Representing Simple Trajectories as Oriented Curves

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Abstract

We present a formal framework for the description of ordering information on directed linear structures like trajectories of moving objects. The foundation of this approach is an axiomatic characterization of oriented curves. They provide a generalized notion of direction. The proposed geometric framework allows a qualitative characterization of oriented curves without commitment to concepts of measurement. The geometry of oriented curves is applied to reasoning about the possibility of meeting of objects that move along intersecting trajectories. To determine the possibility of meeting is part of planning and scheduling tasks, e.g. concerning systems of transport vehicles, as well as in the task of avoiding collision. We show that the reasoning system proposed here needs only one purely temporal notion, namely that of simultaneity.

Introduction

The representation of knowledge about space and time and about spatial and temporal concepts is a central topic for areas of knowledge representation that investigate how the structure of the represented domain affects knowledge representation and how the structure can be exploited in reasoning. One can thereby focus on different structural aspects such as topology, ordering and distance, and, when studying concepts and representations of motion, also explore the interaction of both structures (cf. Galton 1997).

In this article, we propose a calculus of oriented curves that can be embedded in the framework of ordering geometry (cf. Schlieder 1995, Eschenbach & Kulik 1997, Eschenbach et al. 1998).[1] Oriented curves are a general geometric device that can represent a variety of entities that are both linear (i.e., contain no cycles and do not branch) and directed (i.e., distinguish between start and end). Since oriented curves need not to be straight, they capture the idea of a generalized notion of direction.

Oriented curves can be defined in a geometric framework without considering, e.g., mappings from time to space. Therefore, they are basically atemporal. But as will become clear, they allow non-temporal as well as temporal interpretations. Oriented curves can, e.g., be taken as the geometric specification of arrows in maps or diagrams, or any other linear and directed device in diagrammatic reasoning.

The application of oriented curves discussed in this paper is a temporal one, namely, the course of motion or trajectory of an object. An oriented curve represents both the collection of the positions occupied by the moving object and the order of occupation of these places. Although time is not represented explicitly, the effects of temporal order on space can be captured. As will be shown, the only purely temporal notion that is needed in order to reason about the possibilities of meeting of moving objects is the notion of simultaneity.

Representations of Courses of Motion

To represent time is to represent change and the order of changes around us. Measurement of time is neither basic for the notion of change nor for notions such as repetition or periodicity. The formal framework introduced in this article is accordingly not depending on concepts of measurement of time and neither of space. It thereby fits in the area of qualitative spatial reasoning (cf. Cohn 1997). The results obtained on this basis are general, since they also hold when the framework is enriched by information about distance or angle.

Trajectories of moving objects exhibit several spatial properties: They are connected, have shape, do not branch and are directed. Furthermore, the individual positions successively occupied by an object in the course of its motion are represented as points, i.e., as spatial entities that do not exhibit an internal spatial structure. Of course, this assumption is a considerable restriction, since many natural ways of movement—such as walking in contrast to running and the rolling of a ball—are not captured in all respects by this restriction. However, for many purposes like route planning, one can abstract from these complexities as is done by Mohnhaupt & Neumann (1989), Habel (1990), Eisenkolb et al. (1998) as well, or represent the movement of a spatially extended object by several (synchronized) trajectories. Thus, the approach presented here complements the work in the area of qualitative kinematics that focuses on the relative motion of extended object parts.

Representations of trajectories of moving objects are traditionally based on mappings from time to space. In contrast to this, Mohnhaupt and Neumann (1989) use chains of coordinates, while Eisenkolb et al. (1998) propose a qualitative language to specify vector chains. The

[1] The proofs of the theorems mentioned here are omitted in this article (cf. Kulik & Eschenbach 1999).

proposal of specifying trajectories on a geometric level as oriented curves generalizes the approaches of Habel (1990) and of Mohnhaupt and Neumann in that it does not require an explicit representation of time. Oriented curves generalize the idea of vector chains since they allow for smoothly bent courses as well as abrupt changes in direction. However, the representation proposed here does not include any notion of velocity as the ones of Mohnhaupt and Neumann (1989) and of Eisenkolb et al. (1998) do.

One more critical restriction of the approach presented here has to be mentioned. Oriented curves do not have internal cycles and, thus, the simple trajectories represented by them do not allow a point or sub-curve to be passed through more than once. However standstill in one place can be accounted for. More complex trajectories of objects that pass through a point more than once can be represented as chains of simple trajectories. The individuation and order of simple trajectories in a chain that constitutes a complex trajectory is not generally describable on the basis of spatial order and, thus, has to be expressed by additional means.

