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# Representation and Reasoning with Uncertain Temporal Relations 

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#### Abstract

It is often necessary to deal with uncertain knowledge and the area of temporal reasoning is not an exception. Basically uncertainty includes two main aspects: inexactness and inconsistency. We suggest representation which includes explicit probability values of the consistent and the inconsistent parts of a temporal relation. The probability of the consistent part of the relation is divided between the basic temporal relations, i.e. "く" (before), " $=$ " (at the same time), and " $>$ " (after). The inconsistent part of the relation has one probability value which is divided between the three basic relations presenting the percentage values of their support among knowledge sources. Both the probabilities and the percentage values are used in our reasoning mechanism, which consists of three operations: inversion, composition, and addition. These can be used to derive the probability and percentage values for a relation between any two temporal points.


## 1. Introduction

The need to represent and reason using temporal knowledge arises in a wide range of disciplines, including computer science, philosophy, psychology, and linguistics. During the early 80s there were attempts to provide general theories of time and action as McDermott's temporal logic (McDermott 1982), Allen's theory of action and time (Allen 1984), and Vilain's theory of time (Vilain 1982).

These established the two main temporal ontological primitives (point and interval), made the initial proposals on representational issues and reasoning algorithms, pointed out the general problems as reasoning by default, the interaction of actions, and the use of a temporal reasoner in application, and convinced that more basic machinery needs to be built before defining a general theory of time (Vila 1994).

It is often necessary to deal with uncertain knowledge and the area of temporal reasoning is not an exception. Basically uncertainty includes two main aspects:

[^0]inexactness and inconsistency. Most approaches to temporal reasoning suppose that an inexact temporal relation is a disjunction of two or more basic relations and this seems to be a weakness in decision support systems and other systems where some knowledge about inconsistent relations exist and might be useful.

Inconsistencies may occur, for example, due to a source of information that is not fully reliable. Also when opinions of several experts exist these may be inconsistent. It is common to consider knowledge consistent if it supposes the absence of contradictions and inconsistent when knowledge contains contradictions (Roos 1992). Inconsistency in information is the norm, and it needs to be formalized (Gabbay and Hunter 1991). There is a difference between artificial and human intelligent behavior: a human often resolves inconsistencies not by "restoring" consistency as is so often done in AI, but by applying rules telling one how to act when an inconsistency arises (Gabbay and Hunter 1991).

Our approach has similarities with the DempsterShafer theory (Shafer 1976), but it is different in that here we do not divide probabilities for all the subsets, but only distinguish the probability for basic elements and for the whole relation.

Our point of view is that inconsistency is normal and a "good" thing. In this paper we take temporal points as ontological primitives and consider the relations between them. We propose a representation mechanism for uncertain temporal relations including the ability to represent both inexactness and inconsistency, and reason with them. The representation will be described in Section 2 , and the reasoning mechanism which uses the proposed representation will be presented in Section 3. Section 4 provides a conclusion and suggests some further research topics.

## 2. Representation of Uncertain Relations

This section deals with the representation of uncertain relations between temporal points. First we will introduce some basic definitions used throughout the paper.

Definition 1．Basic relations between temporal points are：＂＜＂（before），＂＝＂（at the same time），and＂＞＂（after）． We will call them exact temporal relations．Disjunctions of these relations，＂$\leq "$（＂く＂or＂＝＂），＂$\geq$＂（＂＞＂or＂＝＂），＂$\neq "$ （＂＜＂or＂＞＂）and＂？＂（＂く＂or＂＝＂or＂＞＂）are called inexact temporal relations．

Definition 2．An inconsistent temporal relation is a conjunction of two or more basic temporal relations，and it inherits the temporal meanings of all the basic temporal relations included．Each conjunction is presented by a triple $\left[d^{<}, d^{=}, d^{\prime}\right]$ ，where the values $d^{く}, d^{=}, d^{>}$give the percentages of each basic relation within the inconsistent relation，and $\mathrm{d}^{<}+\mathrm{d}^{=}+\mathrm{d}^{>}=1$ ．

An inconsistent relation includes conflicting meanings of a relation．For example，if an expert says：＂This relation is＂＜＂，and another says：＂This relation is＂$>$＂．Then their common opinion can be presented by the inconsistent relation＂＜and＞＂．In our presentation inconsistent knowledge is stored without trying to restore consistency．

Definition 3．The value of the exactness of a relation （basic or inconsistent）between two temporal points is the probability that exactly this relation holds between them． There are three basic relations（＂＜＂，＂＞＂，＂＝＂）and one inconsistent relation between two temporal points．We present the values of exactness by a 4 －tuple（ $\left.\mathrm{e}^{<}, \mathrm{e}^{=}, \mathrm{e}^{>}, \mathrm{e}^{\mathrm{i}}\right)$ ， where $e^{<}, e^{=}, e^{>}, e^{i}$ are the values of the exactness of the relations＂＜＂，＂＝＂，＂＞＂，and the inconsistent relation， respectively and $e^{i}=1-\left(e^{<}+e^{=}+e^{>}\right)$．Thus we give to the inconsistent relation all the probability that cannot be given to the basic relations．

