A Spatio-Temporal View of Knowledge

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Abstract

In this paper, we extend the well-known multi-modal language for knowledge of agents 'faithfully'. That is, the framework we propose preserves in essence the modal way of describing knowledge. Two desirable properties arise from our approach. First, extending the language gives us clearly more expressive power, which is exploited for the development of a spatio-temporal view of knowledge, in particular. And second, doing so modally enables us to prove some good-natured meta-properties of the new system.

For a start, we integrate two additional features into the commonly used logic of knowledge. The first one is as simple as natural: distinguished states are named, in fact, by nominals from hybrid logic. The second one is a new modal operator associated with every agent. Such a modality ascribes to the agent 'access' to states that are complementary to his or her own view of the world and is, therefore, called a 'distinction operator' (contrasting the knowledge operator which goes together with indistinguishability of states). It turns out that distinction operators and hybrid names perfectly co-operate. In fact, we obtain completeness and decidability of the appearing logical system as well as a corresponding complexity result.

Then we add a spatio-temporal component to the just introduced hybrid logic of knowledge and distinction. To this end, an 'effort operator' is incorporated, and the set of names is structured according to the additional dimension. We argue in favour of a formal basis for spatio-temporal epistemic reasoning, provide a suitable axiomatization, and get a completeness theorem for the system derived from that, too.

KEYWORDS: reasoning about knowledge in multi-agent systems, hybrid logic, spatio-temporal epistemic logic, topological reasoning

Introduction

Since the early eighties, the idea of *knowledge* has been considered more and more significant for both designing and analysing multi-agent systems. In the context of such systems, knowledge is *ascribed externally,* by the designer or analyst, to the agents, and the multi-modal $S5_m$ represents the basic logic of knowledge in case m agents are concerned ($m \in \mathbb{N}$); cf, eg, the standard textbooks Fagin *et al.* or Meyer & van der Hoek.

According to the modal setting, a binary relation R_i on the set S of all states of the system is associated with every agent $i \in \{1,\ldots,m\}$. In fact, we have that $s\,R_i\,t$ holds by definition, iff the states $s,t \in S$ are *indistinguishable* to agent i. For example, in case of a distributed system this means that the i–local components of s and t coincide. More generally, it is always assumed that R_i is an equivalence relation (which explains the appearance of S5). Knowledge of agent i is then defined as validity in all i–indistinguishable states.

There is obviously a second, complementary relation that could (and, because of our external view, is allowed to) be assigned to agent i in a natural way: *distinguishability* of states. Why not use this relation, too, in order to specify more complex properties of multi-agent systems? As far as we know this has not been done up to now, actually. Maybe one of the reasons for this is that distinguishability does not fit in with the modal framework (eg, because it is an *irreflexive* relation and irreflexivity cannot be expressed by a modal formula).

In the present paper we apply *hybrid logic,* HL, to reasoning about knowledge of agents. Basic HL extends modal logic, ML, in such a way that many of the advantageous features of the latter system are retained. And already the most basic system of HL offers much more expressive power than ML; cf, eg, Blackburn or Blackburn, de Rijke, & Venema, Sec. 7.3. For example, irreflexivity can be expressed now. We will show below that we can hybridly get to grips with the relation of distinguishability. Moreover, we can also incorporate *time*.

Now we go a little more into particulars about the content of this paper. In the next section, we define precisely the language underlying the just indicated *hybrid logic of knowledge and distinction,* abbreviated Kn^+. An example showing how the new language can be utilized for specifying properties of multi-agent systems, is also contained in this section. Then, in Section 3, the logic itself is dealt with. We prove a completeness theorem for a corresponding axiomatization, argue towards decidability, and determine the computational complexity of the satisfiability problem for the single-agent case there.

In the second part of the paper we take a step further. We consider certain *temporal* knowledge structures, modelling an *effort operator* for knowledge acquisition. It turns out that a *spatial* component is inherent in such models as well. In fact, we are led to an HL–based characterization of the appropriately generalized Kn$^+$–models as *complement closed spaces of sets*. In this way, structures appear that are 'topological' to an extent. Thus we discovered an interesting connection between reasoning about knowledge and spatial reasoning.

