# Reasoning About Algebraic Answers In Physics

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## Abstract

One of the key components of an Intelligent Tutoring System (ITS) is the mechanism for reasoning about the student's input. This component affects the kinds of questions that can be asked, which in turn affects both the kinds of lesson material supported and the success of the ITS itself. It affects how precisely the system can pinpoint student errors and thus the subsequent help that the system provides. This paper describes an example of a class of Physics problems whose answers are most naturally represented as algebraic equations. We describe how our system, the PHYSICS-TUTOR, supports such a mechanism and the reasoning underlying it. Analyzing such input requires not only an understanding of algebra but also knowledge of physics concepts.

## Introduction

One of the key components of an Intelligent Tutoring System (ITS) is the mechanism for reasoning about the student's input. This component affects the kinds of questions that can be asked, which in turn affects both the kinds of lesson material supported and the success of the ITS itself. It affects how precisely the system can pinpoint student errors and thus the subsequent help that the system provides.

The reasoning mechanism is intimately connected with how the student can submit answers. There are many ways for a tutoring program to accept answers. The type of answer chosen depends greatly on the type of material presented and also has a strong effect on the questions that can be asked of the student.

Most current tutoring systems support a combination of multiple choice, natural language text and numeric answers. These mechanisms are very limiting for an introductory Physics course. Introductory Physics at colleges and universities covers a number of concepts that require varied answer mechanisms and abstract reasoning to be effective. This paper describes an example of a class of Physics problems whose answers are most naturally represented as algebraic equations. We describe how our system, the PHYSICS-TUTOR, supports such a mechanism and the reasoning underlying it. Analyzing such input requires not only an understanding of

algebra but also knowledge of physics concepts. This analysis uses information about the scalar/vector nature of variables and their dimensions to identify errors made by the student.

## An Example Physics Problem

A basic concept taught in elementary physics is that of **Force Balance** in statics. One common example is that of a block $A$ on an inclined plane that is connected via a pulley to another block B (see Figure 1). If the blocks are in equilibrium (i.e., stationary), the forces on each block sum to zero. Students are frequently asked to specify the equation relating the forces acting on the masses. One of the several possible answers is:
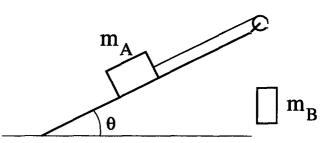
$$m_A * g * \sin(\theta) = m_B * g$$



Figure 1: Example Problem: Blocks in Equilibrium

The objective of this example problem is to determine whether students:

- understand that in equilibrium the net force acting on each block is zero,
- can find equations from that principle *and*
- can put these together to find a single equation relating the masses and $\theta$.

The various quantities, e.g., mass and inclination angle, are given as variables to help the student learn how to reason abstractly, i.e., without relying on numeric quantities and calculators.

## Existing Reasoning Mechanisms

In this section, we describe some commonly used ways of inputting answers and reasoning about them. We also discuss the limitations of these approaches that the PHYSICS-TUTOR system addresses.

- multiple choice

  This mechanism is easily implemented using WWW technology (Novak & Patterson 1998; Hubler & Assad 1995; Novak 1997). Answers are easily graded and the feedback to the student can be immediate, an important consideration.

  The main weakness of this approach is that it does not encourage the development of abstract reasoning and also makes it difficult to determine how the student generated the answer.

- natural language text

  This relies heavily on the ability to use keywords to decipher an answer.

  This is reasonable in some domains (Freedman 1997) but not in Physics. There are too many ways to state an equation to easily use natural language text analysis.

- numeric answers

  This mechanism can be coupled with simulation programs (e.g., written in JAVA) to show the implications of the student's answers (Titus, Martin, & Beichner 1998). A problem could be phrased so that there was one unknown and the other parameters were given numerically.

  However, if the answer is incorrect, it is difficult to determine how the student generated the answer, i.e., there are no intermediate steps/points available. Numerical answers are also less effective than algebraic analysis at building abstract reasoning.

The ANDES system (VanLehn 1996; Gertner 1998) supports student input in the form of algebraic equations but does not use domain knowledge to reason from Physics first principles. The student input is transformed into a canonical form using a standard set of transformations. The canonical form is then compared with a pre-enumerated list of equivalent correct and incorrect answers. Feedback is then provided if the student's answer matches one in the list. The system however does not reason about the answer itself but instead relies on pre-determined error classifications associated with each incorrect answer. This approach has several drawbacks. The main drawback is that for each of the correct and incorrect answers there are many equivalent formulations. Pre-enumerating them does not seem to be a feasible solution and restricts the system's ability to provide useful feedback.

## Reasoning About Algebraic Answers

The algebraic solution to the problem described in the previous section can be stated in multiple ways. Here are some examples:

1. $m_A * g * \sin(\theta) = m_B * g$

2. $m_A * \sin(\theta) = m_B$

3. $m_A * g * \sin(\theta) - m_B * g = 0$

These answers illustrate some of the difficulties involved in being able to accept algebraic answers from students. Some of the difficulties in parsing **correct** answers include:

- *handling special symbols*: There are several special symbols that are associated with concepts, e.g, $\mu$ is usually used to denote the coefficient of friction while $\theta$ is used to denote an arbitrary angle.