We represent a relation between two temporal points a and $\mathbf{b}$ as a vector $\mathbf{L}_{\mathbf{a}, \mathbf{b}}$ which is composed of a 4－tuple including the values of exactness followed by a 3－tuple including the values of percentages within the inconsistent part：
$\mathbf{L}_{\mathbf{a}, \mathbf{b}}=\left(\mathrm{e}^{<}, \mathrm{e}^{=}, \mathrm{e}^{>}, \mathrm{e}^{\mathrm{i}}\left[\mathrm{d}^{<}, \mathrm{d}^{=}, \mathrm{d}^{>}\right]\right)$，where $\mathrm{e}^{<}+\mathrm{e}^{=}+\mathrm{e}^{>}+\mathrm{e}^{\mathrm{i}}=1$ and $\mathrm{d}^{<}+\mathrm{d}^{=}+\mathrm{d}^{>}=1$ ，and the values $\mathrm{d}^{<}, \mathrm{d}^{=}, \mathrm{d}^{>}$（can be nonzero）only in the case when $\mathrm{e}^{\mathrm{i}} \neq 0$ ．
Let us consider three illustrating examples．
Example 1．Let the knowledge source 1 give the relation＂$\leq$＂between two temporal points $\mathbf{a}$ and $\mathbf{b}$ ．Then

$$
\mathbf{L}_{\mathbf{a}, \mathbf{b}}=(0.5,0.5,0,0)
$$

In this example there is only one source of information and the relation provided is inexact．Thus no inconsistency is included and the triple［ $\left.\mathrm{d}^{<}, \mathrm{d}^{=}, \mathrm{d}^{>}\right]$is omitted．

Example 2．Let the knowledge sources 1，2，and 3 correspondingly give relations＂＜＂，＂$="$＂，and＂＞＂about the relation between the two temporal points $\mathbf{a}$ and $\mathbf{b}$ ．Then

$$
\mathbf{L}_{\mathbf{a}, \mathbf{b}}=(0,0,0,1[1 / 3,1 / 3,1 / 3]) .
$$

In this example the relation is totally inconsistent and equally distributed within the inconsistent part．

Example 3．Let the knowledge sources 1 and 2 correspondingly give the relations＂$\leq$＂and＂$\geq$＂about the relation between two temporal points $\mathbf{a}$ and $\mathbf{b}$ ．The vector representation of the relation given by knowledge source 1 is $(0.5,0.5,0,0)$ ，and the corresponding vector of knowledge source 2 is $(0,0.5,0.5,0)$ ．In order to be able to derive the common knowledge a reasoning mechanism is needed．In the next section we define three operations：inversion， composition，and addition for the above relations and after that we finish example 3.

## 3．Reasoning with Uncertain Relations

In this section we propose a reasoning mechanism for the representation presented in Section 2．We define three operations：inversion，composition，and addition．

Inversion and composition，discussed in the first subsection，are classical in all the mechanisms of temporal reasoning．We use addition instead of the commonly used intersection operation．The difference between these two is the way of handling inconsistency．The usual intersection operation finds out the common part in the intersected relations，cleaning inconsistency out．Our addition operation is intended to exploit all the information using the inconsistent part of the representation vector when a contradiction arises．

## 3．1．Inversion and Composition

Definition 4．Let there be a temporal relation $\mathbf{L}$ between two temporal points $\mathbf{a}$ and $\mathbf{b}$ ．A predicate $P$ is defined as follows：
$\mathbf{P}(\mathbf{a}, \mathbf{L}, \mathbf{b})=\left\{\begin{array}{l}\text { true，if the relation } \mathbf{L} \text { holds between } \mathbf{a} \text { and } \mathbf{b} \\ \text { false，otherwise }\end{array}\right.$
Definition 5．The inversion operation denoted by～is defined by the following equation：
$\mathrm{P}\left(\mathbf{a}, \mathbf{L}_{\mathbf{a}, \mathbf{b}}, \mathbf{b}\right) \Leftrightarrow \mathrm{P}\left(\mathbf{b}, \tilde{\mathbf{L}}_{\mathbf{a}, \mathbf{b}}, \mathbf{a}\right)$ ，where $\mathbf{a}, \mathbf{b}$ are temporal points， $\mathbf{L}_{\mathrm{a}, \mathrm{b}}=\left(\mathrm{e}_{1}^{<}, \mathrm{e}_{1}^{=}, \mathrm{e}_{1}^{>}, \mathrm{e}_{1}^{\mathrm{i}}\left[\mathrm{d}_{1}^{<}, \mathrm{d}_{1}^{=}, \mathrm{d}_{1}^{>}\right]\right)$is the relation between the points $\mathbf{a}$ and $\mathbf{b}$ ，and $\tilde{\mathrm{L}}_{\mathrm{a}, \mathrm{b}}$ is the inverted relation represented by the relation $\mathbf{L}_{b, a}=\left(e_{r}^{<}, e_{r}^{=}, e_{r}^{>}, e_{r}^{i}\left[d_{r}^{<}, d_{r}^{=}, d_{r}^{>}\right]\right)$，where $e_{r}^{<}=e_{1}^{>}, e_{r}^{=}=e_{1}^{=}$， $\mathrm{e}_{\mathrm{r}}^{>}=\mathrm{e}_{1}^{<}, \mathrm{e}_{\mathrm{r}}^{\mathrm{i}}=\mathrm{e}_{1}^{\mathrm{i}}, \mathrm{d}_{\mathrm{r}}^{<}=\mathrm{d}_{1}^{>}, \mathrm{d}_{\mathrm{r}}^{=}=\mathrm{d}_{1}^{=}, \mathrm{d}_{\mathrm{r}}^{>}=\mathrm{d}_{1}^{<}$
The inversion operation is presented in Figure 1.