Concluding this introduction we compare our approach with related work. To our knowledge, a combination of hybrid logic and the common logic of knowledge has not been considered up to now (whereas the individual parts of our combined system are well-established; cf the references above). The logic studied in the second part of this paper refers to Dabrowski, Moss, & Parikh. In that paper, a general spatio-epistemic reasoning framework called TOPOLOGIC was proposed. In order to obtain more expressive power with regard to topology, a 'sorted' (i.e., both state- and set-sensitive) hybridization of TOPOLOGIC was developed subsequently; cf Heinemann. A hybrid extension of the classical topological semantics of modal logic was briefly considered in Gabelaia. All in all, hybrid logics in connection with knowledge or topology received amazingly little attention by the relevant communities so far. In contrast, hybrid versions of *temporal* reasoning formalisms are very common; see Blackburn for further references.

An Extension of the Modal Language for Knowledge

Let $m \in \mathbb{N}$ be a natural number. In this section, we extend the basic modal language describing knowledge of m agents by *nominals* and a *distinction operator* for every agent. Nominals are to denote *states* (as in usual hybrid logic), whereas applying the distinction operator associated with $i \in \{1,\ldots,m\}$ means, in particular, moving to any state that is complementary to i's actual *knowledge state* (which equals the set of all states indistinguishable to i, by definition).

Let PROP $= \{p,q,\ldots\}$ and NOM $= \{a,b,\ldots\}$ be two denumerable set of symbols called *proposition letters* and *nominals,* respectively. We assume that these sets are disjoint. Then we define the set WFF of *well-formed formulas* over PROP$\cup$NOM by the rule

$$\alpha ::= p \mid a \mid \neg\alpha \mid \alpha\wedge\beta \mid K_i\alpha \mid D_i\alpha,$$

where $i \in \{1,\ldots,m\}$. The modality K_i represents knowledge of agent i, and D_i the ability to recognize a knowledge state different from the actual one. The duals of K_i and D_i are denoted L_i and C_i, respectively. The missing boolean connectives $\top,\bot,\vee,\rightarrow,\leftrightarrow$ are treated as abbreviations, as needed.

We give next meaning to formulas. This is done with respect to suitable Kripke structures. In the following, let $\mathscr{P}(S)$ denote the powerset of a given set S.

Definition 1 (Extended Kripke structure) *Let $S \neq \emptyset$ be a set (of* states*). For every $i \in \{1,\ldots,m\}$, let R_i, Q_i be binary relations on S such that*

- *R_i is an equivalence, and*
- *for all $s,t \in S : (s\,Q_i\,t \iff \text{not } s\,R_i\,t)$.*

Moreover, let V : PROP $\cup$ NOM $\longrightarrow \mathscr{P}(S)$ be a mapping such that $V(a)$ is a singleton subset of S for all $a \in$ NOM. Then $\mathscr{M} := (S,R_1,\ldots,R_m,Q_1,\ldots,Q_m,V)$ is called an extended Kripke structure (for m agents) *(or, in short, an* EKS*).*

Now let an *EKS* $\mathscr{M} = (S,R_1,\ldots,R_m,Q_1,\ldots,Q_m,V)$ be given. We define the relation of satisfaction , $\models$, between states of $\mathscr{M}$ and formulas.

Definition 2 (Satisfaction and validity) *Let $\mathscr{M}$ be an* EKS *as above, and let $s \in S$ be a state. Then*

$$\begin{array}{lll}
\mathscr{M},s\models p & :\iff & s\in V(p)\\
\mathscr{M},s\models a & :\iff & s\in V(a)\\
\mathscr{M},s\models \neg\alpha & :\iff & \mathscr{M},s\not\models\alpha\\
\mathscr{M},s\models \alpha\wedge\beta & :\iff & \mathscr{M},s\models\alpha \text{ and } \mathscr{M},s\models\beta\\
\mathscr{M},s\models K_i\alpha & :\iff & \forall t:(s\,R_i\,t\Rightarrow \mathscr{M},t\models\alpha)\\
\mathscr{M},s\models D_i\alpha & :\iff & \forall t:(s\,Q_i\,t\Rightarrow \mathscr{M},t\models\alpha),
\end{array}$$

where $i \in \{1,\ldots,m\}$, $p \in$ PROP, $a \in$ NOM, and $\alpha,\beta \in$ WFF. In case $\mathscr{M},s\models\alpha$ is true we say that α holds in *$\mathscr{M}$* at *s. – A formula α is called* valid in *$\mathscr{M}$ iff it holds in $\mathscr{M}$ at all states. (Manner of writing: $\mathscr{M}\models\alpha$.)*

The following remark shows that the just defined language is rather expressive.

