A Theory for Convex Interval Relations including Unbounded Intervals

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Abstract

We extend the basic axiomatization of interval convex relations by Allen and Hayes with *unbounded intervals*. Unbounded intervals include *since intervals* with a finite beginning point and infinite ending point, *until intervals* with an infinite beginning point and finite ending point and the constant *alltime* representing the whole time line, with both extreme points being infinite. A number of results show the adequacy of the axiomatization proposed; in particular, unbounded intervals are proven to contain unbounded sequences of meeting intervals extending towards the past and/or the future. Importantly, the theory is proven to be consistent.

Introduction

The notion of time is inherent in many activities which involve intelligence. Areas where time and, in particular, temporal repetition, are fundamental components include medical diagnosis, scheduling of classes and ecological modeling. In particular, much work is devoted to dealing with time where the intent is to formalize temporal objects, such as time intervals or points, and how they relate, often independent from what occurs during the objects themselves. Influential proposals of modelings for temporal objects include (Allen 1983; Vilain & Kautz 1986; Ladkin 1987; Ligozat 1991). A fundamental and inspiring work about temporal intervals is that of Allen (Allen 1983; Allen & Hayes 1989), which has given rise to very extensive research in temporal reasoning. Here we extend the axiomatization of Allen's interval algebra presented in (Allen & Hayes 1989) with *unbounded* (or *infinite*) *intervals*. Unbounded intervals increase expressibility in an interval-based temporal theory; for example in the time theory developed in (Cukierman 2003) unbounded intervals are the convexification of (i.e. they minimally cover) temporal objects representing infinite repetition. Thus, unbounded intervals allow to refer to the interval covering the whole extent of an infinite or non-ending process. More precisely, unbounded intervals allow that the convexification operation be total and thus defined for both finite and infinite temporal objects. For an intuitive example of expressiveness gained with unbounded intervals, consider the sentence "The sun rises every day". Assuming that there was one "first sun rise" (after the Big Bang and the creation of the Earth) and assuming that the sun will keep rising forever, the whole period during which the sun rises over and over again would be represented as a *since* interval. Very recently (Bouzid & Ladkin 2002) suggests the inclusion of *half infinite intervals* (corresponding to our unbounded intervals) in an algebra of *union of convex intervals*. Infinite intervals are conceived in the referred work for different reasons than the ones presented here. Furthermore, they do not deepen into how such intervals would relate with others with Allen's relations, and rather, they are doubtful about such an extension. Here we fully add unbounded intervals to the axiomatization of temporal relations. Hence the two researches are complementary, while their research provides another example of an area where our results are of interest. What we present in this article is a sub-theory of a time theory developed in (Cukierman 2003). In such theory *structured temporal objects* are defined, during which *atemporal assertions* are true or false. The building blocks of the temporal terms are *convex intervals*, *unbounded intervals* and *qualitative convex interval relations* extending the axiomatization in (Allen & Hayes 1989). In the present paper we define and characterize *unbounded intervals* and how they behave with respect to all the basic convex interval relations. Several results show the adequacy of the axiomatization proposed; the theory is proven to be consistent. It is not a complete theory, inheriting such characteristic from the axiomatization it is extending (Allen & Hayes 1989). We further analyze this below.

One can imagine that *since* and *until* intervals present a "specular symmetry"; *until* intervals extend towards the past and have a finite ending, *since* intervals have a finite beginning and extend towards the future. The symmetry these two type of unbounded intervals have is present in their axioms and associated theorems. Due to space limitations and based on this specular symmetry we mainly include results involving *since* intervals. Results about *until* intervals are specularly symmetric. Also due to lack of space we only outline some proofs of theorems and metatheorems. Full detailed proofs of all the results included in this paper are developed in (Cukierman 2003). We first present the axiomatization, then results which follow from it and conclude with a discussion.