A Geometry of Oriented Curves

The presented framework provides a geometric characterization for curves and ordering information on them. Since one objective is to identify general characteristics of linear structures in space, we develop a description of curves that does not make any particular assumptions about the properties of curves, that is to say, whether they are smoothly bent, have vertices, are rectifiable, etc. The main requirement is that they are linear in the sense that they can be supplied with a total ordering structure. However, the specification of the spatial embedding of curves is unnecessary for our purpose. Therefore, only the necessary part of the geometric framework is presented here (cf. Eschenbach et al., 1998, Eschenbach & Kulik, 1997).

The formal framework is many sorted predicate logic with identity. The geometric structure introduces three types of entities and two primitive relations. The entities are points (denoted by P, Q, P', P_1, …), curves (c, c_1, …), and oriented curves (o, o_1, …). The primitive relations are the binary relation of *incidence* (denoted by ι) and the ternary relation of *precedence* with respect to oriented curves ($\prec$). We first give the characterization of curves (that represent the collection of positions an object moves through), then show how betweenness can be defined on curves, and finally introduce oriented curves such that every curve can be oriented in exactly two ways.

A Geometry of Simple Curves

A collection of definitions supply abbreviations of more complex formulae in the following. A curve c' is *part of* another curve c or its *sub-curve* (in symbols $c' \sqsubseteq c$), if all points of c' are incident with c. A curve c that has exactly the points of curves c_1 and c_2 (see axioms (C8) and (C9)) is called their *sum* $c_1 \sqcup c_2$.

$$c' \sqsubseteq c \quad \Leftrightarrow_{\text{def}} \quad \forall P\, [P\ \iota\ c' \Rightarrow P\ \iota\ c]$$

$$c = c_1 \sqcup c_2 \quad \Leftrightarrow_{\text{def}} \quad \forall Q\, [Q\ \iota\ c \Leftrightarrow (Q\ \iota\ c_1 \vee Q\ \iota\ c_2)]$$

Remark. Sum ($\sqcup$) is a partial operation: Two curves need not have a curve as their sum since curves are connected and do not branch. From this it follows, e.g. that curves without a common point do not form a sum.

An *inner point* of a curve c is a point that is on two sub-curves of c such that none of them is part of the other. An *endpoint* of a curve is on the curve and not an inner point. If a curve c does not have an endpoint, then it is *closed*. Otherwise, it is *open*. Two curves c, c' *meet* at point P, symbolized by $\mathsf{meet}(P, c, c')$, if P is a common point of them and all their common points are endpoints.

$$\mathsf{ipt}(P, c) \quad \Leftrightarrow_{\text{def}} \quad \exists c_1\, c_2\, [c_1 \sqsubseteq c \wedge c_2 \sqsubseteq c \wedge P\ \iota\ c_1 \wedge P\ \iota\ c_2 \wedge \neg c_1 \sqsubseteq c_2 \wedge \neg c_2 \sqsubseteq c_1]$$

$$\mathsf{ept}(P, c) \quad \Leftrightarrow_{\text{def}} \quad P\ \iota\ c \wedge \neg\mathsf{ipt}(P, c)$$

$$\mathsf{cl}(c) \quad \Leftrightarrow_{\text{def}} \quad \neg\exists P\, [\mathsf{ept}(P, c)]$$

$$\mathsf{op}(c) \quad \Leftrightarrow_{\text{def}} \quad \exists P\, [\mathsf{ept}(P, c)]$$

$$\mathsf{meet}(P, c, c') \quad \Leftrightarrow_{\text{def}} \quad P\ \iota\ c \wedge P\ \iota\ c' \wedge \forall Q\, [Q\ \iota\ c \wedge Q\ \iota\ c' \Rightarrow \mathsf{ept}(Q, c) \wedge \mathsf{ept}(Q, c')]$$

Remark. This definition allows two curves to meet at both ends.

Curves are strictly linear in that they do not include internal cycles and do not branch. More precisely, according to (C1) every proper sub-curve of a given curve is open, and if three sub-curves of a given curve have one endpoint in common, then one of the three sub-curves is included in one of the others (C2).