Remark 3 *For every $i \in \{1,\ldots,m\}$, the relation $R_i\cup Q_i$ is obviously* universal *(i.e., $R_i\cup Q_i = S\times S$). This implies that the* global modality A*, cf Blackburn, de Rijke, & Venema, Sec. 7.1, is definable in our language (by $\mathsf{A}\alpha :\equiv K_i\alpha\wedge D_i\alpha$). – Having the global modality to hand, the hybrid* satisfaction operator $@_a$ *belonging to the nominal a, cf Blackburn, de Rijke, & Venema, Sec. 7.3, can be defined as well (by $@_a\alpha :\equiv \mathsf{A}(a\rightarrow\alpha)$). – It follows that a lot of important frame properties (eg, irreflexivity) can be expressed in the language for knowledge and distinction; cf loc cit.*

We illustrate now the newly obtained expressive power by an example.

Example 4 *We remind the reader of the well-known* muddy children puzzle; *cf Fagin* et al.*, Sec. 2.3. In case m children are involved, the domain of the Kripke frame modelling that scenario equals the m–*dimensional unit cube *C_m. The relation of i–indistinguishability is defined by $(x_1,\ldots,x_m)\,R_i\,(y_1,\ldots,y_m) :\iff x_j = y_j$ for all $j\neq i$ ($i,j\in\{1,\ldots,m\}; x_j,y_j\in\{0,1\}$). Now, it is possible to specify this structure completely by a finite set of formulas of the new language. Eg, the formulas $p_i\wedge a\rightarrow K_i(p_i\rightarrow a)$, $\neg p_i\wedge a\rightarrow K_i(\neg p_i\rightarrow a)$ and $L_ip_i\wedge L_i\neg p_i$ express together the fact that every R_i–equivalence class consists of exactly two 'p_i–complementary' points (where p_i represents 'child i has a muddy forehead', and $a\in$ NOM is some nominal). The operator D_i is required to capture the size of C_m. (We omit the details concerning this, but mention that, in general, the new modality enables one to count (knowledge) states.)*

The Logic

First in this section we provide an axiom system, $\mathscr{A}$, for the set of all formulas valid in every *EKS*. Second, we prove the soundness and completeness of the logic Kn$^+$ derived from $\mathscr{A}$, with respect to the class of all the intended structures.

And finally, we show that Kn^+ is decidable and has, in the single-agent case, an NP–complete satisfiability problem.

Apart from all instances of propositional tautologies, the system $\mathscr{A}$ consists of eight formula schemata. For a start, we list the distribution axioms with respect to every modality:

$$\begin{array}{l} 1.\ K_i(\alpha \to \beta) \to (K_i\alpha \to K_i\beta) \\ 2.\ D_i(\alpha \to \beta) \to (D_i\alpha \to D_i\beta), \end{array}$$

where $i \in \{1,\ldots,m\}$ and $\alpha,\beta \in \mathrm{WFF}$. – Because of later requirements, the axioms of the following group are formulated in a purely hybrid way.

$$\begin{array}{ll} 3.\ a \to L_i a & 4.\ L_iL_i a \to L_i a \\ 5.\ L_i a \to K_iL_i a & 6.\ L_i a \vee C_i a \\ 7.\ L_i a \to \neg C_i a & 8.\ a \wedge \alpha \to K(a \to \alpha), \end{array}$$

where $1 \le i \le m$, $a \in \mathrm{NOM}$ and $\alpha \in \mathrm{WFF}$. – Note that the first three schemata of this list represent hybrid versions of the usual axioms of knowledge; eg, the schema 4, expressing the transitivity of agent i's accessibility relation R_i, corresponds to the axiom of positive introspection. – The first-order equivalents to the purely hybrid schemata 3 – 7 (eg, mutual exclusion of the relations R_i and Q_i, captured by Axiom 7) play their part in the proof of Theorem 6 below. The last axiom ensures there that the denotation of every nominal is single-valued.

By adding suitable proof rules we obtain the logic Kn^+. First, we have four commonly known Kn^+–rule schemata: *modus ponens,* both K_i– and D_i–*necessitation,* and the hybrid schema (NAME); as to the latter, cf Blackburn, de Rijke, & Venema, p 440. In addition, we need one new schema:

$$\text{(ENRICHMENT)} \quad \frac{O_i\,(a \wedge P_j(b \wedge \alpha)) \to \beta}{O_i(a \wedge P_j\alpha) \to \beta},$$

where $1 \le i,j \le m$, $O_i,P_j \in \{L_1,\ldots,L_m,C_1,\ldots,C_m\}$, $a,b \in \mathrm{NOM}$, $\alpha,\beta \in \mathrm{WFF}$, and b does not occur in a, α or β. The reader can now easily convince himself or herself that Kn^+ is sound with respect to the class of all *EKS*s.