The axiomatization

We propose that when one (at least) in a pair of intervals is unbounded, then only a subset of all possible Allen's relations is possible between the two. The definition of unbounded intervals and a characterization of such subset of relations are specified in the axiomatization and explained in detail next. The representation formalism used is first order logic (FOL) with equality; sorts are identified with unary predicates (referred to as *sortal predicates*). The logical connectives have the usual precedence rules, where $\dot{\bigvee}$ is used for exclusive or. Convex (bounded) intervals are considered as primitive objects; the sortal predicate identifying interval terms is *interval(_)*. We include within the *interval* sort three kinds of *unbounded intervals*: the (one) constant *alltime* and *since* and *until* intervals. The unary sortal predicates *since(_)* and *until(_)* respectively identify these (sub)-sorts of intervals. The basic Allen's convex interval relations are written $.b.$,$.m.$,$.s.$,$.d.$,$.o.$,$.f.$ and $.eq.$ for *before, meets, starts, during, overlaps, finishes* and *equals* respectively; inverse relations are written for example $.b^{-1}.$ for the inverse of *before* (i.e. *after*). A set of relations represents the disjunction of those relations in the set. For any single basic relation r, r abbreviates $\{r\}$ and for any set of basic relations S, $not[S]$ abbreviates its complement set with respect to all basic relations. The sortal predicate identifying Allen's convex interval relations is written as *allen(_)*.

The time theory in (Cukierman 2003) includes *reified* temporal relations. Syntactically, an expression which would normally be regarded as propositional obtains the status of *term* when *reified* in a first order theory, and it can therefore be an argument to a predicate and/or be quantified. Semantically, when a concept is reified it becomes an entity. For the present (sub-)theory, reification is not strictly needed. Hence, reification in this case should be only considered as an abbreviation of the non-reified notation. For example, to express that the interval i is *before or finishes* j we write $related(i, j, \{.b., .f.\})$ instead of something like $Before(i, j) \bigvee Finishes(i, j)$. In passing, this provides a more concise notation when we want to express disjunctions like $related(i, j, \{.b., .m., .f., .s., .d.\})$.

The *meets* relation: Axioms $\mathrm{M}'.1$ to $\mathrm{M}'.5.1$

Our axiomatization is based on that of the relation *meets* and the *ordered sum* of intervals (Allen & Hayes 1989), extended to unbounded intervals. Axioms $M'.1$, $M'.2$, $M'.3$, $M'.4$ and $M'.5.1$ correspond to axioms *[M1]*,*[M2]*, *[M3]*, *[M4]* and *[M5.1]* respectively in the referred publication; i.e., as a naming convention our corresponding axioms add a prime symbol. Axiom *[M5.1]* is a functional restatement of the existential form in Axiom *[M5]*. The choice between these two forms of axioms is extensively discussed in (Ladkin 1987). We include the functional version in our theory. This makes our theory non-complete, but non-completeness also follows from other more essential design decisions, as discussed below, hence we keep the functional version *[M5.1]* for clarity reasons. Axioms $M'.3$ and $M'.5.1$ include a variation with respect to their corresponding Axioms *[M3]* and *[M5.1]* *excluding* unbounded intervals. Axioms $M''.3$ and $M''.5.1$ below (with two prime symbols in their names) constitute the corresponding axioms for the unbounded interval cases. Axioms $M'.1$, $M'.2$ and $M'.4$ are a straightforward translation from their corresponding axioms *[M1]*,*[M2]* and *[M4]*; in these axioms unbounded intervals falsify some condition in the implication antecedents, making the whole formula trivially true for those cases. The corresponding concepts to axioms *[M1]*,*[M2]* and *[M4]* for unbounded intervals follow from Axioms *i.1* to *i.4* presented below.

All the basic Allen convex interval relations are expressible in terms of the *meets* relation, and this applies to unbounded and bounded intervals. This is proven in (Allen & Hayes 1989) for bounded intervals and we prove it for unbounded intervals. For example, the case of two *since* intervals where one *finishes* the other is as follows[1]:

$$
\begin{aligned}
&(since(s) \bigwedge since(s') \bigwedge related(s, s', .f.)) \Leftrightarrow \\
&\ (\exists k, q.\ interval(k) \bigwedge interval(q) \bigwedge \\
&\ related(q, s', .m.) \bigwedge related(q, k, .m.) \bigwedge related(k, s, .m.) \\
&\ \ \bigwedge (\forall i.\ interval(i) \supset \neg related(s, i, .m.)) \\
&\ \ \bigwedge (\forall i.\ interval(i) \supset \neg related(s', i, .m.)))
\end{aligned}
$$

This case is illustrated in Figure 1.