(C1) $\forall c\, c' \quad [c' \sqsubseteq c \wedge c' \neq c \Rightarrow \mathsf{op}(c')]$

(C2) $\forall c\, c_1\, c_2\, c_3 \quad [c_1 \sqsubseteq c \wedge c_2 \sqsubseteq c \wedge c_3 \sqsubseteq c \wedge \exists P\, [\mathsf{ept}(P, c_1) \wedge \mathsf{ept}(P, c_2) \wedge \mathsf{ept}(P, c_3)] \Rightarrow c_2 \sqsubseteq c_3 \vee c_3 \sqsubseteq c_2 \vee c_1 \sqsubseteq c_2 \vee c_2 \sqsubseteq c_1 \vee c_1 \sqsubseteq c_3 \vee c_3 \sqsubseteq c_1]$

Every curve has at least one inner point, i.e. a point which is not an endpoint of this curve (C3). Axiom (C4) states that every inner point P of a curve divides the curve into two sub-curves meeting at P.

(C3) $\forall c\, \exists P \quad [\mathsf{ipt}(P, c)]$

(C4) $\forall c\, P \quad [\mathsf{ipt}(P, c) \Rightarrow \exists c_1\, c_2\, [\mathsf{meet}(P, c_1, c_2) \wedge c = c_1 \sqcup c_2]]$

Curves have at most two endpoints (C5) and, if a curve has one endpoint, then it has another one (C6). On the other hand, if two curves meet and constitute a closed curve, then they meet at all their endpoints (C7).

(C5) $\forall c\, P\, Q\, R \quad [\mathsf{ept}(P, c) \wedge \mathsf{ept}(Q, c) \wedge \mathsf{ept}(R, c) \Rightarrow (P = Q \vee P = R \vee Q = R)]$

(C6) $\forall c\, P \quad [\mathsf{ept}(P, c) \Rightarrow \exists Q\, [\mathsf{ept}(Q, c) \wedge P \neq Q]]$

(C7) $\forall c\, c_1\, c_2\, P \quad [\mathsf{cl}(c) \wedge \mathsf{meet}(P, c_1, c_2) \wedge c = c_1 \sqcup c_2 \Rightarrow \forall Q\, [\mathsf{ept}(Q, c_1) \Rightarrow \mathsf{meet}(Q, c_1, c_2)]]$

If two curves meet at one endpoint, then there is a curve that has exactly the points of the two given curves (C8). Curves differ in the points they are incident with (C9). Therefore, curves can be represented as sets of points, although such a representation will not be employed here.

(C8) $\forall c_1\, c_2 \quad [\exists P\, [\mathrm{meet}(P, c_1, c_2)] \Rightarrow \exists c\, [c = c_1 \sqcup c_2]]$

(C9) $\forall c\, c' \quad [\forall P\, [P\ \iota\ c \Leftrightarrow P\ \iota\ c'] \Rightarrow c = c']$

The axioms given yield the following consequences. The sub-curve relation is an order relation, and sum is monotone with respect to it. Every open curve has exactly two endpoints. If an endpoint of a given curve lies on a sub-curve then it is also an endpoint of this sub-curve. Consequently, inner points of sub-curves of any curve are inner points of the curve. If two curves meet and their sum is open, then the only point they have in common is their meeting-point and the endpoints of the two curves that are not the meeting-points are also the endpoints of the sum of these curves.

Three theorems are especially important for the following: (T1) For any two points on a curve there is a sub-curve that connects these points, i.e. these points are the endpoints of the sub-curve. (T2) Every proper sub-curve of an open curve that has a common endpoint with the open curve can be complemented by another curve so that their sum constitutes the open curve. (T3) If two sub-curves of a given open curve have a common endpoint, then the sub-curves meet or one of them is included in the other.

(T1) $\forall P\, Q\, c \quad [P\ \iota\ c \wedge Q\ \iota\ c \Rightarrow \exists c'\, [c' \sqsubset c \wedge \mathrm{ept}(P, c') \wedge \mathrm{ept}(Q, c')]]$

(T2) $\forall c\, c_1\, P \quad [c_1 \sqsubset c \wedge c_1 \neq c \wedge \mathrm{op}(c) \wedge \mathrm{ept}(P, c_1) \wedge \mathrm{ept}(P, c) \Rightarrow \exists Q\, c_2\, [\mathrm{meet}(Q, c_1, c_2) \wedge c = c_1 \sqcup c_2]]$

