \forall\varphi \in \Omega$, **if** $\mathcal{A} \models_M (\phi \equiv \varphi)$ **then** $\{\mathcal{A} \models_\alpha \mathbf{Uncert}(\phi) \iff \mathcal{A} \models_\alpha \mathbf{Uncert}(\varphi)\}$.

Let us consider now two non contradictory formulae Ψ and Φ. Knowing the uncertainties on Ψ and Φ, what will be the uncertainty on their conjunction?

If Ψ is true (thus no uncertainty on Ψ) the determination of the truth value of $(\Psi \cap \Phi)$ will depend only on that of Φ. In these terms, the uncertainty on $(\Psi \cap \Phi)$ is equal to the uncertainty on Φ. It follows that, if one of the formulae is ***totally uncertain***, their conjunction is also ***totally uncertain***. So, the resulting uncertainty from $(\Psi \cap \Phi)$ is, according to Shannon theory of entropy (Shannon & Weaver 1949), equal to the ***sum*** individual uncertainties of Φ and Ψ. In the qualitative domain, we use this same property. So, instead the numerical operator "+", we use a qualitative additive operator S defined bellow. Thus we have the fourth axiom:

Axiom 4 $\forall\Phi \in \Omega, \forall\Psi \in \Omega$, **if** $\Psi \not\equiv \neg\Phi$ **and** $\{\mathcal{A} \models_\alpha \mathbf{Uncert}(\Phi), \mathcal{A} \models_\beta \mathbf{Uncert}(\Psi)\}$ **then** $\mathcal{A} \models_\gamma \mathbf{Uncert}(\Psi \cap \Phi)$**with** $\tau_\gamma = S(\tau_\alpha, \tau_\beta)$.

Definition 6 The operator S is defined in order to satisfy the following properties of classical sum: $\forall(\tau_\alpha, \tau_\beta, \tau_\gamma, \tau_\delta) \in [\tau_1, \tau_M]$,
s_1. $S(\tau_\alpha, \tau_1) = \tau_\alpha$,
s_2. $S(\tau_\alpha, \tau_\beta) = S(\tau_\beta, \tau_\alpha)$,
s_3. $S(\tau_\alpha, S(\tau_\beta, \tau_\gamma)) = S(S(\tau_\alpha, \tau_\beta), \tau_\gamma)$
s_4. $\tau_\alpha \leq \tau_\beta$ and $\tau_\gamma \leq \tau_\delta \Rightarrow S(\tau_\alpha, \tau_\gamma) \leq S(\tau_\beta \leq \tau_\delta)$.

The formulae ϕ and $\neg\phi$ are mutually exclusive. However, if the formulae ϕ and $\neg\phi$ are uncertain, then considering the available information, one will choose the most relevant formula (ie. the most certain formula). The last axiom is then:

Axiom 5 $\forall\phi \in \Omega$, **if** $\{\mathcal{A} \models_\alpha \mathbf{Uncert}(\phi)$ **and** $\mathcal{A} \models_\beta \mathbf{Uncert}(\neg\phi)\}$ **then** $\mathcal{A} \models_{\alpha\wedge\beta} \mathbf{Uncert}(V_\chi\phi)$ **with** { **if** $\tau_\alpha \leq \tau_\beta$ **then** $\tau_\chi = \tau_M$, **else** $\tau_\chi = \tau_1$}.

[4] Note that $\mathcal{SF}$ designates the formulae set for the system of many-valued predicates calculus.

[5] Please note that this scale and linguistic terms are chosen subjectively by a human expert.

[6] Note that the truth values "*true*" and "*false*" correspond respectively to τ_M and τ_1.

Uncertainty management

In this section we present and prove the fundamental theorems regarding the management of uncertainty.

Theorem 1 *Conjunction of several formulae.*
$\forall \Phi_i \in \Omega, i \in \{1, ..., n\}$, if the formulae $\Phi_{i=1,...,n}$ are non contradictory and for all i we have Φ_i is u_{β_i}-uncertain, then the conjunction $(\Phi_1 \cap .. \cap \Phi_n)$ is u_χ-uncertain with $\tau_\chi = S(\tau_{\beta_1}, ...S(\tau_{\beta_{n-1}}, \tau_{\beta_n})...)$.