Figure 1. Inversion operation
The composition operation is presented in Figure 3. A temporal relation $\mathbf{L}_{\mathbf{a}, \mathbf{b}}$ exists between the temporal points $\mathbf{a}$ and $\mathbf{b}$, and a temporal relation $\mathbf{L}_{\mathbf{b}, \mathbf{c}}$ between the temporal points $\mathbf{b}$ and $\mathbf{c}$. The temporal relation between the temporal points a and $\mathbf{c}$ can be derived using the composition operation (*). This temporal relation is derived using the composition table proposed by Vilain and Kautz (Vilain and Kautz 1986) represented in Figure 2. A row of this table corresponds to a relation $\mathbf{L}_{\mathbf{a}, \mathbf{b}}$, a column corresponds to a relation $\mathbf{L}_{\mathbf{b}, \mathbf{c}}$, and the cell at the intersection of the row and column includes the result of the composition operation relation $\mathbf{L}_{\mathrm{a}, \mathrm{c}}$. Since we use the basic relations we need only the shaded areas of the composition table.

| $*$ | $=$ | $<$ | $\leq$ | $>$ | $\geq$ | $\neq$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $=$ | $=$ | $<$ | $\leq$ | $>$ | $\geq$ | $\neq$ |
| $<$ | $<$ | $<$ | $<$ | $?$ | $?$ | $?$ |
| $\leq$ | $\leq$ | $<$ | $\leq$ | $?$ | $?$ | $?$ |
| $>$ | $>$ | $?$ | $?$ | $>$ | $>$ | $?$ |
| $\geq$ | $\geq$ | $?$ | $?$ | $>$ | $\geq$ | $?$ |
| $\neq$ | $\neq$ | $?$ | $?$ | $?$ | $?$ | $?$ |

Figure 2. Composition table (Vilain and Kautz 1986)


Figure 3. Composition operation
We define our composition table in Figure 4. The cells of the first row define the vector presenting temporal relation $\mathbf{L}_{\mathrm{a}, \mathbf{b}}$, the cells of the first column define the vector presenting temporal relation $\mathbf{L}_{\mathbf{b}, \mathbf{c}}$, and the contents of a cell at the intersection of a row and a column defines which values of the resulting vector $\mathbf{L}_{a, c}$ are affected.

| * | $\mathrm{e}_{1}^{<}$ | $\mathrm{e}_{1}^{=}$ | $\mathrm{e}_{1}{ }^{\text {P }}$ | $\mathrm{d}_{1}$ | $\mathrm{d}_{1}=$ | $\mathrm{d}_{1}{ }^{\text {d }}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{e}_{2}$ | $\mathrm{e}_{\mathrm{r}}{ }^{<}$ | $\mathrm{e}_{\mathrm{r}}^{<}$ | $\begin{gathered} \hline \mathrm{e}_{\mathrm{r}}^{<} \mathrm{e}_{\mathrm{r}}^{=} \\ \mathrm{e}_{\mathrm{r}}^{>} \end{gathered}$ | $\mathrm{d}_{\mathrm{r}}^{<}$ | $\mathrm{d}_{\mathrm{r}}^{<}$ | $\begin{gathered} \mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{=} \\ \mathrm{d}_{\mathrm{r}}^{>} \end{gathered}$ |
| $\mathrm{e}_{2}=$ | $\mathrm{e}_{\mathrm{r}}$ | $\mathrm{e}_{\mathrm{r}}^{=}$ | $\mathrm{e}_{\mathrm{r}}^{\text {P }}$ | $\mathrm{d}_{\mathrm{r}}{ }^{\text {c }}$ | $\mathrm{d}_{\mathrm{r}}{ }^{-}$ | $\mathrm{d}_{\mathrm{r}}{ }^{\text {P }}$ |
| $\mathrm{e}_{2}{ }^{\text {a }}$ | $\begin{gathered} \mathrm{e}_{\mathrm{r}}^{<} \mathrm{e}_{\mathrm{r}}^{=} \\ \mathrm{e}_{\mathrm{r}}^{>} \\ \hline \end{gathered}$ | $\mathrm{e}_{\mathrm{r}}^{\text {P }}$ | $\mathrm{e}_{\mathrm{r}}^{\text {P }}$ | $\begin{gathered} \mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{=} \\ \mathrm{d}_{\mathrm{r}}^{>} \end{gathered}$ | $\mathrm{d}_{\mathrm{r}}$ | $\mathrm{d}_{\mathrm{r}}$ |
| $\mathrm{d}_{2}$ | $\mathrm{d}_{\mathrm{r}}^{<}$ | $\mathrm{d}_{\mathrm{r}}$ | $\begin{gathered} \hline \mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{=} \\ \mathrm{d}_{\mathrm{r}}^{>} \end{gathered}$ | $\mathrm{d}_{\mathrm{r}}^{<}$ | $\mathrm{d}_{\mathrm{r}}$ | $\begin{gathered} \mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{=} \\ \mathrm{d}_{\mathrm{r}}^{>} \end{gathered}$ |
| $\mathrm{d}_{2}{ }^{\text {a }}$ | $\mathrm{d}_{\mathrm{r}}{ }^{\text {r }}$ | $\mathrm{d}_{\mathrm{r}}^{=}$ | $\mathrm{d}_{\mathrm{r}}{ }^{\text {P }}$ | $\mathrm{d}_{\mathrm{r}}^{<}$ | $\mathrm{d}_{\mathrm{r}}{ }^{-}$ | $\mathrm{d}_{\mathrm{r}}^{\text {P }}$ |
| $\mathrm{d}_{2}$ | $\begin{gathered} \mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{=} \\ \mathrm{d}_{\mathrm{r}}^{>} \\ \hline \end{gathered}$ | $\mathrm{d}_{\mathrm{r}}{ }^{\text {P }}$ | $\mathrm{d}_{\mathrm{r}}^{\text {P }}$ | $\begin{gathered} \mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{=} \\ \mathrm{d}_{\mathrm{r}}^{>} \end{gathered}$ | $\mathrm{d}_{\mathrm{r}}$ | $\mathrm{d}_{\mathrm{r}}^{\text {P }}$ |