(T3) $\forall c\, c_1\, c_2\, P \quad [\mathrm{ept}(P, c_1) \wedge \mathrm{ept}(P, c_2) \wedge c_1 \sqsubset c \wedge c_2 \sqsubset c \wedge \mathrm{op}(c) \Rightarrow (\mathrm{meet}(P, c_1, c_2) \vee c_1 \sqsubset c_2 \vee c_2 \sqsubset c_1)]$

Betweenness and Ordering on Simple Curves

The main idea for the definition of betweenness on curves is that a point Q is between two other points with respect to a curve if there is a sub-curve that connects the two points and has Q as an inner point:

$$\beta(c, P, Q, R) \quad \Leftrightarrow_{\mathrm{def}} \quad P \neq R \wedge \exists c'\, [c' \sqsubset c \wedge \mathrm{ept}(P, c') \wedge \mathrm{ept}(R, c') \wedge \mathrm{ipt}(Q, c')]$$

A simple consequence is that endpoints of curves are not between any point of this curve (wrt. this curve). The next theorems show that all fundamental properties of orderings on linear structures are satisfied for betweenness on open curves (cf. Huntington 1938, Eschenbach et al. 1998). Let c denote a curve and P, Q and R points: (T4) If Q is between P and R wrt. c, then P, Q and R are incident with c and are distinct. (T5) If Q is between P and R wrt. c, then Q is between R and P wrt. c. (T6) If c is open and Q is between P and R wrt. c, then P is not between Q and R wrt. c. (T7) If P, Q and R are distinct and on c then one of the points is between the others wrt. c. Finally, (T8) if Q is between P and R wrt. c and Q' another point distinct from Q and lying on c then Q is either between P and Q' or between Q' and R wrt. c.

(T4) $\forall c\, P\, Q\, R \quad [\beta(c, P, Q, R) \Rightarrow P\ \iota\ c \wedge Q\ \iota\ c \wedge R\ \iota\ c \wedge P \neq Q \wedge Q \neq R \wedge P \neq R]$

(T5) $\forall c\, P\, Q\, R \quad [\beta(c, P, Q, R) \Rightarrow \beta(c, R, Q, P)]$

(T6) $\forall c\, P\, Q\, R \quad [\mathrm{op}(c) \wedge \beta(c, P, Q, R) \Rightarrow \neg\beta(c, Q, P, R)]$

(T7) $\forall c\, P\, Q\, R \quad [P\ \iota\ c \wedge Q\ \iota\ c \wedge R\ \iota\ c \wedge P \neq Q \wedge Q \neq R \wedge P \neq R \Rightarrow \beta(c, P, Q, R) \vee \beta(c, Q, P, R) \vee \beta(c, P, R, Q)]$

(T8) $\forall c\, P\, Q\, R\, Q' \quad [\beta(c, P, Q, R) \wedge Q'\ \iota\ c \wedge Q \neq Q' \Rightarrow (\beta(c, P, Q, Q') \vee \beta(c, Q', Q, R))]$

The last result worth mentioning here is that on closed curves every point is between any pair of different points. Therefore, the development of oriented curves is based on open curves only.

Oriented Curves and Precedence

The axioms for oriented curves and the precedence structure are closely related to the axioms of ordering for oriented straight lines in Eschenbach & Kulik (1997). Oriented curves constitute a more general way than oriented straight lines of describing directions in space. The description of oriented curves is given on the basis of a primitive ternary relation of precedence ($\prec$) that distinguishes the order of points on an oriented curve and is compatible with the relation of betweenness as defined above.

A point *is between two other points on an oriented curve* if and only if one of them precedes it and the other one is preceded by it. A *starting point* of an oriented curve precedes any other point on it.

$$\beta_o(o, P, Q, R) \quad \Leftrightarrow_{\mathrm{def}} \quad \prec(o, P, Q) \wedge \prec(o, Q, R) \vee \prec(o, R, Q) \wedge \prec(o, Q, P)$$

$$\mathrm{stpt}(P, o) \quad \Leftrightarrow_{\mathrm{def}} \quad \forall Q\, [P \neq Q \wedge Q\ \iota\ o \Rightarrow \prec(o, P, Q)]$$

Points that are ordered by an oriented curve are incident with the curve (O1). Since the basic spatial structure of the oriented curves shall correspond to the structure of open curves, every oriented curve coincides with an open curve in all points (O2), and betweenness on an oriented curve is compatible with betweenness on the underlying curve (O3). Additionally, every oriented curve has a starting point (O4). For every open curve and any two points on it there is an oriented curve that coincides with the curve at all points and orders the two points in a predefined way (O5). Finally, oriented curves are identical if they totally agree in the ordering of points (O6).

