

A Class of Star-Algebras for Point-Based Qualitative Reasoning in Two-Dimensional Space

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Abstract: In this article we have presented a new scheme for reasoning with points in 2D-space, called Star-ontology(6), and presented the current results of our study on complexity of reasoning with incomplete/disjunctive information using this new ontology. In this paper we have also proposed a generalized framework Star-ontology(α) for an integer α , that could be specialized to many ontologies including some of the known ones like the 2D Cardinal-directions ontology. This generalization also points to an interesting direction for investigation in the field of spatio-temporal reasoning.

Key Words: Spatial and temporal reasoning; Constraint-based reasoning; Qualitative reasoning; Geometrical reasoning; Knowledge Representation

1. Introduction

Starting from the early studies of simple point-based ontology in linear time, spatio-temporal constraint-based reasoning has matured into a discipline with its own agenda and methodology. The study of an *ontology* starts with an underlying ‘space’ and develops a set of (mutually exclusive) jointly exhaustive and pairwise disjoint (JEPD) ‘basic relations’ with respect to a reference object located in that space. Basic relations correspond to the equivalent regions in the space for the purpose of placing a second object there with respect to the first one. For example, a second point B can be at ‘East’ of a reference point A in the Cardinal directions-ontology (see the Figure 1 below), where the space is ‘zoned’ with respect to the point A. The underlying space and such a relative ‘zoning’ scheme of the space with respect to a reference object forms an ‘ontology.’

Qualitative reasoning using an *ontology* involves a given set of objects and binary disjunctive relations (subset of the set of basic relations) between some of those objects. The

satisfiability question in the reasoning problem is - whether the relations are consistent with respect to each other or not. The power set of the set of basic relations is closed with respect to the primary reasoning operators like composition, inversion, set union and set intersection, and thus, forming an algebra. In most cases such algebras are happened to be relational algebras. In the literature on this area, the term ‘algebra’ (or ‘calculus’) is more frequently used while referring to the concept of ‘ontology’ as described here. Thus, the “reasoning in interval-algebra” will mean the “reasoning in interval ontology” in our terminology.

In the last few decades many such ontologies have been invented. In this work we have proposed a new one, called Star-ontology, for reasoning with points (objects) in two-dimensional space. Our main results presented here comprise of a study of its properties and some complexity issues of doing reasoning in it. There are many real life situations where qualitative reasoning with the proposed Star-ontology is important. For example, consider a set of mobile agents who have only imprecise (disjunctive) information regarding their relative angular directions with respect to each other and yet want to locate their possible relative positions.

We have also developed a generalized scheme (Star-ontology(α)), for an integer α , for a class of similar ontologies. The generalization not only encompasses the new ontology that we are proposing here (for $\alpha=6$), but also includes another one (2D-Cardinal directions ontology) studied before (for $\alpha=4$), and provides directions for many new and interesting other ontologies for different values of α and further works on them.

We will first introduce the 2D-Cardinal directions ontology of Ligozat (1998) and then develop the new Star-ontology(6). Subsequently we will generalize above to the Star-ontology(α) and then briefly conclude the paper.

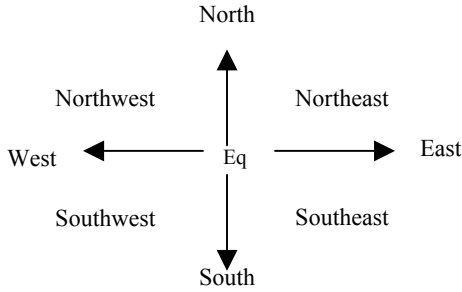


Figure 1: 2D-Cardinal directions ontology

2. 2D-Cardinal directions ontology

Gerard Ligozat (1998) introduced a point-based disjunctive qualitative reasoning scheme in two-dimensional space. The set of nine JEPD basic spatial-relations in this ontology could be represented as (Figure 1) {Equal, East, North, West, South, Northeast, Northwest, Southwest, Southeast}. The first relation 'Equal' is the point region on the reference point itself (identity relation, e.g., a point B is 'Equal' to the reference point A). The next four ('East' through 'South') are one-dimensional semi-infinite regions fanning out from the reference point. The subsequent four regions are two-dimensional open regions enclosed within those four lines.