Thus, given that all the axioms ultimately are expressible with the *meets* relation and that such relation is axiomatized for both unbounded and bounded intervals (with axioms named M', M'' and i), so are all Allen's basic relations in this axiomatization.

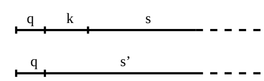

Figure 1: Expressing the *finishes* relation between two *since* intervals in terms of *meets*.

Axioms *[M.1]* to *[M.5.1]* and their translations follow. The notation from (Allen & Hayes 1989) is kept when referring to their axioms, where ":" represents the relation *meets*, except that we write "$\bigwedge$" instead of "$\&$" for conjunction.

Axiom *[M1]* states (where i, j, k, l are intervals):

$[M1]$ $\forall i, j, k, l.\ ((i : j) \bigwedge (i : k) \bigwedge (l : j)) \supset (l : k)$.

The intuition of this axiom is that the "place" where two intervals meet is unique. In the present formalization this axiom is expressed as:

$$
\begin{aligned}
\mathbf{M'.1}\ &\forall i, j, k, l.\ (interval(i) \bigwedge interval(j) \bigwedge interval(k) \\
&\bigwedge interval(l) \bigwedge related(i, j, .m.) \bigwedge related(i, k, .m.) \\
&\bigwedge related(l, j, .m.)) \supset related(l, k, .m.)
\end{aligned}
$$

Given that unbounded intervals may not *meet* or be *met by* other intervals this translation does not apply to unbounded intervals extreme points (although this axiom is trivially true in the case of unbounded intervals because of a false antecedent). Yet, uniqueness of such infinite points (even though they are not directly formalized) is a consequence of Axioms *i.1* to *i.4* presented below.

Axiom *[M2]* ensures that meeting places are totally ordered. The exclusive-or connective is used in the referred

[1] Notice that this is not one of the axioms we have chosen for our system (although it is consistent with them). Rather, this is one formula showing how the relation *finishes* is expressible purely based on the *meets* relation, when involving unbounded intervals.

publication ($\oplus$, their notation). The intent is that exactly one of three possibilities is true for the meeting points of intervals i and j, and k and l. We straightforwardly translate this axiom in two axioms: Axiom *M'.2* and Axiom *M'.2.1*. Only Axiom *M'.2* is included here[2]. Similarly to Axiom $M'.1$, this translation does not apply to unbounded intervals extreme points (although this axiom is trivially true in such case). The ordering of such infinite points with respect to finite points is a consequence of Axioms *i.1* to *i.4*, see below.
$[M2]\ \forall i,j,k,l.\ ((i:j) \bigwedge (k:l)) \supset ((i:l) \oplus\ (\exists q.\ (i:q:l)) \oplus (\exists r.\ (k:r:j)))$.

M$'$.2 $\forall i,j,k,l.\ (interval(i) \bigwedge interval(j) \bigwedge interval(k) \bigwedge interval(l) \bigwedge related(i,j,.m.) \bigwedge related(k,l,.m.)) \supset (related(i,l,.m.) \bigvee (\exists q.\ (interval(q) \bigwedge related(i,q,.m.) \bigwedge related(q,l,.m.))) \bigvee (\exists r.\ (interval(r) \bigwedge related(k,r,.m.) \bigwedge related(r,j,.m.))))$

Axiom *[M3]* in (Allen & Hayes 1989) specifies that given any interval, there is always an interval immediately *before* and an interval immediately *after* such any interval. *Unbounded intervals* may not have an interval immediately before nor immediately after like any "normal" interval does. Hence the translation of Axiom *[M3]* must take this into account. We do this in two "steps". We *exclude unbounded intervals* from the otherwise straightforward translation of Axiom *[M3]* (Axiom $M'.3$) and we add an additional axiom (Axiom $M''.3$ below) to specify how each type of unbounded interval is restricted with respect to the *meets* relation (this technique is also used for the translation of Axiom *[M5.1]* below). Axiom *[M3]* is as follows:
$[M3]\ \ \forall i.\exists j,k.\ (j:i:k)$, which we translate (excluding unbounded intervals) as:

M$'$.3 $\forall i.\ (interval(i) \bigwedge \neg unbounded\_interval(i)) \Leftrightarrow \exists j,k.\ (interval(j) \bigwedge interval(k) \bigwedge related(j,i,.m.) \bigwedge related(i,k,.m.))$.