(O1) $\forall o\, P\, Q \quad [\prec(o, P, Q) \Rightarrow P\ \iota\ o \wedge Q\ \iota\ o]$

(O2) $\forall o\, \exists c \quad [\mathrm{op}(c) \wedge \forall P\, [P\ \iota\ o \Leftrightarrow P\ \iota\ c]]$

(O3) $\forall P\, Q\, R\, o \quad [\beta_o(o, P, Q, R) \Leftrightarrow \exists c\, [\forall P\, [P\ \iota\ o \Leftrightarrow P\ \iota\ c] \wedge \beta(c, P, Q, R)]]$

(O4) $\forall o\, \exists P \quad [\mathrm{stpt}(P, o)]$

(O5) $\forall P\, Q\, c \quad [\mathrm{op}(c) \wedge P \neq Q \wedge P\ \iota\ c \wedge Q\ \iota\ c \Rightarrow \exists o\, [\forall R\, [R\ \iota\ o \Leftrightarrow R\ \iota\ c] \wedge \prec(o, P, Q)]]$

(O6) $\forall o_1\, o_2 \quad [\forall P\, Q\, [\prec(o_1, P, Q) \Leftrightarrow \prec(o_2, P, Q)] \Rightarrow o_1 = o_2]$

Precedence on oriented curves is a total ordering relation. Since theorem (T9) holds, incidence on oriented curves could also be defined in terms of precedence, replacing (O1) and (O4).

(T9) $\forall o\, P \quad [P\ \iota\ o \Leftrightarrow \exists Q\, [\prec(o, P, Q) \vee \prec(o, Q, P)]]$

Theorem (T10) shows how the ordering of any pair of points R and S on an oriented curve o can be determined on the basis of a given pair of points P and Q using betweenness and incidence only. Thus, the underlying curve and one pair of points are sufficient for the ordering of the points on the oriented curve.

$$\begin{aligned}
&\text{(T10)}\ \forall o\, P\, Q \quad [\prec(o, P, Q) \Rightarrow \\
&\quad \forall R\, S \quad [\prec(o, R, S) \Leftrightarrow S\ \iota\ o \wedge R \neq S \wedge \\
&\quad\quad ([(\beta_o(o, R, P, Q) \vee P = R) \wedge \neg\beta_o(o, S, R, Q)] \\
&\quad\quad \vee [\beta_o(o, P, R, S) \wedge \neg\beta_o(o, Q, P, R)])]]
\end{aligned}$$

If oriented curves have exactly the same points and order at least two of them in the same way, then they are identical.

$$\begin{aligned}
&\text{(T11)}\ \forall o_1\, o_2 [\forall P\, [P\ \iota\ o_1 \Leftrightarrow P\ \iota\ o_2] \wedge \\
&\quad \exists P\, Q \quad [\prec(o_1, P, Q) \wedge \prec(o_2, P, Q)] \Rightarrow o_1 = o_2]
\end{aligned}$$

Accordingly, for every curve there are exactly two oriented curves that are coincident with the curve and order the points in opposite manner. Similarly, the ordering can be determined on the basis of the starting point and the underlying curve.

Motion, Trajectories and Time

The structure of oriented curves shall be tested on the task of evaluating collections of intersecting trajectories with respect to the question whether the common points can be meeting points of the objects. This general setting has many specific instances, especially in the planning of systems of transport vehicles (where meeting points represent the option of exchange or connection) or the scheduling of tracks (where meeting points represent collision). But since the formal background is purely qualitative, the answers it provides abstract from distance, duration and velocity. Thus, the theorems mainly state under which condition meeting is possible but not, whether and when it occurs.

This section presents the additional means for making statements about changing positions of objects in space. It specifies the postulates necessary for the newly introduced terms and collects some inferences that can be obtained. The formal system is augmented with an a train scenario, depicted in Figure 1 below, similar to the one of Gerevini (1997). It consists of four cities (C_1, …, C_4), four intermediary stations (S_1, …, S_4), and of four trains (l_1, …, l_4) whose trajectory is represented by oriented curves. In contrast to Gerevini's description we do not have to refer to temporal entities and their ordering explicitly, since the relevant ordering information is given by oriented curves.