Ligozat named the relevant algebra (formed by the power set of these nine basic relations, along with the standard operators needed for constraint propagation, e.g., disjunctive-composition, inverse, set-intersection, and set-union) related to this ontology as Cardinal directions-algebra. A one-dimensional version of this ontology is the simple point-based reasoning scheme that is studied extensively within the spatio-temporal qualitative reasoning community. A higher dimensional version of the 2D-Cardinal directions ontology (called n-D Cardinal directions Algebra) has also been studied recently by Condotta et al (2001) and Mitra et al (2001).

3. Star-ontology for 60-degree division

In this work we are proposing a new ontology in two-dimensional space. Instead of using the traditional Cartesian system (as in the 2D-Cardinal directions ontology) we are proposing an ontology with six lines fanning out from the reference point with sixty-degree angle between any adjacent pair of lines, as shown in the Figure

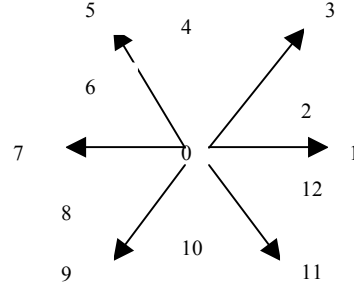


Figure 2: Star-ontology (6)

2. As a convention, the first of such six lines (instead of four lines in the 2D-Cardinal directions ontology) is aligned to the positive X-axis ("East") in a Cartesian space. This reference orientation of the underlying space is absolute.

For the lack of any natural language terms, we will call the basic relations corresponding to these regions as {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}. The relation 0 is the standard 'Equality' with respect to the reference point. The odd numbered relations represent the six lines fanning out from the reference point and the even numbered relations correspond to the two-dimensional open regions in between the consecutive lines. The regions numbered higher than six are inverse of the respective regions numbered lower than six, 0 being inverse of itself. Thus, 7 is inverse of 1, 8 is inverse of 2, and so on. We will call this ontology as Star-ontology(6), for a reason to be explained later, and the corresponding algebra as Star-algebra(6).

The Table 1 is the composition table (ct) between these basic relations. Each row in the table corresponds to a basic relation (say, r) between two points y to x , while each column indicates the basic relation (say, l) between points z to y , each entry in the table is the resulting relationship from z to x (r compose l , or $r.l$, where '.' indicates the composition operation). They are derived by explicitly drawing such points in the 2D-space. For example, if a point y is at the region '2' with respect to z , expressed as $(y(2)z)$, and $x(4)y$, then x could be at any of the regions '2', '3', or '4' with respect to the point z , $(x(2,3,4)z)$. 'T' indicates 'tautology' (disjunction of all thirteen basic relations) in the table. The row and the column corresponding to the 'Equality' or the '0' relation is omitted because: for any basic relation r , $r.0 = 0.r = r$.

The following properties can be observed from the table: \forall basic relations r and l , (1) $r.r = r$, (2) $r.r^\cup = r^\cup.r = \text{either T, when } r \text{ is a}$

two-dimensional region, or $\{r, 0, r^\cup\}$, when r is a one-dimensional region, with r^\cup being the inverse of r , (3) $r.l = l.r$ (commutative), (4) $r.l = \text{inverse}(r^\cup.l^\cup)$, where inverse of a set comprises of inverse of the elements in the original set.

Above four properties are observed in the Cardinal directions-ontology as well, and indicate very nicely behaved algebras. These properties originate from the symmetry of the underlying space that is not true in many varieties of spatio-temporal ontology studied so far.

Figure 3 provides a graphical representation G of the Star-ontology(6). The

graph represents the basic relations as its nodes and their connectivity as arcs. Apart from the connectivity information, the regions indicated by the nodes in the graph have their own dimensionality. In the Figure 3 a dark node indicates a 1D-region (line) and an open circle indicates an open region of two-dimensions surrounded by two such lines. Of course, the center is the '0' region with zero-dimension. This is very similar to the lattice representation of Ligozat for studying the maximal tractable sub-algebras of the corresponding time-interval algebra (Ligozat, 1996) or 2D-Cardinal directions algebra (Ligozat, 1998).