Proposition 5 (Soundness) *Let* $\alpha \in \mathrm{WFF}$ *be a formula which is* Kn^+*–derivable. Then* α *is valid in all* EKS*s.*

The converse of Proposition 5 is also valid. Due to limited space, we cannot give a detailed proof of this result (neither of the issues following below).

Theorem 6 (Completeness) *Let* $\alpha \in \mathrm{WFF}$ *be valid in all* EKS*s. Then* α *is* Kn^+*–derivable.*

Proof. (Sketch) Let s be a maximal Kn^+–consistent set of formulas. Then s is called *named,* iff s contains some $a \in \mathrm{NOM}$. And s is called *enriched,* iff for all $i,j \in \{1,\ldots,m\}$, $O_i,P_j \in \{L_1,\ldots L_m,C_1,\ldots,C_m\}$, $a \in \mathrm{NOM}$, and $\alpha \in \mathrm{WFF}$, $O_i(a \wedge P_j\alpha) \in s$ implies $O_i\,(a \wedge P_j(b \wedge \alpha)) \in s$ for some $b \in \mathrm{NOM}$. – Let $\widetilde{\mathrm{N}}$ be a denumerable set of *new* nominals, and $\widetilde{\mathrm{F}}$ the set of formulas extended accordingly. Then we obtain the subsequent *Modified Lindenbaum Lemma*.

Lemma 7 *Every maximal consistent set* $s \subseteq \mathrm{WFF}$ *can be extended to a named and enriched maximal consistent set* $\widetilde{s} \subseteq \widetilde{\mathrm{F}}$.

Let a structure $\widetilde{\mathscr{M}}$ be defined as follows. The domain S of $\widetilde{\mathscr{M}}$ consists of all named points that are yielded from $\widetilde{s}$ by enrichment, and the relations of $\widetilde{\mathscr{M}}$ are the induced ones. Then we have the following *Existence Lemma*.

Lemma 8 *Let* $O \in \{L_i,C_i \mid i = 1,\ldots,m\}$*. Assume that* $s \in S$ *contains the formula* $O\alpha$*. Then some* $t \in S$ *exists that is* O*–accessible from* s *and contains* α*.*

Both lemmata can be proved with the aid of the new rule (among other things). – Now, because of the axioms of the second group above, the model $\widetilde{\mathscr{M}}$ turns out to be an *EKS*. Moreover, an appropriate *Truth Lemma* is valid for $\widetilde{\mathscr{M}}$. Letting α be a non-derivable formula and s contain $\neg\alpha$, we have, therefore, found a model falsifying α. In this way Theorem 6 is proved. □

Our next topic is decidability. In order to prove this property for Kn^+ we would like to use the standard tool applicable to many other logics of knowledge, viz *filtration;* cf Fagin *et al.*, Sec. 3.2. However, trying this we get very soon into serious difficulties with the distinguishability relation Q_i, for it is not possible to filtrate this relation in such a way that its disjointness with R_i can be preserved ($i \in \{1,\ldots,m\}$). The way out of this dilemma is to do some model surgery by changing the filter set suitably and, in particular, eliminating the distinction operators temporarily; cf the proceeding in case of the modal *difference operator* in Blackburn, de Rijke, & Venema, proof of Theorem 7.8.[1]

Theorem 9 (Decidability) *The set of all formulas satisfiable in some* EKS *is decidable.*

Proof. (Sketch) It suffices to establish the finite model property for Kn^+. So, let $\mathscr{M}$ be some *EKS* that realizes a given satisfiable formula α, for which we want to find a finite model, at some state. Let Σ be the set of all subformulas of α. Moreover, let $\sim_\Sigma$ be the usual filtration relation induced by Σ on the domain S of $\mathscr{M}$. The equivalence class of a point $s \in S$ with respect to $\sim_\Sigma$ is denoted $\bar{s}$. We choose an injective mapping ι from the (finite) set of all such classes into the set of all proposition letters not occurring in Σ. Furthermore, a new proposition letter p^i_β is assigned to every formula $D_i\beta \in \Sigma$ in such a way that $p^i_\beta \neq p^j_\gamma$ whenever $i \neq j$ or $\beta \neq \gamma$. For any $\delta \in \Sigma$, let δ' denote the result of substituting every subformula $D_i\beta$ of δ with p^i_β. Then we let

$$\begin{array}{rcl} \Sigma' & := & \{\delta' \mid \delta \in \Sigma\} \cup \\ & & \{p_{\iota(\bar{s})}, K_i p_{\iota(\bar{s})} \mid s \in S, 1 \le i \le m\} \cup \\ & & \{p^i_\beta \mid D_i\beta \in \Sigma, 1 \le i \le m\}. \end{array}$$

Σ' will be used as a filter set in a moment. Obviously, Σ' is D_i–free.