The formalization can be used to derive conclusions that correspond to common-sense knowledge about the possibilities of meeting according to the geometrical properties of the trajectories. For the case of the simple train scenario this can be interpreted as common-sense reasoning about direct train connections.

Simultaneity and Localization

Any propositional formalism that represents changes in the world has to provide two kinds of expressions: Expressions that stand for time-independent propositions, i.e., expressions that can be true or false without any additional anchoring in time and expressions that stand for states, i.e., whose truth or falsity depends on time. Time-independent expressions can be used to represent atemporal statements such as the statements of mathematics or statements that are completely anchored in time. In the context provided here geometric statements such as 'point P is on the trajectory of x' or 'point P precedes point Q with respect to the trajectory of x' belong to this class.

The mapping of objects to their trajectories is also considered time-independent. Since any object under consideration shall have exactly one trajectory that is an oriented curve, '$\mathrm{t}(x)$' is used to refer to the trajectory of x. Therefore the expression '$P\ \iota\ \mathrm{t}(x)$' states that a point P is on the trajectory of x, i.e. x passes through P.

Expressions whose truth depends on time are, e.g., 'object x is at position P' or 'x occupies P.' It expresses a state that is classified by Galton (1990) as a 'state of position.' This expression is subsequently symbolized by '$\mathsf{at}(x, P)$.' Time-dependent expressions need not be true at any time (under consideration). That such an expression A is true at least once, is symbolized by '$\mathsf{once}(A)$'. Therefore '$\mathsf{once}(\mathsf{at}(x, P))$' means that x is at some time at P. This expression is completely anchored in time. Thus, the operator 'once' combined with time-dependent expressions yields time-independent ones. In this function it is comparable to operators such as occurs, holds or holds-in in the calculi of McDermott (1982), Allen (1984) or Galton (1990).

Time-independent expressions can be combined by truth-functional operators of propositional logic. The operator '&' for time-dependent expressions is comparable to conjunction. It combines two expressions yielding an expression that is true exactly at those times when both of the sub-expressions are true. Thus, simultaneity can be expressed via 'once' and '&'. $\mathsf{once}(A\ \&\ B)$ means that A and B are at some time both true. Therefore '$\mathsf{once}(\mathsf{at}(x_1, P_1)\ \&\ \mathsf{at}(x_2, P_2))$' represents a proposition that states that at some time when x_1 is at P_1, x_2 is at P_2.

The relation of simultaneous collocation (connect) is an abbreviation based on 'once', 'at' and '&'. It is fundamental for the train scenario—with the interpretation of 'immediate connection between trains at a station'.

$$\mathsf{connect}(x, y, P) \Leftrightarrow_{\mathrm{def}} \mathsf{once}(\mathsf{at}(x, P)\ \&\ \mathsf{at}(y, P))$$

The interaction of the symbols 'once' and '&' is formalized using the axioms (A1)–(A4). They state that—in the context of once—& is commutative, associative, idempotent and if a combined expression is true at one time, then the sub-expression are true at some time as well.

(A1) $\forall A\, B \quad [\mathsf{once}(A\ \&\ B) \Leftrightarrow \mathsf{once}(B\ \&\ A)]$

(A2) $\forall A\, B\, C \quad [\mathsf{once}((A\ \&\ B)\ \&\ C) \Leftrightarrow \mathsf{once}(A\ \&\ (B\ \&\ C))]$

(A3) $\forall A \quad [\mathsf{once}(A) \Rightarrow \mathsf{once}(A\ \&\ A)]$

(A4) $\forall A\, B \quad [\mathsf{once}(A\ \&\ B) \Rightarrow \mathsf{once}(A) \wedge \mathsf{once}(B)]$

The axioms for once have as a consequence that connect is symmetric with respect to the first and second argument.

(T12) $\forall\ x\, y\, P \quad [\mathsf{connect}(x, y, P) \Leftrightarrow \mathsf{connect}(y, x, P)]$

On the other hand, transitivity of immediate connections is not deducible. l_1 may wait for l_4 at station S_3 while l_2 leaves.