Figure 4. Composition table
Let us, for example, analyze support for the $\mathrm{e}_{\mathrm{r}}^{<}$value in the relation $\mathbf{L}_{\mathbf{a}, \mathbf{c}}$. It can be seen from Figure 4, that the $e_{r}^{<}$value of the relation $\mathbf{L}_{\mathbf{a}, \mathbf{c}}$ is fully supported by the three combinations of the values in $\mathbf{L}_{\mathbf{a}, \mathbf{b}}$ and $\mathbf{L}_{\mathbf{b}, \mathbf{c}}: \mathrm{e}_{1}^{<} \& \mathrm{e}_{2}^{<}$, $\mathrm{e}_{1}^{<} \& \mathrm{e}_{2}^{=}$, and $\mathrm{e}_{1}^{=} \& \mathrm{e}_{2}^{<}$. The two combination: $\mathrm{e}_{1}^{<} \& \mathrm{e}_{2}^{>}$and $\mathrm{e}_{1}^{>} \& \mathrm{e}_{2}^{<}$partly support the value $\mathrm{e}_{\mathrm{r}}^{<}$of the composition result. Since three values exist which these two combinations partly support we divide the support equally between them. Thus the value $\mathrm{e}_{\mathrm{r}}^{<}$is equal to the sum of the supporting combinations:

$$
e_{r}^{<}=e_{1}^{<} \cdot e_{2}^{<}+e_{1}^{<} \cdot e_{2}^{=}+e_{1}^{=} \cdot e_{2}^{<}+e_{1}^{<} \cdot e_{2}^{>} \cdot \frac{1}{3}+e_{1}^{>} \cdot e_{2}^{<} \cdot \frac{1}{3} .
$$

Analogously $\mathrm{e}_{\mathrm{r}}^{=}$and $\mathrm{e}_{\mathrm{r}}^{>}$are:

$$
\begin{gathered}
\mathrm{e}_{\mathrm{r}}^{=}=\mathrm{e}_{1}^{=} \cdot \mathrm{e}_{2}^{=}+\mathrm{e}_{1}^{<} \cdot \mathrm{e}_{2}^{>} \cdot \frac{1}{3}+\mathrm{e}_{1}^{>} \cdot \mathrm{e}_{2}^{<} \cdot \frac{1}{3}, \\
\mathrm{e}_{\mathrm{r}}^{>}=\mathrm{e}_{1}^{>} \cdot \mathrm{e}_{2}^{>}+\mathrm{e}_{1}^{=} \cdot \mathrm{e}_{2}^{>}+\mathrm{e}_{1}^{>} \cdot \mathrm{e}_{2}^{=}+\mathrm{e}_{1}^{<} \cdot \mathrm{e}_{2}^{>} \cdot \frac{1}{3}+\mathrm{e}_{1}^{>} \cdot \mathrm{e}_{2}^{<} \cdot \frac{1}{3} .
\end{gathered}
$$

The percentage values $d_{r}^{<}, d_{r}^{=}$, and $d_{r}^{>}$are obtained analogously. Notice, that when any value $\mathrm{d}_{1}$ or $\mathrm{d}_{2}$ is taken it needs to be multiplied by the corresponding value of exactness $e_{1}^{i}$ or $e_{2}^{i}$. After the values $d_{r}^{<}, d_{r}^{=}$, and $d_{r}^{>}$are calculated analogously accordingly to the above formulas they need to be normalized to represent the percentage values inside the inconsistent part of $\mathbf{L}_{\mathbf{a}, \mathbf{c}}$.