The model $\mathscr{M}$ is modified to the effect that the valuation is to make $p_{\iota(\bar{s})}$ true at exactly one point of $\bar{s}$, and p^i_α at exactly

[1]The distinction operator can, in fact, be viewed as a generalized difference operator: it allows to jump to points *outside the equivalence class of the actual one* for evaluating a given formula there (instead of jumping to a point *different from the actual one* for the same purpose).

the points where $D_i\alpha$ is true in model $\mathscr{M}$. Let this variant of $\mathscr{M}$ be designated $\mathscr{M}'$.

Now, $\mathscr{M}'$ is filtrated through Σ' as it is standard of the logic of knowledge. It ensues that the filtration $\bar{R}_i$ of the accessibility relation on $\mathscr{M}$ belonging to K_i, is an equivalence. Let $\sim_{\Sigma'}$ denote the filtration relation induced by Σ' on S. Then every equivalence class with respect to $\sim_\Sigma$ is divided into exactly two equivalence classes with respect to $\sim'_\Sigma$ (if the former class consists of more than one point). Moreover, we can prove that these two smaller classes are not $\bar{R}_i$–connected.

Finally, we can show by a suitable induction that the filtrated model just described is semantically equivalent to $\mathscr{M}$ with respect to Σ. This proves the desired finite model property. $\square$

In case of at least two knowers the ordinary logic of knowledge is PSPACE–complete; cf Halpern & Moses, Th. 6.17. In contrast, the satisfiability problem is NP–complete in the single-agent case; cf Fagin *et al.*, Sec. 3.6. It is now natural to ask whether NP–completeness is the complexity of the DL–satisfiability problem in case $m = 1$ as well. The answer to this question turns out to be 'yes'. – For the rest of this section we assume that $m = 1$.

Theorem 10 (Complexity) *The set of all formulas satisfiable in some* EKS *for one agent, is* NP*–complete.*

Proof. (Sketch) Let $\alpha \in$ WFF be a satisfiable formula. That is, there exists a finite *EKS* $\mathscr{M} = (S,R,Q,V)$[2] and a state $s \in S$ such that $\mathscr{M},s \models \alpha$. We construct now an *EKS* $\mathscr{M}' = (S',R',Q',V')$ such that α holds at some state $s \in S'$ and the size of S' is polynomial in the length $|\alpha|$ of α. This suffices to prove the theorem.

Let $\mathscr{K} = \{K\beta_1,\dots,K\beta_k\}$ be the set of all subformulas of α prefixed by K. Correspondingly, let $\mathscr{D} = \{D\gamma_1,\dots,D\gamma_l\}$ be the set of all subformulas of α prefixed by D. We generate S' in steps by the following procedure:

BEGIN
$S_0 := \{s\}$; STOP:=FALSE;
WHILE NOT STOP DO
  IF there are $x \in S_0$ and $i \in \{1,\dots,l\}$ such that
    $\mathscr{M},x \not\models D\gamma_i$
  AND there is no $y \in S_0$ such that $x\,Q\,y$ and
    $\mathscr{M},y \not\models \gamma_i$
    THEN CHOOSE $z \in S$ such that $\mathscr{M},z \not\models \gamma_i$;
      $S_0 := S_0 \cup \{z\}$
    ELSE STOP:=TRUE;
FOR ALL $j \in \{1,\dots,k\}$ AND $x \in S_0$ such that $\mathscr{M},x \not\models K\beta_j$ DO
  CHOOSE some $z \in S$ such that $x\,R\,z$ and
  $\mathscr{M},z \not\models \beta_j$;
  $S_0 := S_0 \cup \{z\}$
END

Then we define $S' := S_0$, $R' := (S' \times S') \cap R$, and Q' the relation complementary to R' in $S' \times S'$. Moreover, V' is to be the restriction of V to $\mathscr{P}(S')$ in the range.[3] Obviously, the structure $\mathscr{M}' := (S',R',Q',V')$ is an *EKS*.

For the proof of the desired bound on the size of S' it is important to note that every formula from $\mathscr{D}$ contributes at most two new points to S_0 during the WHILE-loop. In fact, it is not hard to convince oneself that the second conjunct in the condition of the IF–clause can never become true afterwards. Thus S' contains at most $2l \cdot k$ points. That is, the cardinality of S' can be estimated by $2 \cdot |\alpha|^2$.