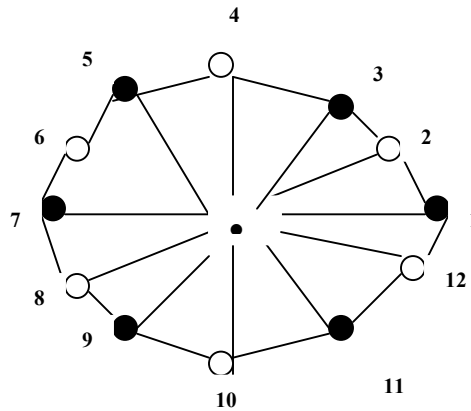


Figure 3: Graphical representation G of Star-ontology(6)

As discussed before, a reasoning problem in the Star-ontology(6) would involve a set of point-variables in 2D space, and a set of disjunctive binary spatial-relationships between some pairs of them using this ontology. The number of such possible disjunctive binary spatial-relationships is 2^{13} over 13 basic relationships, including the *tautology* and the *null* relationship. We often use the term “region” or “relation” for an element of this disjunctive power set. Standard reasoning algorithms use the disjunctive composition that is derived from the basic composition table. Other standard operations needed for the purpose of doing reasoning are inverse, set union, and set intersection. For a review on such standard operations and how they are used in spatio-temporal reasoning algorithms – see Chittaro and Montanari (2000). The power set is closed under these operations forming the Star-algebra(6).

One of the important sub-sets of the full 2^{13} elements Star-algebra(6) is the set of convex relations. *Convex relations* are the disjunctive set of basic relations that constitute a

convex region in two dimensions. Note that a region R is convex iff every point on a line joining any two points x and y (shortest path between x and y) in R also lies within R . For example, regions (2, 3, 4) or (3, 0, 9) are convex relations, but (2, 4) or (2, 3, 4, 5, 6, 7, 8) are not. Note that the region (2, 3, 4) means a union of individual regions 2, 3 and 4. Obviously any convex relation must be comprised of contiguous basic relations in G , but the contiguous nature is not enough to guarantee the convexity. Every basic relation is a convex relation also. Convex relations may or may not include the relation 0 (equality), however, if the region extends from a one-dimensional region to its inverse, both inclusive, then the relation 0 must be included in the relation. For example, excluding 0 from the convex relation (0, 1, 2, 3, 4, 5, 6, 7) would make it non-convex. However, (1, 2, 3, 4, 5, 6) is a convex relation, because it does not include 7, the inverse of 1. Also, (1, 0, 7) is a convex relation but (1, 7) is not. Thus, a convex relation could be expressed as $(a - b, [0])$, by an interval (of length from zero through six) from relation a

through relation b in the graphical representation G (Figure 3) of Star-ontology(6), optionally including 0, except that when a and b are one-

dimensional relations inverse to each other the relation 0 must be included.

Table 1: Composition table between basic relations in Star-ontology(6)

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	2	2,3,4	2,3,4	2,3,4,5,6	<u>1,0,7</u>	12,11,10,9,8	12,11,10	12,11,10	12	12
2	2	2	2	2,3,4	2,3,4	2,3,4,5,6	2,3,4,5,6	<u>T</u>	2,1,12,11,10	2,1,12,11,10	2,1,12	2,1,12
3	2	2	3	4	4	4,5,6	4,5,6	4,5,6,7,8	<u>3,0,9</u>	2,1,12,11,10	2,1,12	2,1,12
4	4,3,2	4,3,2	4	4	4	4,5,6	4,5,6	4,5,6,7,8	4,5,6,7,8	<u>T</u>	4,3,2,1,12	4,3,2,1,12
5	4,3,2	4,3,2	4	4	5	6	6	6,7,8	6,7,8	6,7,8,9,10	<u>5,0,11</u>	4,3,2,1,12
6	6,5,4,3,2	6,5,4,3,2	6,5,4	6,5,4	6	6	6	6,7,8	6,7,8	6,7,8,9,10	6,7,8,9,10	<u>T</u>
7	<u>7,0,1</u>	6,5,4,3,2	6,5,4	6,5,4	6	6	7	8	8	8,9,10	8,9,10	8,9,10,11,12
8	8,9,10,11,12	<u>T</u>	8,7,6,5,4	8,7,6,5,4	8,7,6	8,7,6	8	8	8	8,9,10	8,9,10	8,9,10,11,12
9	10,11,12	10,11,12,1,2	<u>9,0,3</u>	8,7,6,5,4	8,7,6	8,7,6	8	8	9	10	10	10,11,12
10	10,11,12	10,11,12,1,2	10,11,12,1,2	<u>T</u>	10,9,8,7,6	10,9,8,7,6	10,9,8	10,9,8	10	10	10	10,11,12
11	12	12,1,2	12,1,2	12,1,2,3,4	<u>11,0,5</u>	10,9,8,7,6	10,9,8	10,9,8	10	10	11	12
12	12	12,1,2	12,1,2	12,1,2,3,4	12,1,2,3,4	<u>T</u>	12,11,10,9,8	12,11,10,9,8	12,11,10	12,11,10	12	12