Axiom *[M4]* in (Allen & Hayes 1989) specifies the uniqueness of meeting points.
$[M4]\ \ \forall j,k.\ \exists i,l.\ ((i:j:l) \bigwedge (i:k:l)) \supset j=k$. The translation is not included here due to space limitations but it is analogous to the translation of Axiom *[M1]*.

Similarly to Axioms $M'.1$ and $M'.2$, Axiom $M'.4$ does not apply to *unbounded intervals* since they may not have a *meeting* or *met by* interval. Again, in such cases this axiom is trivially true. On the other hand, equality among *unbounded intervals* (or their corresponding extreme points) is not left undefined and it follows from Axioms *i.3* and *i.4*.

Axiom *[M5]* in (Allen & Hayes 1989) guarantees the existence of a *union* or *ordered sum* interval of two meeting intervals. Axiom *[M5.1]* (also in the referred publication) is the corresponding functional version, including the *union* or *ordered sum* functor (written $+$).

$[M5]\ \ \forall i,j.\ (i:j) \supset\ (\exists k.\ \forall m,n.\ (m:i:j:n) \bigwedge (m:k:n))$
$[M5.1]\ \ \forall i,j.\ (i:j) \supset\ (\exists m,n.\ (m:i:j:n) \bigwedge (m:(i+j):n))$

These axioms make use of auxiliary intervals which are guaranteed to exist in case of "normal" (i.e. bounded) intervals by Axiom *[M3]*. Given the restrictions with respect to the *meets* relation for *unbounded intervals*, we need to consider the ordered sum of intervals only for those cases when unbounded intervals *meet* or are *met by* other intervals and without using such auxiliary intervals. Hence, we translate Axiom *[M5.1]* straightforwardly into Axiom $M'.5.1$, but similarly to Axiom *[M3]* we exclude unbounded intervals. Axiom $M''.5.1$ below additionally specifies how the ordered sum is defined when involving unbounded intervals.

Composition of relations holds in our axiomatization analogously as it holds in Allen and Hayes' axiomatization. For example from $related(i,j,.m.) \bigwedge related(j,k,.b.)$ it follows that $related(i,k,.b.)$. Some cases of composition will not be possible when involving unbounded intervals, but this only is falsifying antecedents in implications, hence there is no special consideration needed for unbounded intervals in these cases. The *equality relation* ($.eq.$) can be considered as the logical symbol "$=$" (for which the equality axioms hold) and it can also be defined for bounded intervals by Axiom *[M4]* (whose corresponding translation is $M'.4$ here) if strengthened to a biconditional (see (Allen & Hayes 1989), page 228). Equality of unbounded intervals is axiomatized as part of Axioms *i.1* to *i.4*, $M''.3$ and $M''.5.1$.

Unbounded intervals axioms

Axiom names prefixed with "*i*", *i.1* to *i.4* below axiomatize unbounded intervals with respect to the *start* and *finishes* relations ($.s.$, $.f.$). We use these relations to make it more intuitively clear how the infinite points where unbounded intervals *start* or *finish* are involved (even though such points are not explicitly defined). Recall however that all of Allen's relations are expressible in terms of the *meets* relation, regardless of the intervals being unbounded or not, hence these axioms could be so expressed.

As well, we include axioms whose names are prefixed with "M''", $M''.3$ and $M''.5.1$. These axioms correspond to unbounded intervals as they relate with the *meets* relation and *ordered sum*, and thus they complement the axiomatization of unbounded intervals. They also complement the *meets* relation axioms above (Axioms $M'.1$ to $M'.5.1$) which were restricted to apply to bounded intervals.

Axioms *i.1.a* and *i.1.b* define unbounded intervals as three different (and disjoint) terms: a constant *alltime* and two sub-sorts of unbounded intervals: *since* and *until*. (The two axioms, *a* and *b*, are imposed given the nature of exclusive disjunction, only axiom *a* is included here.)