Because of the above choice of states, we obtain the following assertion by an induction argument (not carried out here): for all subformulas δ of α and states $s' \in S'$ it holds that

$$\mathscr{M}',s' \models \delta \iff \mathscr{M},s' \models \delta.$$

From that we conclude $\mathscr{M},s \models \alpha$. Now, the proof of the theorem is completed by an obvious guess-and-check algorithm, as usual. $\square$

Note that the size of (the domain of) the model we have just constructed, is quadratic in $|\alpha|$. In contrast, a model of linear size exists for every satisfiable formula of the usual logic of knowledge of one agent; cf Fagin *et al.*, Prop. 3.6.2. We suspect that this is not always true if the distinction operator is present.

Incorporating the effort operator

In this section we extend the logic of knowledge and distinction by a spatio-temporal dimension.

Two alternative ways of temporalizing a language $\mathscr{L}$ for knowledge appeared in the literature up to now: time can be integrated into $\mathscr{L}$ either *explicitly* or *implicitly*. To a great extent, the existing formalisms for reasoning about knowledge follow the first variant; cf Fagin *et al.*, Sec. 4.3, or (more comprehensively) Halpern, van der Meyden, & Vardi. Because of our interest in the *spatial* properties of knowledge (see below) we focus on the second approach here, which is due to Dabrowski, Moss, & Parikh.

It is basically assumed in the latter paper that spending effort to acquire knowledge (eg, by a computation or an experiment) can never result in a loss of knowledge.[4] Consequently, the knowledge state of an agent, i.e., the set of states indistinguishable to him or her, keeps on decreasing, or at least not increasing, during the acquisition procedure. The effort operator is then expressed by an (implicitly temporal) modality $\Box$ interacting with the knowledge operator in an easily comprehensible way. Actually, the interplay of these two connectives is described by the axiom schema $K\Box\alpha \to \Box K\alpha$ (in case of a single agent).

Dabrowski, Moss, & Parikh called their logical system of knowledge and effort TOPOLOGIC, because it can also

[2]We omit the index '1' here and in the following.

[3]Nominals not denoting an element of S' are interpreted arbitrarily.

[4]This natural requirement is, therefore, closely related to the notion of *perfect recall* from classical logic of knowledge; cf Fagin *et al.*, Sec. 4.4.4. In fact, perfect recall means that agents do not forget, or, in other words, knowledge cannot be lost in the course of time. Assuming perfect recall is quite reasonable in many contexts, at least from a theoretical point of view; cf, eg, van der Meyden & Shilov, or Dixon & Fisher.

be applied to qualitative spatial modelling. In fact, certain elementary properties of points in a space can be treated formally with the aid of TOPOLOGIC. By way of example we consider *closeness*. This notion is connected with knowledge according to the fact that acquiring knowledge means *shrinking* within a system of sets (viz the various knowledge states of the agent), thus *approximating* points (viz the states of complete knowledge).

While some important classes of spaces, eg, topological, treelike, or directed ones, can be captured by TOPOLOGIC (cf Georgatos 94, Georgatos 97, and Weiss & Parikh, respectively), more expressive languages are needed for other naturally arising spatial structures. In the following, we are concerned with *complement closed spaces,* which prove to be just the models for the desired spatio-temporalized logic of knowledge and distinction. (Note that the systems mentioned in the final section of the introduction are not adequate for complement closed spaces.)

By adding nominals and a distinction operator we extend now the language underlying TOPOLOGIC, $\mathcal{L}_t$, to the framework of evolving knowledge. Or, to say it the other way round, we extend the language for knowledge and distinction by a spatio-temporal operator $\Box$ representing effort. For the sake of a clear presentation we confine ourselves to the single-agent case.

According to the additional dimension the set of nominals is divided into two disjoint subsets, $\mathrm{NOM} = \mathrm{NOM}_1 \cup \mathrm{NOM}_2$. The elements $a, b, \ldots \in \mathrm{NOM}_1$ denote *states* as above (thus the static entities of our models), whereas the elements $A, B, \ldots \in \mathrm{NOM}_2$ name *sets* (which prove to be the objects changing in the course of time). Thus we define the set $\widetilde{\mathrm{F}}$ of formulas by the rule

$$\alpha ::= p \mid a \mid A \mid \neg\alpha \mid \alpha \wedge \beta \mid K\alpha \mid D\alpha \mid \Box\alpha.$$

The dual of $\Box$ is denoted $\Diamond$. – In order to define the semantics of our combined language we first specify precisely the appropriate domains.