- *implicit variables and quantities*: Beyond the Greek symbols, there are also several variables (single letter names) that have a standard semantics when used in Physics. In our example $g$ (acceleration of gravity) was not in the statement of the problem but is needed in the answer.

- *removal of common subexpressions and factors*: In the second answer above, $g$ has been factored out of the equation.

Some of the possible mistakes that a good Physics ITS should handle include:

- invalid dimensions, $m_B * g - m_A * \sin(\theta) = 0$

- missing factor, $m_B * g = m_A * g$

- missing term, $m_B * g * \sin(\theta) = 0$

- an incorrect alignment of vectors such as

$$m_B * g + m_A * g * \sin(\theta) \;-\; 0$$
$$\text{or} \quad m_B * g - m_A * g * \cos(\theta) \;=\; 0$$

## The PHYSICS-TUTOR Project

We have developed a mechanism for reasoning about algebraic answers in Physics. Our approach is strongly biased towards the domain of introductory Physics because it uses strong domain knowledge to reason about the answers and perform credit-blame assignment. In this section, we provide a brief overview of the PHYSICS-TUTOR project and describe a technique that we developed for reasoning about algebraic answers.

The PHYSICS-TUTOR project is developing an ITS that is to be used in introductory Physics courses. One of us has extensive experience teaching introductory courses in this area. The aim of the project is to utilize AI techniques to help develop tutoring systems that are more flexible and "intelligent" than previous systems.

There are many existing Physics Tutoring systems that use some combination of multiple choice selection and numeric inputs as the answer mechanism. We believe there are important reasons to go beyond this to include mechanisms for analysing algebraic input.

## Physics Domain Knowledge

The biggest discovery that we made when searching through Physics textbooks is that there is a manageable number of basic concepts that can be built into an ITS to enable it to delimit the analysis of a student's reasoning process

Algebraic answers in Physics only make sense if they relate properly to physical concepts. For example, it makes sense to take the component of a Force at some angle but the same does not hold true for a Mass. That is, $m_A * g * \sin(\theta)$ has a direct interpretation as a component of a Force, while $m_A * \sin(\theta)$ indicates either that a factor has been extracted from an expression involving forces, or that the student has erroneously projected a scalar quantity.

To process and make sense of answers like

$$m_A * g * \sin(\theta) = m_B * g$$

a system must be able to recognize that the $m$ implicitly stands for a Mass and that $g$ is implicitly an Acceleration and the combination constitutes a Force.

The PHYSICS-TUTOR system includes a domain knowledge base about basic Physics concepts, the dimensions associated with each and the operations that can be applied to each and whether or not the result is also an instance of another Physics concept. This knowledge is represented as a sequence of patterns as shown in Figure 2.
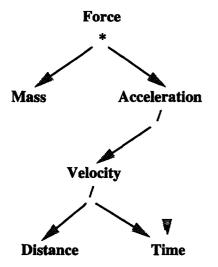


Figure 2: Physics Concepts and Their Relationship To One Another

The example pattern specifies that Mass can be multiplied with an Acceleration to form a Force, and Distance can be divided by Time to form a Velocity. A particular operation between two concepts may be disallowed (Mass cannot be added to Force) or may be possible but not lead to a concept. For example, Mass times Time cannot directly arise from analyzing a physical situation.

In addition, the PHYSICS-TUTOR system has a class hierarchy (Fig 3) whereby the inheritance structure of the Physics concepts can be specified, e.g., Force is a subclass of Vector. This hierarchy allows us to efficiently specify the operations that are permitted on each Physics concept and its subconcepts. An example of this is that we can extract a component of a Vector using a trigonometric function but the same cannot be applied to a scalar. Since Force is a Vector but Mass is not, the expression $m * g * \sin(\theta)$ can have a direct physical interpretation while $m * \sin(\theta)$ cannot.
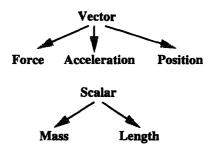


Figure 3: Example Hierarchies of Physics Concepts

To summarize, our knowledge base consists of

- concepts (names) pertaining to each variable, the associated dimensions and how they are composed from other concepts

- operations on concepts and their effects

- variables that are commonly associated with each concept, e.g., $m$ for Mass and $g$ for Acceleration

- a class hierarchy of concepts

## Credit-Blame Assignment

When the instructor supplies the correct answer to a question, the answer is parsed by the PHYSICS-TUTOR. For example, the answer

$$m_A * g * \sin(\theta) = m_B * g$$
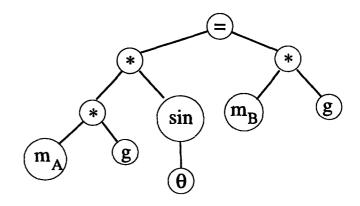
is parsed into the tree shown in Figure 4.