The preconvex subset (of the full 2^{13} element set) is a superset of the convex subset such that some lower dimensional relations from the interval $(a - b, [0])$ are allowed to be absent. Thus, (2, 4, 5, 6) or (3, 9) are preconvex, but not convex relations. It could be easily shown that the set of convex relations and the set of preconvex relations are closed under inverse, composition, and intersection operations, thus, forming the convex sub-algebra and *preconvex sub-algebra* of the Star-algebra(6).

There are 156 convex relations (including *null* and *tautology* relations) and 508 preconvex relations out of the total 2^{13} elements in the Star-algebra(6). The notion of preconvexity is very useful in finding a maximal tractable sub-algebra in many ontologies, where path-consistency (with polynomial algorithms) guarantees global consistency. Ligozat has

developed these notions for the purpose in time-interval ontology (Ligozat, 1996) and later for the 2D-Cardinal directions ontology (Ligozat, 1998). We suspect that is true for Star-algebra(6) as well but we do not yet have any proof for that.

Path-consistency does not imply global consistency. The following example illustrates that. Consider four points (p, q, r, s) having relations: $s(2, 6)p$, $s(6, 10)q$, $s(10, 2)r$, $q(3)p$, $r(7)q$, and $r(5)p$. Any path-consistency algorithm will not detect inconsistency here. However, one could easily verify that there does not exist any global solution for this problem instance. Lack of space prevents us from depicting the relationships in a figure.

The Star-ontology(6) has some similarity to the Cyclic-time ontology developed by Balbiani and Osmani (2001), where even a problem instance with only basic relations may

not be tractable (path-consistent but not globally consistent). This raises some doubt whether path-consistency will really imply global consistency for preconvex sub-algebra of the Star-algebra(6).

The following theorem proves the intractability of the full Star-algebra(6). The proof uses similar technique as used by Ligozat (1998) for proving NP-hardness of 2D-Cardinal directions algebra, but avoids utilizing the projections on the two axes.

Theorem 1: Reasoning with full Star-algebra(6) is NP-hard.

Proof: Construction of a Star-ontology problem instance from an arbitrary 3-SAT problem instance. (1) For every literal l_{ij} (in the 3-SAT source problem), create two points P_{ij} and R_{ij} such that $P_{ij} [2 - 8] R_{ij}$, and (2) for every clause C_i we have $P_{i1} [8 - 12] R_{i2}$ and $P_{i2} [8 - 12] R_{i3}$ and $P_{i3} [8 - 12] R_{i1}$. Also, (3) for every literal l_{ij} that has a complementary literal l_{gh} we have two relations between their corresponding points: $P_{ij} [6 - 12] R_{gh}$ and $P_{gh} [6 - 12] R_{ij}$. Note, a disjunctive relation between two points P and R , $P [a - b] R$ indicates $P (a, a+1, a+2, \dots, b) R$, where a and b

are basic relations of Star-ontology(6) $\{0, 1, 2, \dots, 12\}$.

We assign $P_{ij} [8] R_{ij}$ whenever any literal l_{ij} is true.