Definition 6. Let us define operation $\otimes$ between two vectors $\overline{\mathrm{v}}=\left[\mathrm{v}_{1}, \mathrm{v}_{2}, \mathrm{v}_{3}\right]$ and $\overline{\mathrm{u}}=\left[\mathrm{u}_{1}, \mathrm{u}_{2}, \mathrm{u}_{3}\right]$ so that the result of the operation is also a three valued vector $\overline{\mathrm{t}}=\left[\mathrm{t}_{1}, \mathrm{t}_{2}, \mathrm{t}_{3}\right]$, where

$$
\mathrm{t}_{1}=\mathrm{v}_{1} \cdot \mathrm{u}_{1}+\mathrm{v}_{1} \cdot \mathrm{u}_{2}+\mathrm{v}_{2} \cdot \mathrm{u}_{1}+\frac{1}{3}\left(\mathrm{v}_{1} \cdot \mathrm{u}_{3}+\mathrm{v}_{3} \cdot \mathrm{u}_{1}\right)
$$

$$
\begin{gathered}
\mathrm{t}_{2}=\mathrm{v}_{2} \cdot \mathrm{u}_{2}+\frac{1}{3}\left(\mathrm{v}_{1} \cdot \mathrm{u}_{3}+\mathrm{v}_{3} \cdot \mathrm{u}_{1}\right) \text {, and } \\
\mathrm{t}_{3}=\mathrm{v}_{3} \cdot \mathrm{u}_{3}+\mathrm{v}_{3} \cdot \mathrm{u}_{2}+\mathrm{v}_{2} \cdot \mathrm{u}_{3}+\frac{1}{3}\left(\mathrm{v}_{1} \cdot \mathrm{u}_{3}+\mathrm{v}_{3} \cdot \mathrm{u}_{1}\right) \cdot
\end{gathered}
$$

Definition 7. Let there be three temporal points a,b, and $\mathbf{c}$ and two temporal relations
$\mathbf{L}_{\mathbf{a}, \mathbf{b}}=\left(\mathrm{e}_{1}^{<}, \mathrm{e}_{1}^{=}, \mathrm{e}_{1}^{>}, \mathrm{e}_{1}^{\mathrm{i}}\left[\mathrm{d}_{1}^{<}, \mathrm{d}_{1}^{=}, \mathrm{d}_{1}^{>}\right]\right)$and
$\mathbf{L}_{\mathbf{b}, \mathbf{c}}=\left(\mathrm{e}_{2}^{<}, \mathrm{e}_{2}^{=}, \mathrm{e}_{2}^{>}, \mathrm{e}_{2}^{\mathrm{i}}\left[\mathrm{d}_{2}^{<}, \mathrm{d}_{2}^{=}, \mathrm{d}_{2}^{>}\right]\right)$.
The composition operation $\left({ }^{*}\right)$ is defined by the following equation:

$$
P\left(\mathbf{a}, \mathbf{L}_{\mathbf{a}, \mathbf{b}}, \mathbf{b}\right) \wedge \mathrm{P}\left(\mathbf{b}, \mathbf{L}_{\mathbf{b}, \mathbf{c}}, \mathbf{c}\right) \Leftrightarrow \mathrm{P}\left(\mathbf{a}, \mathbf{L}_{\mathbf{a}, \mathbf{c}}, \mathbf{c}\right)
$$

where $\mathbf{L}_{\mathbf{a}, \mathbf{c}}=\mathbf{L}_{\mathbf{a}, \mathbf{b}} * \mathbf{L}_{\mathbf{b}, \mathbf{c}}$ and $\mathbf{L}_{\mathbf{a}, \mathbf{c}}=\left(\mathrm{e}_{\mathrm{r}}^{<}, \mathrm{e}_{\mathrm{r}}^{=}, \mathrm{e}_{\mathrm{r}}^{>}, \mathrm{e}_{\mathrm{r}}^{\mathrm{i}}\left[\mathrm{d}_{\mathrm{r}}^{<}, \mathrm{d}_{\mathrm{r}}^{=}, \mathrm{d}_{\mathrm{r}}^{>}\right]\right)$.
The values of $\mathbf{L}_{\mathbf{a}, \mathbf{c}}$ are defined as follows:
$\bar{e}_{\mathrm{r}}=\overline{\mathrm{e}}_{1} \otimes \bar{e}_{2}, \mathrm{e}_{\mathrm{r}}^{\mathrm{i}}=1-\left(\mathrm{e}_{\mathrm{r}}^{<}+\mathrm{e}_{\mathrm{r}}^{=}+\mathrm{e}_{\mathrm{r}}^{>}\right)$,
$\mathrm{d}_{\mathrm{r}}^{<}=\left(\mathrm{f}^{<}+\mathrm{g}^{<}+\mathrm{h}^{<}\right) /\left(\mathrm{j}^{<}+\mathrm{j}^{=}+\mathrm{j}^{>}\right)$,
$d_{r}^{=}=\left(f^{=}+g^{=}+h^{=}\right) /\left(j^{<}+j^{=}+j^{>}\right)$,
$d_{r}^{>}=\left(f^{>}+g^{>}+h^{>}\right) /\left(j^{<}+j^{=}+j^{>}\right)$, where
$\overline{\mathrm{f}}=\overline{\mathrm{e}}_{1} \otimes\left(\mathrm{e}_{2}^{\mathrm{i}} \cdot \overline{\mathrm{d}}_{2}\right), \overline{\mathrm{g}}=\left(\mathrm{e}_{1}^{\mathrm{i}} \cdot \overline{\mathrm{d}}_{1}\right) \otimes \overline{\mathrm{e}}_{2}$, and
$\overline{\mathrm{h}}=\left(\mathrm{e}_{1}^{\mathrm{i}} \cdot \overline{\mathrm{d}}_{1}\right) \otimes\left(\mathrm{e}_{2}^{\mathrm{i}} \cdot \overline{\mathrm{d}}_{2}\right)$, and $\overline{\mathrm{j}}=(\overline{\mathrm{f}}+\overline{\mathrm{g}}+\overline{\mathrm{h}})$.
Example 4. Let there be two relations $\mathbf{L}_{\mathbf{a}, \mathbf{b}}=(1,0,0,0)$ and $\mathbf{L}_{\mathbf{b}, \mathbf{c}}=(0.5,0.5,0,0)$. The composition of them accordingly to Definition 7 results to the relation $\mathbf{L}_{\mathbf{a}, \mathbf{c}}=\mathbf{L}_{\mathbf{a}, \mathbf{b}} * \mathbf{L}_{\mathbf{b}, \mathbf{c}}=(1,0,0,0)$.