Proof 1 *Consequence of the axiom 4.*

Corollary 1 *If $\forall \Phi_{i \in \{1,...,n\}} \in \Omega$,
$\mathcal{A} \models_1 Uncert(\Phi_{i \neq a})$ and $\mathcal{A} \models_\alpha Uncert(\Phi_{a \in \{1,..,n\}})$.
Using the theorem 6.1 one obtains:
$\mathcal{A} \models_\chi Uncert(\Phi_1 \cap ... \cap \Phi_a \cap ... \cap \Phi_n)$ with $\tau_\chi = \tau_\alpha$.*

In other terms, the conjunction of several formulae is u_α-uncertain if none of them is less uncertain than u_α-uncertain.

Theorem 2 *Disjunction of two formulae.*
**$\forall \phi \in \Omega, \forall \varphi \in \Omega$, if we have:
φ is u_α-uncertain and Φ is u_β-uncertain, then $(\varphi \cup \phi)$ is u_γ-uncertain, with $\tau_\gamma \leq \tau_\alpha \wedge \tau_\beta$.**

Proof 2 $\forall \phi \in \Omega, \forall \varphi \in \Omega$, *we have:
(1) $\mathcal{A} \models_M (\phi \equiv (\phi \cup \neg\varphi) \cap (\phi \cup \varphi))$ and
(2) $\mathcal{A} \models_M (\varphi \equiv (\varphi \cup \neg\phi) \cap (\phi \cup \varphi))$
Let suppose that: $\mathcal{A} \models_\alpha Uncert(\varphi)$,
x $\mathcal{A} \models_\beta Uncert(\phi)$, $\mathcal{A} \models_\gamma Uncert(\varphi \cup \phi)$,
$\mathcal{A} \models_{\sigma_1} Uncert(\phi \cup \neg\varphi)$, $\mathcal{A} \models_{\sigma_2} Uncert(\varphi \cup \neg\phi)$.
The axioms 3, 4 and the equations (1) and (2) gives:
$\tau_\alpha = (\tau_{\sigma_1}, \tau_\gamma)$ and $\tau_\beta = (\tau_{\sigma_2}, \tau_\gamma) \Rightarrow \tau_\gamma \leq \tau_\alpha \wedge \tau_\beta$.*

Theorem 3 *Disjunction of several formulae .*
$\forall \Phi_i \in \Omega, i \in \{1, ..., n\}$ if we have several formulae $\Phi_{i=1...,n}$ such that, for all i, Φ_i is u_{β_i}-uncertain, then the disjunction $(\Phi_1 \cup ... \cup \Phi_n)$ is u_χ-uncertain with $\tau_\chi \leq \tau_{\beta_1} \wedge ... \wedge \tau_{\beta_n}$.

Proof 3 *Consequence of the theorem 2.*

Corollary 2 $\forall \Phi_{i=1,...,n} \in \Omega$, *if $\exists \Phi_a \mid \Phi_a \in \Omega$, and $\mathcal{A} \models_1 Uncert(\Phi_a)$, then using the theorem 3 one has:
$\mathcal{A} \models_\lambda Uncert(\Phi_1 \cup \Phi_a \cup ... \cup \Phi_n)$ with $\tau_\lambda = \tau_1$.
So, the disjunction of formulae is* certain *if only one of them is* certain.

Theorem 4 *Generalized Modus Ponens rule.*
**$\forall \phi \in \Omega, \forall \varphi \in \Omega$, if we have:
φ is u_α-uncertain and $(\varphi \supset \phi)$ is u_β-uncertain, then ϕ is u_λ-uncertain with $\tau_\lambda \leq S(\tau_\alpha, \tau_\beta)$.**