For any truth assignment that makes a clause false (with all literals in it being false) we cannot have the corresponding six points located in a two dimensional space satisfying the relations as in the first two set of constructions. On the other hand if any literal in a clause is true we can have assignments if and only if the corresponding complementary literal is false in another clause. Thus, the constructed problem instance in the Star-ontology(6) can have a solution if and only if there exists a satisfying truth assignment for the source 3-SAT problem instance.

The construction is polynomial: six points per clause, six relations per clause from construction (1), three relations per clause from (2), and at the most three relations per pair of clauses from (3). Hence, the above construction is a polynomial transformation from 3-SAT problem to the Star-algebra(6) problem proving the later to be NP-hard. *End proof.*



Figure 4. Star-ontology(3), basic relations do not have unique composition

4. Generalized Star-ontology

The Star-ontology could be extended beyond six divisions with $2*6+1=13$ basic relations. Consider dividing the space into eight regions instead of six in a similar angular fashion. The basic relations would be $\{0, 1, 2, \dots, 17\}$, where 0 indicate 'Equality' with respect to the reference point, 1 indicates the 'East,' every odd relation correspond to a semi-infinite line from the origin (the reference point), and even relations indicate a 2D-conic section bounded between two such consecutive semi-infinite lines with $(360/\alpha) = 45$ -degree angles between them. One can easily generalize the concept to a Star-ontology(α), where α stands for any even integer indicating

the number of divisions of the 2D-space. The set of basic relations would be $\{0, 1, 2, 3, \dots, 2*\alpha\}$, and the angle between each pair of consecutive lines is $(360/\alpha)$. Ligozat's (1998) 2D-Cardinal directions ontology is a special case with $\alpha=4$ in this framework. It is self-evident now why the ontology proposed in the last few sections is called Star-ontology(6).

Star-ontology does not form any algebra when α is an odd integer, and thus, is useless for any reasoning purpose. See the Figure 4 for the Star-ontology(3) with basic relations $\{0, 1, 2, 3, 4, 5, 6\}$. A major problem here is the difficulty in defining inverse of any basic relation (other than that for the region 0), and also, the result of composition operations are not unique. For example, composition operation (2.4), between

three points x , y , and z , with y (2) x , and z (4) y , would result in both z (2, 3, 4) x , and z (0, 1, 2, 3, 4, 5, 6) x , depending on where the point y is located in the supposedly equivalent region 2 with respect to the point x (Figures 4). A composition table could not be formed for this ontology, and doing any reasoning is impossible. This observation is true for any Star-ontology(α) where α is an odd integer.

Most of the complexity results discussed in the previous sections will remain valid in the generalized Star-ontology(α). However, we know that reasoning with the preconvex relations in Star-ontology(4) or 2D-Cardinal directions algebra is a maximal-tractable algebra (Ligozat, 1998).

An interesting case is that of the Star-ontology(2) when α is 2. The five basic relations here could be semantically described as $\{Equality, Front, Above/Left, Back, Below/Right\}$. This ontology may find interesting applications. Studying the corresponding simple algebra would be a future direction to our work. Star-ontology(0) with two basic relations $\{Equality, Non-equality\}$ is also of some theoretical interest for a broad study of the spatio-temporal reasoning.

5. Conclusion

In this short paper we have proposed a new ontology named Star-ontology(6) for reasoning with angular directions in two-dimensional space. We have discussed complexity issues in reasoning with this ontology and proposed a generalized framework for the Star-ontology(α) that includes the former ontology for $\alpha=6$, and Ligozat's 2D-Cardinal directions ontology for $\alpha=4$. We did not make any claim that Star-ontology(6) is computationally "harder" than the Star-ontology(4). Our primary result is in generalizing the latter into an explicitly non-Cartesian system. Our hope is that such a generalization will provide some flexibility to an appropriate spatial-reasoning practitioner.

Some interesting other ontologies that could be developed out of such a generalized framework (for different values of α) are also being suggested here. A new methodology for studying the complexity issues that completely avoids using projections on co-ordinate axes is being introduced here, which may have some broader implications than what is being achieved in this work.

Acknowledgement: This work has been supported by the NSF grant IIS-0296042. Some unknown reviewers' comments were very helpful.

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