Definition 11 (Complement closed spaces) *1. Let S be a non-empty set and $\mathcal{O} \subseteq \mathcal{P}(S)$ a set of subsets of S satisfying*

- $S \in \mathcal{O}$, *and* $X \in \mathcal{O}$ *implies* $S \setminus X \in \mathcal{O}$.

Then the pair $(S, \mathcal{O})$ is called a complement closed *or* cc set frame*.*

2. *Let $\mathcal{S} := (S, \mathcal{O})$ be a cc set frame. Then the set $\mathcal{N}_{\mathcal{S}} := \{s, U \mid s \in U \text{ and } U \in \mathcal{O}\}$ is called the* set of neighbourhood situations *of $\mathcal{S}$.*
3. *Let $\mathcal{S} = (S, \mathcal{O})$ be a cc set frame. A mapping $V : \mathrm{PROP} \cup \mathrm{NOM} \longrightarrow \mathcal{P}(S)$ such that*
   - *$V(a)$ is a singleton subset of S for all $a \in \mathrm{NOM}_1$ and*
   - *$V(A) \in \mathcal{O}$ for all $A \in \mathrm{NOM}_2$*

   *is called an $\mathcal{S}$–*valuation*.*
4. *A* complement closed *or* cc space *is a triple $(S, \mathcal{O}, V)$ where $(S, \mathcal{O})$ is a cc set frame and V an $\mathcal{S}$–valuation.*

It is now clear how a cc space emerges from a given *EKS:* as every element of $\mathcal{O}$ is to represent some knowledge state of the agent and spending effort means shrinking the knowledge state, time is 'encoded' in the spatial structure $(\mathcal{O}, \subseteq)$. In other words, the effort operator is reflected by the set inclusion relation on $\mathcal{O}$. This is made explicit through the last clause of the next definition, in which the relation of satisfaction in cc spaces is introduced. This relation holds between neighbourhood situations and formulas (as it is common for $\mathcal{L}_t$).

Definition 12 (Satisfaction and validity II) *Let a cc space $\mathcal{M} = (S, \mathcal{O}, V)$ and a neighbourhood situation s, U of the cc set frame $(S, \mathcal{O})$ be given. Then*

$$\begin{array}{lll}
s,U \models_{\mathcal{M}} p & :\Longleftrightarrow & s \in V(p) \\
s,U \models_{\mathcal{M}} a & :\Longleftrightarrow & s \in V(a) \\
s,U \models_{\mathcal{M}} A & :\Longleftrightarrow & V(A) = U \\
s,U \models_{\mathcal{M}} \neg\alpha & :\Longleftrightarrow & s,U \not\models_{\mathcal{M}} \alpha \\
s,U \models_{\mathcal{M}} \alpha \wedge \beta & :\Longleftrightarrow & s,U \models_{\mathcal{M}} \alpha \text{ and } s,U \models_{\mathcal{M}} \beta \\
s,U \models_{\mathcal{M}} K\alpha & :\Longleftrightarrow & \forall t \in U : t,U \models_{\mathcal{M}} \alpha \\
s,U \models_{\mathcal{M}} D\alpha & :\Longleftrightarrow & \forall t \in S \setminus U : t, S \setminus U \models_{\mathcal{M}} \alpha \\
s,U \models_{\mathcal{M}} \Box\alpha & :\Longleftrightarrow & \forall U' \in \mathcal{O} : \left\{ \begin{array}{l} s \in U' \subseteq U \\ \Rightarrow \\ s,U' \models_{\mathcal{M}} \alpha, \end{array} \right.
\end{array}$$

for all $p \in \mathrm{PROP}$, $a \in \mathrm{NOM}_1$, $A \in \mathrm{NOM}_2$, and $\alpha, \beta \in \widetilde{\mathrm{F}}$. In case $s, U \models_{\mathcal{M}} \alpha$ is true we say that α holds *in $\mathcal{M}$* at *the neighbourhood situation s, U. – A formula α is called* valid in *$\mathcal{M}$ iff it holds in $\mathcal{M}$ at every neighbourhood situation.*

We should point to a a peculiarity of the just defined language here. Note that the meaning of both proposition letters and nominals depends only on states (as it is the case for the standard logic of knowledge and time, too). This fact is mirrored in two special axioms of the logic presented below (Axioms 10 and 11).