Proof 4 $\forall (\phi, \varphi) \in \Omega$, $\mathcal{A} \models_M ((\phi \cap \varphi) \equiv ((\varphi \supset \phi) \cap \varphi))$
*According to the axiom 3 if $\mathcal{A} \models_\gamma Uncert(\phi \cap \varphi)$
then $\mathcal{A} \models_\gamma Uncert((\varphi \supset \phi) \cap \varphi))$.
Using the axiom 4 one can write:
$\mathcal{A} \models_\gamma Uncert((\varphi \supset \phi) \cap \varphi) \Rightarrow \mathcal{A} \models_\beta Uncert(\varphi \supset \phi)$ and
$\mathcal{A} \models_\alpha Uncert(\varphi)$ such as:
$\tau_\gamma = S(\tau_\beta, \tau_\alpha)$. Always according the axiom 4:
$\forall \phi \in \Omega, \forall \varphi \in \Omega$, if $\mathcal{A} \models_\lambda Uncert(\phi)$, then $\tau_\lambda \leq \tau_\gamma$.
Finally $\tau_\lambda \leq S(\tau_\alpha, \tau_\beta)$.*

Theorem 5 *Combination of uncertainties*
**$\forall \phi \in \Omega, \forall \varphi \in \Omega$, if we have:
ϕ is u_α-uncertain, and $(\phi \supset \Phi)$ is u_β-uncertain
φ is u_γ-uncertain and $(\varphi \supset \Phi)$ is u_σ-uncertain
then Φ is u_μ-uncertain
with $\tau_\mu \leq S(\tau_\alpha, \tau_\beta) \wedge S(\tau_\gamma, \tau_\sigma)$.**

Proof 5 *According to the Modus Ponens the equations (1) and (2) give respectively:
$\mathcal{A} \models_\mu Uncert(\Phi)$ with $\tau_\mu \leq S(\tau_\alpha, \tau_\beta)$ and
$\tau_\mu \leq S(\tau_\gamma, \tau_\sigma)$. Therefore $\tau_\mu \leq S(\tau_\alpha, \tau_\beta) \wedge S(\tau_\gamma, \tau_\sigma)$.*

Corollary 3 $\forall \phi \in \Omega, \forall \varphi \in \Omega, \forall \Phi \in \Omega$ *if we have:
$\mathcal{A} \models_\alpha Uncert(\phi)$, and $\mathcal{A} \models_\beta Uncert(\phi \supset \Phi)$ (1)
$\mathcal{A} \models_\gamma Uncert(\varphi)$ and $\mathcal{A} \models_\sigma Uncert(\varphi \supset \neg\Phi)$ (2)
Then, using the Modus Ponens, the equations (1) and (2) give respectively: $\mathcal{A} \models_\mu Uncert(\Phi)$ and $\mathcal{A} \models_\nu Uncert(\neg\Phi)$ with $\tau_\mu \leq S(\tau_\alpha, \tau_\beta)$ and
$\tau_\nu \leq S(\tau_\gamma, \tau_\sigma)$. According the axiom 5, one obtains:
$\mathcal{A} \models_\theta Uncert(V_\chi \Phi)$ with $\tau_\theta \leq S(\tau_\gamma, \tau_\sigma) \wedge S(\tau_\alpha, \tau_\beta)$ and
$\tau_\chi = \tau_M$ if $S(\tau_\alpha, \tau_\beta) \leq S(\tau_\gamma, \tau_\sigma)$ else $\tau_\chi = \tau_1$.*

Theorem 6 *Propagation of uncertainty.*
$\forall \phi \in \Omega, \forall \varphi \in \Omega, \forall \Phi \in \Omega$, if $(\phi \supset \Phi)$ is u_β-uncertain and $(\Phi \supset \varphi)$ is u_σ-uncertain then $(\phi \supset \varphi)$ is u_μ-uncertain with $\tau_\mu \leq S(\tau_\beta, \tau_\sigma)$.

Proof 6 $\forall \phi, \varphi, \Phi \in \Omega$ *if $\mathcal{A} \models_\beta Uncert(\phi \supset \Phi)$ and $\mathcal{A} \models_\sigma Uncert(\Phi \supset \varphi)$, then according to the axiom 4 we have: $\mathcal{A} \models_\delta Uncert((\phi \supset \Phi) \cap (\Phi \supset \varphi))$with $\tau_\delta = S(\tau_\beta, \tau_\sigma)$ and $\mathcal{A} \models_M (((\phi \supset \Phi) \cap (\Phi \supset \varphi)) \supset (\phi \supset \varphi))$
$\Rightarrow \mathcal{A} \models_1 Uncert(((\phi \supset \Phi) \cap (\Phi \supset \varphi)) \supset (\phi \supset \varphi))$
Using the rule of Modus Ponens, one obtains:
$\mathcal{A} \models_\mu Uncert(\phi \supset \varphi)$ with $\tau_\mu \leq S(\tau_\delta, \tau_1) = \tau_\delta$.
Finally, $\tau_\mu \leq S(\tau_\beta, \tau_\sigma)$.*

Strategy

In the previous theorems, by using the maximum entropy principle (Jaynes 1982), one can replace the sign "$\leq$" by "$=$" . As Jaynes suggests it, this choice is a kind of assurance which protects against the false predictions. This principle enables us to palliate the difficulty of the management of the uncertainties intervals.