[2] Substituting the *or* connective ($\bigvee$) in Axiom $M'.2$ by an *exclusive or* would not accomplish expressing disjointedness; the *exclusive or* connective will accomplish this in the case of *two* logical subformulas, but not when the formula involves more than two subformulas. Hence we propose separate axioms to this effect. For example, to express that exactly a or b or c is true we have $(a \bigvee b \bigvee c)$ and $(a \Leftrightarrow (\neg b \bigwedge \neg c))$ and so on.

i.1 unbounded intervals

a. $\forall w.\ unbounded\_interval(w) \Leftrightarrow (interval(w) \bigwedge (w = alltime \bigvee since(w) \bigvee until(w)))$.

Axiom *i.2.a* states that every interval is $.in.$ *alltime*, and that "$.in.$" is the only relation any interval may relate with this unbounded interval, where $.in. = \{.d.,.eq.,.s.,.f.\}$. Axiom *i.2.b* states that the only intervals that *finish* ($.f.$) *alltime* are *since* intervals and the only intervals that *start* ($.s.$) *alltime* are *until* intervals. See Figure 2 for a graphic illustration. Axioms *i.2.a* and *i.2.b* are not included here due to lack of space.

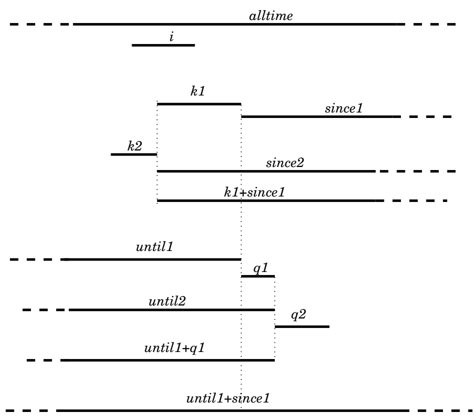

Figure 2: Graphical layout of one consistent scenario involving unbounded intervals.

Axiom *i.3.a* states that those intervals that *finish, are finished* by or *equal* a *since* interval are *since* intervals and *since* intervals finish *alltime*. Axiom *i.3.b* states that two *since* intervals only relate with the relations *finish, finished by* or *equal*. Intuitively, they have the same infinite ending point and may or may not have the same beginning point. Axiom *i.3.c* states that *since* intervals and *alltime* are the *only* intervals to relate with the relations *finish, finished by* or *equal* with a *since* interval. Refer to Figure 2. Symmetrically to Axiom *i.3*, Axiom *i.4* specifies how *until* intervals relate. Axiom *i.3.a* is included next.

i.3 since

a. $\forall w, j.\ (since(w) \bigwedge interval(j)) \supset$
$((related(j, w, .f.) \supset since(j)) \bigwedge$
$(related(j, w, .f^{-1}.) \supset (since(j) \bigvee$
$j = alltime)) \bigwedge$
$(related(j, w, .eq.) \Leftrightarrow j = w))$

Axiom $M''.3.a$ states that there exists an interval *meeting* any *since* interval and there exists an interval *met by* any *until* interval. Axiom $M''.3.b$ states that no interval is met by a *since* interval and that no interval can *meet* an *until* interval. Note that the intervals *meeting* a *since* interval or those *met by* an *until* interval may be unbounded or not. Observation 1 then shows that no interval can *meet* or *be met* by *alltime*. Hence we have completely described how *unbounded intervals* behave with respect to the relation *meets* ($.m.$).

M$'$.3 unbounded intervals and meets

a. $\forall s, u.\ (since(s) \bigwedge until(u)) \supset$
$\exists k.\ interval(k) \bigwedge related(k, s, .m.) \bigwedge$
$\exists q.\ interval(q) \bigwedge related(u, q, .m.)$

b. $\forall s, u, i.\ (since(s) \bigwedge until(u) \bigwedge interval(i)) \supset$
$related(s, i, not[.m.]) \bigwedge related(i, u, not[.m.]))$

A note about unboundedness Axiom $M''.3$ states that *since* intervals are *bounded towards the past* and *unbounded towards the future* and symmetrically *until* intervals are *bounded towards the future* and *unbounded towards the past*. This notion of *boundedness* is based on the existence or not of a *meeting* or *met by* interval. It is akin to the notion of number sequences being *bounded* by a number in the mathematical analysis realm.