Figure 4: Parse Tree of An Answer

When a student enters an answer, the system first constructs a parse tree and checks it for dimensional consistency. The system uses its knowledge base to assign dimensions to variables, e.g., $kg$ for $m$ and $m/s^2$ for $g$. If the answer is dimensionally inconsistent, the student will have to change the equation or the dimensions assigned to the variables. This parse tree is then compared to the solution to determine if the student's response is correct. This is not a simple procedure.

**Matching Correct Answers** One correct answer that the student could enter is:

$$m_A * \sin(\theta) - m_B = 0$$

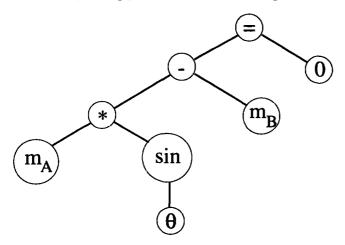The corresponding parse tree is shown in Figure 5.



Figure 5: A Correct Answer From A Student

In this case the student has (1) rearranged terms and (2) simplified the answer by removing a common factor ($g$ - the acceleration due to gravity). The answer is correct but does not have a direct physical manifestation because it appears to involve taking a component of a Mass, not a Force. To correctly match the answer with the solution, the system executes the following algorithm:

1. Starting from the bottom up, it tries to ensure that all the terms match physical concepts in its knowledge base. When it finds a mismatch, it constructs a list of possible concepts that match. For this example there is one match, that of $Force = Mass * Acceleration$ for both terms in the equation.

2. The next step is to find a consistent modification for all the terms, e.g., injecting a factor ($g$) into all terms of the equation, so that the terms have meaningful physical quantities and the answer has not been changed.

3. The final step is to apply algebraic transformations to the parse tree to find a match between it and the solution tree. The answers contain few enough factors that exhaustive search is feasible. The system is provided with rewrite rules so that it understands the rules of commutativity etc., in algebraic operations.

**Isolating Errors In Answers** The previous section described how the system tries to match a correct response with the provided solution. This section describes how the PHYSICS-TUTOR system performs credit-blame assignment with an incorrect answer. What the system does once the errors are found, e.g., generate hints or a simpler similar problem, is beyond the scope of this paper.

An example of an incorrect answer is:

$$m_B * g + m_A * g * \sin(\theta) = 0$$

This answer is dimensionally correct and all the terms correspond to physical quantities. The corresponding parse tree is shown in Figure 6.
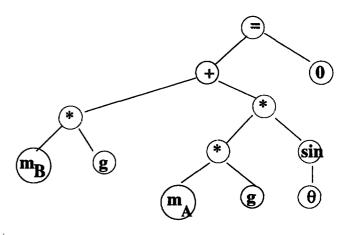


Figure 6: Parse Tree Of An Incorrect Answer

The system will try to match the generated parse tree with the solution tree and will not succeed. The closest match to the solution (Figure 4) is a tree where the + operator is replaced by a −.

The PHYSICS-TUTOR system possesses knowledge about different types of errors that the student can make, e.g., reversed operators (as in the example above) and incorrect operators (if the student had used a * operator instead of + or −). Associated with each of these error types is an error handling template that specifies the strategy to be employed if that particular error type is encountered.

## Limitations and Open Issues

The PHYSICS-TUTOR system is still in the early stages of development. We have and are continuing to evaluate the effectiveness and efficiency of our method on various simple statics problems. We are in the process of expanding and extending our system to cover sufficient problems for a three week period of an introductory college Physics course. This section describes the current limitations of our approach and the open issues that we are facing.

- *Time-dependent quantities:* Many of the concepts in Physics refer to quantities that are functions of Time,

but appear in problems in introductory courses primarily as the values at a very few discrete times. For example, Velocity is a Time dependent quantity but is often initially treated as a constant. In subsequent lessons, the student may be asked to calculate acceleration and this will involve being able to specify and reason about the Velocity of an object at two different times, $t_1$ and $t_2$. To reason about these quantities, the system must be able to associate and map them to the appropriate objects, e.g., $v_1$ and $v_2$ can only be associated with Mass $m_1$ while $v_3$ and $v_4$ can only be associated with Mass $m_2$.

- *periodic quantities:* A further complication arises from the equivalence of periodic quantities. For example, the expressions $\sin(\omega * t)$ and $\sin(2\pi + \omega * t)$ could be considered to be equivalent. Reasoning about these equations can only be done with a knowledge of the periodic nature of the trigonometric functions, knowledge that the PHYSICS-TUTOR system currently does not possess.

- *differentiation and integration:* The PHYSICS-TUTOR system does not support input of differential and integral operations and consequently cannot reason about them. We plan to address these issues in the future.

## Conclusion

This paper has described a class of Physics problems whose answers are most naturally represented as algebraic equations. We have described how the PHYSICS-TUTOR system uses domain knowledge about the concepts from introductory Physics to reason about the equations. The reasoning mechanism is used to perform credit-blame assignment so that help can be effectively directed at specific errors.

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