Some differences with other approaches

The axiomatic of our approach is built to process the uncertainty expressed under ignorance form. However qualitative probabilities approaches (Savage 1954; Gärdenfors 1975; Wellman 1988; Aleliunas 1988; Pacholczyk 1992; Darwiche & Ginsberg 1992; Pearl & Goldszmidt 1996; Lehmann 1996) and the qualitative approach of the evidence theory (Parsons & Mamdani 1993) allow rather to process especially the expressed uncertainty in the belief form. So, in uncertainty representation point of view, our approach presents the advantage to represent explicitly the situation of total ignorance such that it is expressed in the natural language. In the uncertainty management, one can make emerge main two differences between our approach and others qualitative approaches. The first concerns the implication (or the inclusion in set terms). So, if A and

B are two formulae of Ω, one has: $A \supset B \Longrightarrow C(A) \leq C(B)$, where $C()$ can designate a measure of, probability, possibility, necessity or credibility. However, in our approach one has: $A \supset B \Longrightarrow I(A) \geq I(B)$, where $I()$ designate a mesure of ignorance (ie the truth degree of the formulae Uncert(A) and Uncert(B)).

The second difference is relative to the conjunction of formulae (or intersection in set terms). So, generally in all qualitative approaches quoted in the former, one has: $C(A \cap B) \approx C(A) \otimes C(B)$, where $\otimes$ designates a multiplicative qualitative operator. In our approach one has: $I(A \cap B) = I(A) \oplus I(B)$, where $\oplus$ designates an qualitative additive operator.

As we have underlined it, the qualitative processing of the uncertainty ends very often to intervals of uncertainties. However, when the uncertainty is represented in the belief form (propability, credibility..), the exploitation of these intervals is problematical. Thus, when the uncertainty is represented in the ignorance form, one can choose the maximal value of the interval.

Application

In this example, we will use the uncertainty scale defined in the example 4 and in order to lighten the notation, in the following example we will use:
Uncert(Φ) $\equiv u_\beta$*−uncertain* instead of $\mathcal{A} \models_\beta$Uncert($\Phi$).

Example 4 Let us the following knowledge:
(1) Uncert(Patty is Canadian$\supset$Patty doesn't speaks French) $\equiv$ *almost certain.*
(2) Uncert(Patty is Quebecer$\supset$Patty is Canadian) $\equiv$ *certain.*
(3) Uncert(Patty is Quebecer$\supset$Patty speaks French) $\equiv$ *certain.*

Let us assume now that we obtained the following information: It is *certain* that Patty is Quebecer, thus:
(4) Uncert(Patty is Quebecer)$\equiv$*certain.*

Using the knowledge (2), (3) and (4) with the Modus Ponens we obtain:
d_1. Uncert(Patty is Canadian)$\equiv$*certain.*
d_2. Uncert(Patty speaks French)$\equiv$*certain.*

Beside, the knowledge (1) and [d_1] give with the Modus Ponens:
d_3. Uncert(Patty doesn't speaks French)$\equiv$*almost certain.*

Finally, using the combination of uncertainties (*corollary 8*) of the knowledge [d_2] and [d_3] we obtain finally:
d_4. Uncert(Patty speaks French)$\equiv$*certain.*

Conclusion

In this paper we were interested to the processing of uncertainty encoded into a qualitative way. We have explored the uncertainty concept and we have presented the last results obtained with a new qualitative approach of uncertainty processing. This approach offers the possibility to represent explicitly the uncertainty, as it could be evaluated subjectively, by using linguistic values. The graduation scale used here allows us to handle several special cases of uncertainty, including the situation of total ignorance.

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