Trajectories are meant to subsume the positions an object passes through in the course of motion. Therefore, any position that is occupied by the object at some time belongs to its trajectory and vice versa (A5). That trajectories are oriented curves is expressed by axiom (A6), assuming that the sorted logic accounts for the sorting of the variables. The order on the trajectories represents the temporal order. Consequently, an additional principle (A7) has to account for the interaction of simultaneity and the direction on different oriented curves. In addition, all objects under consideration have to behave homogeneously with respect to time, i.e., all such entities are permanently localized. Since we cannot access temporal indices directly, this is expressed by assuming that if some time-dependent expression is true at some time, then every object is localized at some time when this expression is true (A8).

(A5) $\forall x\, P \quad [\mathrm{once}(\mathrm{at}(x, P)) \Leftrightarrow P\ \iota\ \mathrm{t}(x)]$

(A6) $\forall x\, \exists o \quad [\mathrm{t}(x) = o]$

(A7) $\forall P_1\, P_2\, Q_1\, Q_2\, x\, y \quad [\mathrm{once}(\mathrm{at}(x, P_1)\ \&\ \mathrm{at}(y, P_2)) \wedge \mathrm{once}(\mathrm{at}(x, Q_1)\ \&\ \mathrm{at}(y, Q_2)) \Rightarrow \neg(\prec(\mathrm{t}(x), P_1, Q_1) \wedge \prec(\mathrm{t}(y), Q_2, P_2))]$

(A8) $\forall A \quad [\mathrm{once}(A) \Rightarrow \forall x\, \exists P\, [\mathrm{once}(A\ \&\ \mathrm{at}(x, P))]]$

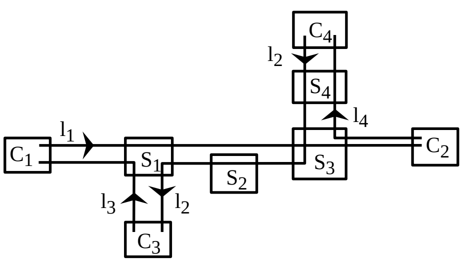

Figure 1. Layout of a simple train system

Valid Inferences about Meeting of Objects

Based on the axioms several theorems can be derived. The ones presented correspond to common-sense knowledge about the possibilities of meeting according to the purely geometrical properties of the trajectories.

The application of (A5) yields that meeting is only possible, if the trajectories have a position in common.

(T13) $\forall P\, x\, y \quad [\mathrm{connect}(x, y, P) \Rightarrow P\ \iota\ \mathrm{t}(x) \wedge P\ \iota\ \mathrm{t}(y)]$

I.e., two trains can have a connection only if there is a station or city, which belongs to the trajectory of both trains.

A simple transformation of (A7) is that a point can only be a meeting point of two moving objects if it is not the case that one object (y) already passed through it when the other object was still moving towards it (T14). Similarly, since trajectories are assumed to be oriented curves, two objects can meet at two points only if they are ordered in the same way on both trajectories (T15).

(T14) $\forall P\, x\, y \quad [\mathrm{connect}(x, y, P) \Rightarrow \forall Q_1\, Q_2\, [\prec(\mathrm{t}(y), Q_2, P) \wedge \prec(\mathrm{t}(x), P, Q_1) \Rightarrow \neg\mathrm{once}(\mathrm{at}(x, Q_1)\ \&\ \mathrm{at}(y, Q_2))]]$

(T15) $\forall P\, Q\, x\, y \quad [\mathrm{connect}(x, y, P) \wedge \mathrm{connect}(x, y, Q) \wedge \prec(\mathrm{t}(x), P, Q) \Rightarrow \prec(\mathrm{t}(y), P, Q)]$

The common-sense knowledge corresponding to theorem (T14) can successfully be used in reasoning about train connections.

$$\mathrm{connect}(l_1, l_2, S_1) \Rightarrow \neg\mathrm{connect}(l_1, l_2, S_3)$$

If l_1 and l_2 meet at S_1, then they cannot be at S_3 at the same time. Since any journey from C_1 to S_2 has to leave S_3 using l_2, (T14) corresponds to the knowledge that if l_1 and l_2 meet at S_1, then S_2 cannot be arrived anymore.

$$\mathrm{connect}(l_1, l_3, S_1) \wedge \mathrm{connect}(l_1, l_2, S_3) \Rightarrow \neg\exists S\, [\mathrm{connect}(l_2, l_3, S)]$$

Correspondingly there is no connection of the trains l_2 and l_3, if l_1 and l_2 meet at S_2 and l_2 and l_2 meet at S_2. I.e., the connection at S_1—relevant for the journey from C_4 to C_1—does not exist.