The next observation follows from Axiom *i.2*: *alltime* is unbounded towards both the future and the past:

Observation 1 (Alltime and meets) $\forall i.\ \ interval(i) \supset \neg related(i, alltime, \{.m., .m^{-1}.\})$

The *ordered sum* involving *unbounded intervals* is restricted to those cases where unbounded intervals can meet (according to Axiom $M''.3$ and Observation 1). Hence the *ordered sum* involving *unbounded intervals* is defined when an *until* interval meets a *since* interval (Axiom $M''.5.1.a$), an *until* interval meets a bounded interval, and when a bounded interval meets a *since* interval (Axioms $M''.5.1.b$ and *c* respectively). Axioms *a* and *b* are presented next.

M$'$.5.1 unbounded intervals ordered sum

a.since and until

$\forall i,j.\ (until(u) \bigwedge since(s)) \supset$
$(related(u, s, .m.) \supset u + s = alltime)$

b. since and bounded

$\forall i,j.\ (since(s) \bigwedge interval(i) \bigwedge \neg unbounded\_interval(i)) \supset$
$(related(i, s, .m.) \Leftrightarrow$
$(\exists w.\ interval(w) \bigwedge w = i + s \bigwedge since(w) \bigwedge$
$(\forall q.\ (interval(q) \bigwedge related(q, i, .m.)) \supset$
$related(q, i + s, .m.))))$

Results

We include some of the results proven in this theory. The next theorem intuitively states that the (infinite) ending point of *since* intervals and of *alltime* is unique. It essentially follows from Axioms *i.2* and *i.3*. Symmetrically the (infinite) beginning point of *until* intervals and of *alltime* is unique [3].

Theorem 1 (Infinite ending point of unbounded intervals)

$\forall w_1, w_2, w_3.\ ((since(w_1) \bigvee w_1 = alltime) \bigwedge$
$interval(w_2) \bigwedge interval(w_3)) \supset$
$((related(w_1, w_2, \{.f., .f^{-1}., .eq.\}) \bigwedge$
$related(w_1, w_3, \{.f., .f^{-1}., .eq.\})) \supset$
$related(w_2, w_3, \{.f., .f^{-1}., .eq.\}))$

The next theorem states that no unbounded interval can *start* nor be *started by* a *since* interval. Recall that an interval *starts* another ($.s.$) when they have the same *beginning* point

[3]Recall that two intervals related with the *finishes* or *equality* relations have the same ending point.

and the former *ends before* the latter. The proof relies of Axioms *i.1* to *i.4*.

Theorem 2

$\forall i, w.\ (unbounded\_interval(i) \bigwedge since(w)) \supset$
$\neg related(i, w, \{.s., .s^{-1}.\})$

Proof outline If i is a unbounded interval, it can be *alltime*, an *until* or a *since* interval (those are the only three possibilities for a unbounded interval, Axiom *i.1*). *Alltime* cannot *start* nor *be started by* a *since* interval: given Axiom *i.3* any *since* interval *finishes alltime*. An *until* interval cannot possible either *start* nor *be started by* a *since* interval: given Axiom *i.4* an *until* interval only *starts* another *until* interval and *alltime*. The only possibility left is that i is a *since* interval. But a *since* interval *finishes*, *is finished by* or *is equal* to a *since* interval (Axiom *i.3.b*). Hence i cannot be a *unbounded interval* and also *start* or *be started by* a *since* interval ■

We next introduce the concept of a temporal term *subcomponent*. More results about unbounded intervals involve their subcomponents and are presented after this definition. We refer to those intervals "in" an interval as *subcomponents* of the interval maintaining the terminology from the whole theory in (Cukierman 2003). We write $subc(t', t)$ when t and t' are intervals and t' is a *subcomponent* of t.

Definition 1 (Interval subcomponents)
$\forall i, i'.\ (interval(i) \bigwedge interval(i')) \supset (subc(i', i) \Leftrightarrow related(i', i, .in.))$.