Example 5. Let there be two relations $\mathbf{L}_{\mathbf{a}, \mathbf{b}}=(0.5,0,0,0.5[0.3,0.2,0.5])$ and
$\mathbf{L}_{\mathbf{b}, \mathbf{c}}=(0,0.5,0,0.5[0.6,0.1,0.3])$. Then the composition of them results to the relation
$\mathbf{L}_{\mathbf{a}, \mathbf{c}}=\mathbf{L}_{\mathbf{a}, \mathbf{b}} * \mathbf{L}_{\mathbf{b}, \mathbf{c}}=(0.25,0,0,0.75[0.524,0.144,0.332])$.

### 3.2. Addition

In this subsection we consider the addition operation, which is described in Figure 5.


Figure 5. Addition operation

We define the addition table in Figure 6. The cells of the first row define the vector representing temporal relation $\mathbf{L}_{\mathbf{1}}$, the cells of the first column define the vector representing temporal relation $\mathbf{L}_{\mathbf{2}}$, and the contents of the cell at the intersection of a row and a column defines which values of the resulting vector $\mathbf{L}_{\mathbf{a}, \mathbf{b}}$ are affected.

| + | $\mathrm{e}_{1}^{<}$ | $\mathrm{e}_{1}^{=}$ | $\mathrm{e}_{1}^{>}$ | $\mathrm{d}_{1}^{<}$ | $\mathrm{d}_{1}^{=}$ | $\mathrm{d}_{1}^{>}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{e}_{2}^{<}$ | $\mathrm{e}_{\mathrm{r}}^{<}$ | $\mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{=}$ | $\mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{>}$ | $\mathrm{d}_{\mathrm{r}}^{<}$ | $\mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{=}$ | $\mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{>}$ |
| $\mathrm{e}_{2}^{=}$ | $\mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{=}$ | $\mathrm{e}_{\mathrm{r}}^{-}$ | $\mathrm{d}_{\mathrm{r}}^{>} \mathrm{d}_{\mathrm{r}}^{=}$ | $\mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{=}$ | $\mathrm{d}_{\mathrm{r}}^{=}$ | $\mathrm{d}_{\mathrm{r}}^{>} \mathrm{d}_{\mathrm{r}}^{=}$ |
| $\mathrm{e}_{2}^{>}$ | $\mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{>}$ | $\mathrm{d}_{\mathrm{r}}^{>} \mathrm{d}_{\mathrm{r}}^{=}$ | $\mathrm{e}_{\mathrm{r}}^{>}$ | $\mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{>}$ | $\mathrm{d}_{\mathrm{r}}^{>} \mathrm{d}_{\mathrm{r}}^{=}$ | $\mathrm{d}_{\mathrm{r}}^{>}$ |
| $\mathrm{d}_{2}^{<}$ | $\mathrm{d}_{\mathrm{r}}^{<}$ | $\mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{=}$ | $\mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{>}$ | $\mathrm{d}_{\mathrm{r}}^{<}$ | $\mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{=}$ | $\mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{>}$ |
| $\mathrm{d}_{2}^{=}$ | $\mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{=}$ | $\mathrm{d}_{\mathrm{r}}^{=}$ | $\mathrm{d}_{\mathrm{r}}^{>} \mathrm{d}_{\mathrm{r}}^{=}$ | $\mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{=}$ | $\mathrm{d}_{\mathrm{r}}^{=}$ | $\mathrm{d}_{\mathrm{r}}^{>} \mathrm{d}_{\mathrm{r}}^{=}$ |
| $\mathrm{d}_{2}^{>}$ | $\mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{>}$ | $\mathrm{d}_{\mathrm{r}}^{>} \mathrm{d}_{\mathrm{r}}^{=}$ | $\mathrm{d}_{\mathrm{r}}^{>}$ | $\mathrm{d}_{\mathrm{r}}^{<} \mathrm{d}_{\mathrm{r}}^{>}$ | $\mathrm{d}_{\mathrm{r}}^{>} \mathrm{d}_{\mathrm{r}}^{=}$ | $\mathrm{d}_{\mathrm{r}}^{>}$ |