(A5), (A6) and (A7) yield that the objects under consideration cannot be at two different places simultaneously.

(T16) $\forall P\, Q\, x \quad [\mathrm{once}(\mathrm{at}(x, P)\ \&\ \mathrm{at}(x, Q)) \Rightarrow P = Q]$

All axioms interact in the proof of the theorem that if an object is in a position before and after another object moves, then it stays in this position while the other one moves. This also shows that the formalization can cope with standstill of objects and waiting trains. Of course, if the place where x stays is on the trajectory of y in this move, then the two objects meet at that place.

(T17) $\forall P\, Q_1\, Q_2\, Q_3\, x\, y\, [\mathrm{once}(\mathrm{at}(x, P)\ \&\ \mathrm{at}(y, Q_1)) \wedge \mathrm{once}(\mathrm{at}(x, P)\ \&\ \mathrm{at}(y, Q_3)) \wedge \beta_o(\mathrm{t}(x), Q_1, Q_2, Q_3) \Rightarrow \mathrm{once}(\mathrm{at}(x, P)\ \&\ \mathrm{at}(y, Q_2))]$

Similarly, these axioms are sufficient to prove more complicated interactions of trajectories as exemplified by the last theorem. It says that if three objects meet in pairs such that the meeting place of x and z precedes that of x and y on the trajectory of x and the meeting place of x and y precedes that of y and z on $\mathrm{t}(y)$, then the meeting place of y and z does not precede that of x and z on $\mathrm{t}(z)$. This result can be generalized for any number of trajectories that meet in pairs.

(T18) $\forall P\, Q\, R\, x\, y\, z\ [\mathrm{connect}(x, z, P) \wedge \mathrm{connect}(x, y, Q) \wedge \mathrm{connect}(y, z, R) \wedge \prec(\mathrm{t}(x), P, Q) \wedge \prec(\mathrm{t}(y), Q, R) \Rightarrow \neg\prec(\mathrm{t}(z), R, P)]$

This is also applicable in the scenario given.

$$\mathrm{connect}(l_2, l_4, S_4) \wedge \mathrm{connect}(l_1, l_2, S_1) \Rightarrow \neg\exists S\, [\mathrm{connect}(l_1, l_4, S)]$$

If travelers want to know, whether it is possible to get an immediate connection from C_1 to C_4, they have to reason about the possible connections between l_1 and l_4. The routing of l_1 and l_4 gives C_2 and S_3 as candidates but theorem (T18) rules out these possibilities due to the ordering of the connection points of l_2 with the two trains.

Some final remarks to the train scenario: The concept of immediate connection is based on simultaneous collocation. Since the ordering is coded in the oriented curves, reasoning about connections that incorporate the waiting of the traveler at the stations can also be modeled. However, measures of durations and exact dating is not in the scope of this formalization that focuses on a qualitative basis of reasoning about spatial change.

Conclusion

In this paper we presented a geometric formalization of curves and oriented curves as general representatives for linear structures that can be embedded in a higher-dimensional context. While several approaches of formalizing linear or serial orders exist especially in the area of the formalization of time (cf. Huntington 1938, Hamblin 1971, Needham 1981, van Benthem 1983, Allen & Hayes 1985, Eschenbach & Heydrich 1995), these approaches do not account for the possibility of embedding them in a more comprehensive structure or the option of coping with several such orderings at the same time. In the approach presented here curves and oriented curves are representatives on the same levels as the other geometric entities (points, lines, etc.) One consequence of this method is that they are linear in a strict sense, since they do not include cycles and do not branch, although they do not need to be straight.

As an application of oriented curves we considered the problem of possible meetings of objects that move through space. Assuming that their trajectories can be modeled as oriented curves, we found that the representation of time can be restricted to the representation of certain expressions that depend on time, i.e., can have different truth values at different times, and can be true simultaneously. Based on means of modeling states developed in AI, this can be done without using expressions that explicitly refer to points or periods of time. The direction of time is implicitly represented in the direction of the trajectories, which can at the same time be seen as representing the structural influence of time on space.

Of course, the graph-theoretic nature of the problems and results presented in the previous section is evident. This stems from at least two facts. On the one hand, graph theory developed from problems of formalizing trajectories as the Königsberg bridge problem. On the other hand, the two types of curves can serve as geometric interpretations of the two types of edges of graphs and as formalizations of the graphical elements (lines and arrows) which are commonly used to visualize graphs.

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