Unbounded intervals are related to their subcomponents next. What we prove is essential in the sense that it reflects the very nature of unbounded intervals and thus confirms the adequacy of our theory. We first show that every interval is a *subcomponent* of *alltime*, whereas any *bounded* interval is *strictly* contained in *alltime*. As a corollary we prove that the ordered sum of two bounded intervals is contained in alltime. This corollary intuitively proves that any finite point (meeting point of two bounded intervals) is after the infinite point *beginning alltime* and before the infinite point *ending alltime*.

The next theorem follows from Axiom *i.2.a* and it outlines an essential result: those intervals whose existence is guaranteed (meeting or met by any interval subinterval of *alltime*) *are also subintervals of alltime*, thus reflecting that *alltime* extends towards "the future". Symmetrically we prove that *alltime* extends towards the past. With an analogous formulation we also prove that *since* intervals extend towards the future and *until* intervals extend towards the past.

Theorem 3 (Alltime - extending towards future)

$\forall i.\ (interval(i) \bigwedge \neg(unbounded\_interval(i))) \supset$
$(\exists j.\ (interval(j) \bigwedge subc(j, alltime) \bigwedge related(i, j, .m.)))$

More about unboundedness The previous theorem (Theorem 3) and the analogous for *alltime* extending towards the past and those for *since* and *until* intervals reflect unboundedness of the intervals based on containing unbounded sequences of meeting intervals. However a distinction should be made in this respect. In the case of discrete models there are intervals which are non-decomposable (referred to as *time moments* in (Allen & Hayes 1989)), therefore an infinite sequence of such meeting intervals is unbounded. An infinite sequence of meeting intervals in a continuous model could be such that the length of intervals shortens as it "evolves" towards the future, and hence there would exist a bound to such sequence (an interval which is after *all* those intervals). Certainly, as the previous theorem shows, in both cases all the intervals in the meeting sequences are contained in the unbounded interval. Furthermore, the case when *the sequence* of meeting intervals may be bounded does not contradict the fact that *unbounded intervals* are indeed *unbounded* (which is guaranteed by Axiom $M''.3$). This rather indicates that these theorems do not necessarily reflect unboundedness (of sequences) but rather containment of any sequence of intervals extending towards the past and/or future, as explained.

The next two theorems summarize how *since* intervals can relate with respect to another interval (unbounded or not), considering all the 13 Allen basic relations. Theorem 4 establishes that a *since* interval can only *be after*, *met by* or *overlapped by* an *until* interval. Theorem 5 shows how a *since* interval relates with a bounded interval. Symmetric results are proven for *until* intervals. See Figure 2.

Theorem 4 $\forall u, s.\ (until(u) \bigwedge since(s)) \supset related(u, s, \{.b., .m., .o.\})$

Theorem 5 (since and bounded interval)

$\forall w, i.\ (since(w) \bigwedge interval(i) \bigwedge \neg unbounded\_interval(i)) \supset$
$(related(i, w, \{.b., .m., .o., .s., .d.\}) \bigwedge$
$\neg related(i, w, not[\{.b., .m., .o., .s., .d.\}]))$

Consistency of the theory

We prove the consistency of the whole theory in (Cukierman 2003) in two separate parts. This proof applies to the subtheory presented here. The first part proves the consistency of all axioms based on the *PC-transform method* (Hughes & Cresswell 1968). The second part of the proof deals with the subset of axioms and definitions involving *reified relations*, which given the nature of the PC-transform method are not checked to be consistent in the general proof.

In (Hughes & Cresswell 1968) a method is presented to check consistency of propositional and predicate modal systems. The method is based on *PC-transforms*. For every formula in the language it is possible to construct its associated *PC-transform* such that the detail of modal operators, quantifiers and predicates arguments is lost. The mentioned method applies to us in the following way: the classic first order inference rules are already proven to be preserved with the transformation in (Hughes & Cresswell 1968). Using Allen's constraint propagation algorithm (which we use in our theorems proofs) amounts to applying first order classical inference rules. In (Cukierman 2003) we prove that the PC-transformed axioms are consistent, thus our (original) system is proven to be consistent.