Figure 6. Addition table
Let us, for example, analyze support for the $\mathrm{e}_{\mathrm{r}}^{<}$ relation in $\mathbf{L}_{\mathbf{a}, \mathbf{b}}$. The value $\mathrm{e}_{\mathrm{r}}^{<}$is fully supported only by the values $\mathrm{e}_{1}^{<}$and $\mathrm{e}_{2}^{<}$. Since their support is at least as strong as their minimum value we define the value of $e_{r}^{<}$as equal to:

$$
\mathrm{e}_{\mathrm{r}}^{<}=\min \left(\mathrm{e}_{1}^{<}, \mathrm{e}_{2}^{<}\right) .
$$

Analogously $\mathrm{e}_{\mathrm{r}}^{=}$and $\mathrm{e}_{\mathrm{r}}^{>}$are:

$$
\mathrm{e}_{\mathrm{r}}^{=}=\min \left(\mathrm{e}_{1}^{=}, \mathrm{e}_{2}^{=}\right), \mathrm{e}_{\mathrm{r}}^{>}=\min \left(\mathrm{e}_{1}^{>}, \mathrm{e}_{2}^{>}\right) .
$$

Definition 8. Let us define operation $\oplus$ between two vectors $\overline{\mathrm{v}}=\left[\mathrm{v}_{1}, \mathrm{v}_{2}, \mathrm{v}_{3}\right]$ and $\overline{\mathrm{u}}=\left[\mathrm{u}_{1}, \mathrm{u}_{2}, \mathrm{u}_{3}\right]$ so that the result of the operation is also a three valued vector $\overline{\mathrm{t}}=\left[\mathrm{t}_{1}, \mathrm{t}_{2}, \mathrm{t}_{3}\right]$,
where
$\mathrm{t}_{1}=\mathrm{v}_{1} \cdot \mathrm{u}_{1}+\mathrm{v}_{1} \cdot\left(\frac{\mathrm{v}_{1} \cdot \mathrm{u}_{2}}{\mathrm{v}_{1}+\mathrm{u}_{2}}+\frac{\mathrm{v}_{1} \cdot \mathrm{u}_{3}}{\mathrm{v}_{1}+\mathrm{u}_{3}}\right)+\mathrm{u}_{1} \cdot\left(\frac{\mathrm{v}_{3} \cdot \mathrm{u}_{1}}{\mathrm{v}_{3}+\mathrm{u}_{1}}+\frac{\mathrm{v}_{2} \cdot \mathrm{u}_{1}}{\mathrm{v}_{2}+\mathrm{u}_{1}}\right)$,
$t_{2}=v_{2} \cdot u_{2}+v_{2} \cdot\left(\frac{v_{2} \cdot u_{1}}{v_{2}+u_{1}}+\frac{v_{2} \cdot u_{3}}{v_{2}+u_{3}}\right)+u_{2} \cdot\left(\frac{v_{1} \cdot u_{2}}{v_{1}+u_{2}}+\frac{v_{3} \cdot u_{2}}{v_{3}+u_{2}}\right)$, and
$t_{3}=v_{3} \cdot u_{3}+v_{3} \cdot\left(\frac{v_{3} \cdot u_{3}}{v_{3}+u_{3}}+\frac{v_{3} \cdot u_{2}}{v_{3}+u_{2}}\right)+u_{3} \cdot\left(\frac{v_{1} \cdot u_{3}}{v_{1}+u_{3}}+\frac{v_{2} \cdot u_{3}}{v_{2}+u_{3}}\right)$.
Definition 9. Let there be two temporal points a and b. and two temporal relations
$\mathbf{L}_{\mathbf{1}}=\left(\mathrm{e}_{1}^{<}, \mathrm{e}_{1}^{=}, \mathrm{e}_{1}^{>}, \mathrm{e}_{1}^{\mathrm{i}}\left[\mathrm{d}_{1}^{<}, \mathrm{d}_{1}^{=}, \mathrm{d}_{1}^{>}\right]\right)$and
$\mathbf{L}_{\mathbf{2}}=\left(\mathrm{e}_{2}^{<}, \mathrm{e}_{2}^{=}, \mathrm{e}_{2}^{>}, \mathrm{e}_{2}^{\mathrm{i}}\left[\mathrm{d}_{2}^{<}, \mathrm{d}_{2}^{=}, \mathrm{d}_{2}^{>}\right]\right)$.
The addition operation $(+)$ is defined by the following equation:

$$
P\left(\mathbf{a}, \mathbf{L}_{1}, \mathbf{b}\right) \wedge P\left(\mathbf{b}, \mathbf{L}_{2}, \mathbf{c}\right) \Leftrightarrow P\left(\mathbf{a}, \mathbf{L}_{\mathbf{a}, \mathbf{b}}, \mathbf{c}\right)
$$