Example 13 *Let $\mathcal{C}$ be the set of all infinite 0–1–sequences. A basis $\mathcal{B}$ for the distinguished topology on $\mathcal{C}$ is determined by the set of all finite initial segments of elements of $\mathcal{C}$. Let $\mathcal{O} := \{X \subseteq \mathcal{C} \mid X \in \mathcal{B} \text{ or } \mathcal{C} \setminus X \in \mathcal{B}\}$. Then, $\mathcal{S} := (\mathcal{C}, \mathcal{O})$ is obviously a cc set frame, which can be viewed as a spatio-temporal structure in a natural way. In fact, $\mathcal{S}$ can be depicted as the full infinite binary tree such that every $X \in \mathcal{B}$ is associated with the node by which it is determined. The levels of the tree represent now the temporal dimension, while the sets attributed to the nodes as well as their complements, represent the spatial one. This view is justified since $\mathcal{S}$ models, in particular, procedures computing binary streams. And with the aid of the formulas of our language one can specify certain properties of such computations. For example, if a procedure P computes some real number ρ (i.e., the output of P encodes a fast-converging Cauchy sequence having limit ρ; cf Weihrauch) and ρ is different from, eg, π, then* one will know this eventually. *Thus the formula $\Diamond K C\pi$ is valid in a suitable cc space based on $\mathcal{S}$.*

In the remaining part of this section we first give a system of axioms for complement closed spaces. Afterwards we touch on the question of completeness of the logic arising from that.

The axiom schemata are arranged in four groups. The first one consists of all the axioms from the previous section (where $m = 1$ and the index '1' is omitted). – By the second group, the effort operator is axiomatized. It turns out that $\Box$

is, in particular, an S4–modality (due to Axioms 9, 12 and 13).

9. $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$
10. $(p \to \Box p) \wedge (\Diamond p \to p)$
11. $(a \to \Box a) \wedge (\Diamond a \to a)$
12. $\Box\alpha \to \alpha$
13. $\Box\alpha \to \Box\Box\alpha$,

where $p \in \mathrm{PROP}$, $a \in \mathrm{NOM}_1$ and $\alpha, \beta \in \widetilde{\mathrm{F}}$. – All the axioms contained in the third group concern the interaction of the modal operators.

14. $K\Box\alpha \to \Box K\alpha$
15. $K\Box\alpha \wedge D\alpha \to \Box D\alpha$
16. $\Box(K\alpha \wedge D\alpha) \to D\Box\alpha$,

where $\alpha \in \widetilde{\mathrm{F}}$. Note that 14 is the commutation axiom for knowledge and effort mentioned already in the introduction to this section. Axioms 15 and 16 describe how effort and distinction commute. – Finally, some further axioms are needed in the proof of the Completeness Theorem below. This is the place where names of sets come into play.

17. $A \to KA$
18. $K\Box(A \wedge L\alpha \to L\beta) \vee K\Box(A \wedge L\beta \to L\alpha)$
19. $K(\Diamond B \to \Diamond A) \wedge L\Diamond B \to \Box(A \to L\Diamond B)$
20. $K\Diamond A \to A$,

where $A, B \in \mathrm{NOM}_2$ and $\alpha, \beta \in \widetilde{\mathrm{F}}$. – The axioms of this last group are not easy to understand at first glance. Actually, they provide for the necessary properties of the accessibility relations belonging to the modalities K and $\Box$ on the canonical model, so that really a cc set frame structure can be guaranteed there.

A logical system called STKn^+ (where 'ST' designates the additional spatio-temporal component), is now obtained from this list by adding appropriate proof rules (notably, suitable modifications of the (ENRICHMENT)–rule above). This system proves to be sound and complete with respect to the class of all cc spaces.

Theorem 14 (Completeness II) *A formula $\alpha \in \widetilde{\mathrm{F}}$ is valid in all cc spaces, iff it is STKn^+–derivable.*

The proof of Theorem 14 will be contained in the full version of this paper.

Concluding remarks

In this final section, we give a short summary of the paper and point to future research. – We extended the logic of knowledge of m agents by *nominals* and a *distinction operator* for every agent. We proved then a corresponding completeness, decidability and complexity result, respectively. After that, we augmented the system by an *effort operator* having both a strong temporal and spatial flavour. We modelled distinguishability in this richer framework as well. In fact, we obtained a completeness theorem for the logic STKn^+ with respect to the class of all *complement closed spaces*. – A further-reaching study of STKn^+ and, in particular, its effectivity properties, has to be postponed. But we believe that a refinement of the methods applied to Kn^+ will show that STKn^+ is also a decidable logic (though of rather high complexity).

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