Our theory also includes axioms dealing with *reified relations*, but the consistency proof loses the information necessary to check their consistency as it eliminates the predicates arguments. Hence, additionally and independently of the general PC-transform based proof we prove the consistency of those axioms involving reified propositions (i.e. involving the *related* predicate). We use an *interval* constraint propagation algorithm to do this. We input the intervals and relations involved in those axioms to a constraint propagation algorithm (Kautz 1991). This problem is of a very small size (13 intervals), where for example 1 interval represents a generic bounded interval. Allen's (polynomial run time) propagation algorithm is not complete for the full interval algebra if relations involve more than three intervals (Allen 1983). However, since the problem size is very small, we graphically represented a consistent scenario given the results from the propagation algorithm, and indeed found a consistent scenario. Thus we corroborated the correctness of the result provided by the algorithm and hence the correctness of the proof. We do not claim that this problem belongs to any specific tractable sub-algebra, but again, this is not affecting us given the size of the problem. See Figure 2.

Models for the theory

Axioms *[M1]* to *[M5] (and [M5.1])* characterize (convex finite) intervals and Allen's relations and allow for both *discrete* and *continuous* models (Allen & Hayes 1989). With such axioms, *moments* are not formally distinguished as different from intervals, but they would intuitively correspond to some intervals in discrete models, such that they are non-decomposable or likewise they are of the minimal length possible in the model. This is what we opt for as well, by translating axioms *[M1]* to *[M5]* in the referred publication and then adding axioms applying to *unbounded intervals*. In choosing this axiomatization and having both continuous and discrete models our theory inherits the non-completeness of Allen's and Hayes axiomatization. There are models to their and our theory which are not isomorphic differing in the density property. (Ladkin 1987) extensively deals with this issue and proposes a complete theory as well as a completion of Allen and Hayes' theory having only continuous models. We have chosen to extend Allen and Hayes for expressivity reasons and leave for future work extending a complete theory such as what is proposed by Ladkin.

One discrete model of our theory and including unbounded intervals would have *intervals* interpreted as pairs of integers $n_1, n_2 \in Z$, $\langle n_1, n_2 \rangle$, where $n_1 < n_2$. When $n_2 = n_1 + 1$ the interval is non-decomposable. $\langle n_1, n_2 \rangle$ *meets* $\langle n_2, n_3 \rangle$. The constant *alltime* is interpreted as the interval $\langle -\infty, \infty \rangle$, a *since* interval is interpreted as $\langle n, \infty \rangle$ for some integer number n and $\langle -\infty, n \rangle$ is the interpretation of an *until* interval. A continuous model would be analogous to the previous one except that the interval extremes are for example rational numbers and there would not be non-decomposable intervals. Certainly these models also show the consistency of the sub-theory presented in this article.

Points are defined in (Allen & Hayes 1989) based on Axioms *[M1]* to *[M5]* with additional definitions. We do not include such in this theory nor do we include axioms about moments comparable to Allen and Hayes' Axiom *[M6]*. We leave this as future work.

Discussion

We have extended the basic axiomatization of interval convex relations (as presented in (Allen & Hayes 1989)) to take unbounded intervals into account. We use finite convex intervals as primitive objects. *Unbounded intervals* are defined; they include *until* intervals, *since* intervals and the constant *alltime*. A number of results show the adequacy of the axiomatization proposed; in particular, unbounded intervals are proven to contain unbounded sequences of meeting intervals extending towards the past and/or the future. Furthermore, in (Cukierman 2003) unbounded intervals are proven to be the limit of some temporal terms representing repetition, paralleling the notion of limit of unbounded number series in the Mathematical Analysis realm.

Very recently, (Bouzid & Ladkin 2002) suggest the inclusion of *half infinite intervals* in an algebra of *union of convex intervals*. These half infinite intervals correspond to our *since* and *until* intervals. However, they do not deepen into how such intervals would relate with others with Allen's relations, and rather, they are doubtful about such an extension. Here we fully add unbounded intervals to the axiomatization of temporal relations. Hence their and our research are complementary, while their research provides an example of an area where our results are of interest.

Future work includes studying the minimality of the axiomatization. It would also be interesting to explore other possible models, applications and parallels to other formalisms aside from the intended models which inspired it. As well, an analogous extension with unbounded intervals to the (complete) theory for convex interval relations presented in (Ladkin 1987) would be interesting to pursue.

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