where $\mathbf{L}_{\mathbf{a}, \mathbf{b}}=\mathbf{L}_{\mathbf{1}}+\mathbf{L}_{\mathbf{2}}$ and $\mathbf{L}_{\mathbf{a}, \mathbf{b}}=\left(\mathrm{e}_{\mathrm{r}}^{<}, \mathrm{e}_{\mathrm{r}}^{=}, \mathrm{e}_{\mathrm{r}}^{>}, \mathrm{e}_{\mathrm{r}}^{\mathrm{i}}\left[\mathrm{d}_{\mathrm{r}}^{<}, \mathrm{d}_{\mathrm{r}}^{=}, \mathrm{d}_{\mathrm{r}}^{>}\right]\right)$.
The values of $\mathbf{L}_{\mathbf{a}, \mathbf{b}}$ are defined as follows:
$\mathrm{e}_{\mathrm{r}}^{<}=\min \left(\mathrm{e}_{1}^{<}, \mathrm{e}_{2}^{<}\right), \mathrm{e}_{\mathrm{r}}^{=}=\min \left(\mathrm{e}_{1}^{=}, \mathrm{e}_{2}^{=}\right), \mathrm{e}_{\mathrm{r}}^{>}=\min \left(\mathrm{e}_{1}^{>}, \mathrm{e}_{2}^{>}\right)$,
$e_{r}^{i}=1-\left(e_{r}^{<}+e_{r}^{=}+e_{r}^{>}\right)$,
$\mathrm{d}_{\mathrm{r}}^{<}=\left(\mathrm{f}^{<}+\mathrm{g}^{<}+\mathrm{h}^{<}+\mathrm{k}^{<}-\mathrm{e}_{\mathrm{r}}^{<}\right) /\left(\mathrm{j}^{<}+\mathrm{j}^{=}+\mathrm{j}^{>}-\mathrm{e}_{\mathrm{r}}^{<}-\mathrm{e}_{\mathrm{r}}^{=}-\mathrm{e}_{\mathrm{r}}^{>}\right)$
$d_{r}^{=}=\left(f^{=}+g^{=}+h^{=}+k^{=}-e_{r}^{=}\right) /\left(j^{<}+j^{=}+j^{>}-e_{r}^{<}-e_{r}^{=}-e_{r}^{>}\right)$
$d_{r}^{>}=\left(f^{>}+g^{>}+h^{>}+k^{>}-e_{r}^{>}\right) /\left(j^{<}+j^{=}+j^{>}-e_{r}^{<}-e_{r}^{=}-e_{r}^{>}\right)$
where $\overline{\mathrm{f}}=\overline{\mathrm{e}}_{1} \oplus\left(\mathrm{e}_{2}^{\mathrm{i}} \cdot \overline{\mathrm{d}}_{2}\right), \overline{\mathrm{g}}=\left(\mathrm{e}_{1}^{\mathrm{i}} \cdot \overline{\mathrm{d}}_{1}\right) \oplus \overline{\mathrm{e}}_{2}$, and
$\overline{\mathrm{h}}=\left(\mathrm{e}_{1}^{\mathrm{i}} \cdot \overline{\mathrm{d}}_{1}\right) \oplus\left(\mathrm{e}_{2}^{\mathrm{i}} \cdot \overline{\mathrm{d}}_{2}\right), \overline{\mathrm{k}}=\overline{\mathrm{e}}_{1} \oplus \overline{\mathrm{e}}_{2}, \overline{\mathrm{j}}=(\overline{\mathrm{f}}+\overline{\mathrm{g}}+\overline{\mathrm{h}}+\overline{\mathrm{k}})$.
Example 3 (continued). Let us now finish the example given in Section 2. Let us recall that we have two vectors $(0.5,0.5,0,0)$ and $(0,0.5,0.5,0)$ that represent the relation between two temporal points. The addition operation of Definition 9 give as the resulting vector $(0,0.5,0,0.5[1 / 3,1 / 3,1 / 3])$. This means, that with a probability of 0.5 the relation between $\mathbf{a}$ and $\mathbf{b}$ is " $=$ ", with a probability of 0.5 it is an inconsistent relation, and the percentage of each of the basic relations inside the inconsistent one is equal to $1 / 3$.

## 4. Conclusion

It is often necessary to deal with uncertain knowledge and the area of temporal reasoning is not an exception. Basically uncertainty includes two main aspects: inexactness and inconsistency. In this paper we proposed a representation and reasoning mechanism for uncertain temporal relations. This is one way to take into account all the knowledge obtained from conflicting knowledge sources about the values of a temporal relation using the probabilities of the basic relations. This also includes a new version which takes into account the inconsistency in a temporal relation.

We use a basic vector presentation that includes seven parameters which represent a relation between two temporal points by means of two parts: the inexact part and the inconsistent part. The first part presents the distribution of probabilities among the three basic relations "<", ">", " $=$ " and the inconsistent relation. The second part represents the composition of the inconsistent relation as the percentages of "<", ">" and " $=$ " within it. The reasoning mechanism is based on three operations: inversion, composition, and addition. These are used to derive the vector representation of the temporal relation
between temporal points for which this relation has not been given.

Further research is needed to analyze the behavior of the suggested reasoning mechanism in real applications. Especially the growth of the probability of the inconsistent part and its use in practical reasoning compared to human approaches towards inconsistency in different domain areas requires further study. The applicability of the suggested reasoning approach with temporal intervals also needs further research.

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