Non-Uniform Belief in Expected Utilities in Interval Decision Analysis

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Abstract

This paper demonstrates that second-order calculations add information about expected utilities when modeling imprecise information in decision models as intervals and employing the principle of maximizing the expected utility. Furthermore, due to the resulting warp in the distribution of belief over the intervals of expected utilities, the conservative Γ-maximin decision rule seems to be unnecessarily conservative and pessimistic as the belief in neighborhoods of points near interval boundaries is significantly lower than in neighborhoods near the centre. Due to this, a generalized expected utility is proposed.

Introduction

In the basic model of Bayesian decision analysis, a decision maker is to choose an alternative/action from a non-empty, finite set $\mathcal{A} = \{A_1, \ldots, A_n\}$ of possible alternatives. Each alternative may end up with a finite set of consequences, and the resulting consequence of each alternative depends on the true (but probably unknown) state of nature $\theta \in \Theta = \{\theta_1, \ldots, \theta_i, \ldots, \theta_k\}$. The corresponding outcome is then evaluated by means of a utility function u satisfying

$$u(\mathcal{A} \times \Theta) \to \mathbb{R}$$
$$(A, \theta) \mapsto u(A, \theta)$$

Since the true state of nature is unknown, the model asserts the knowledge of the probability distribution $P(.)$ on $(\Theta, \mathcal{P}o(\Theta))$. The alternative A to choose is then the alternative which *maximizes the expected utility*, for all $A_i \in \mathcal{A}$. This selection procedure is commonly referred to as *the principle of maximizing the expected utility*, and is argued for in (von Neumann and Morgenstern 1947) and (Savage 1972), as it is implied from widely accepted axiom systems defining formal models of rationality. Thus, a preference ordering relation $\succeq$ on $\mathcal{A}$ is implied from the magnitudes of the different alternatives' expected utility.

Definition 1 *The principle of maximizing the expected utility is accepted if a decision making agent chooses the alternative A^*, whenever*

$$A^* = \arg\max_A(\mathbf{E}(A))$$

where

$$\mathbf{E}(A) = \sum_{\theta \in \Theta} u(A, \theta) \cdot P(\theta)$$

for all $A \in \mathcal{A}$.

An important relation in Bayesian decision analysis is then $A_i \succeq A_j \iff \mathbf{E}(A_i) \geq \mathbf{E}(A_j)$ for any two alternatives $A_i, A_j \in \mathcal{A}$. However, even if a decision maker is able to discriminate between different probabilities, very often complete, adequate, and precise information is missing. The requirement to provide numerically precise information in such models has often been considered unrealistic in real-life decision situations.

Due to this lack of information in real-life decisions, later years of rather intense activities in the decision analysis area have been focused on developing models and frameworks handling this imprecision. Approaches based on upper and lower probabilities (Dempster 1967), sets of probability measures (Levi 1974; Walley 1991), and interval probabilities (Weichselberger 1999), have prevailed. However, the latter still do not admit for discrimination between different beliefs in different values, and selection problems emerge when intervals of expected values are overlapping.

The expected utility of the alternatives are straightforwardly calculated when all components are numerically precise. When the domains of the terms are given in convex sets of probability and utility measures, this is not as straightforward. Proposed methods mainly involve finding lower and upper bounds on expected utilities, derived from lower and upper bounds on the probability and utility variables, yielding interval-valued expected utilities to compare. However, when comparing these directly, many decision problems will lead to a partial preference order on $\mathcal{A}$ while the expected utility intervals are overlapping. Due to this problem, some authors suggest the Γ-maximin[1] principle (Berger 1984; Augustin 2001; Vidakovic 2000).

Definition 2 *The Γ-maximin principle is accepted if a decision making agent chooses the alternative A^*, whenever*

$$A^* = \arg\max_A(\inf(\mathbf{E}(A)))$$

[1]When loss functions are used instead of utility functions, the rule is labeled Γ-minimax.

where

$$\inf \mathbf{E}(A) = \inf \sum_{\theta \in \Theta} u(A,\theta) \cdot P(\theta)$$

for all $A \in \mathcal{A}$.

Belief Distributions

To model imprecision, probability and value estimates can be expressed by sets of probability distributions and utility functions.

Definition 3 *Given a finite sample space Θ and a σ-field Γ of random events in Θ, the probability $P(\theta_i)$ of state θ_i is expressed as the variable p_i bounded by the following constraints*

$$\sup P(\theta_i) = 1 - \inf P(\neg\theta_i)$$
$$\sum_{\theta_i \in \Theta} p_i = 1$$

Definition 4 *Let L be a set of mappings $L = \{u(\mathcal{A} \times \Theta) \to [0,1]\}$ where all $u \in L$ are increasing. Given a subset $U \subset L$ such that $u_{ij} = \{u(A_i,\theta_j)u \in U\}$ is a closed interval, then the interval valued utilities are defined in terms of the closed intervals u_{ij}.*

However, to enable a better qualification of the various possible functions, second-order estimates, such as distributions expressing various beliefs, can be defined over an n-dimensional space, where each dimension corresponds to possible probabilities of an event or utilities of a consequence. In this way, the distributions can be used to express varying strength of beliefs in different probability or utility vectors.

Definition 5 *Let a unit cube be represented by $B = [0,1]^k$. A belief distribution over B is a positive distribution F defined on B such that*

$$\int_B F(x)\, dV_b(x) = 1$$

where V_B is a k-dimensional Lebesgue measure on B. The set of all belief distributions over B is denoted by $BD(B)$. In some cases, we will denote a unit cube by ${}^kB = (b_1,\ldots,b_k)$ to make the number of dimensions and the labels of the dimensions clearer.

It is useful to distinguish between unit cubes representing possible first-order probability distributions on a set of mutually exclusive events, and unit cubes representing first-order utility distributions.

Definition 6 *A P-unit cube is a unit cube $B_P = [0,1]^k$ where $F(x) > 0 \Rightarrow \sum_{i=1}^k x_i = 1$. A V-unit cube B_V lacks this condition.*

If there is positive support on a convex set in a B_P-cube, the set of possible probability distributions correspond to coherent probability (Walley 1991) and feasible interval probability (Weichselberger 1999).

Local Distributions

The only information available in a given decision situation often is local over a subset of lower dimension. As is argued in (Ekenberg, Danielson, and Larsson 2005), this is captured by the central concept of S-projections.

Definition 7 *Let $B = (b_1,\ldots,b_k)$ and $A = (b_{i_1},\ldots,b_{i_s}), i_j \in \{1,\ldots,k\}$ be unit cubes. Let $F \in BD(B)$, and let*

$$f_A(x) = \int_{B\setminus A} F(x)\, dV_{B\setminus A}(x)$$

Then f_A is the S-projection of F on A.

It can be shown that an S-projection of a belief distribution always is a belief distribution (Ekenberg, Danielson, and Thorbiörnson 2005). A special kind of projection is when belief distributions over the axes of a unit cube B are S-projections of a belief distribution over B.

Definition 8 *Given a unit cube $B = (b_1,\ldots,b_k)$ and a distribution $F \in BD(B)$. Then the distribution $f_i(x_i)$ obtained by*

$$f_i(x_i) = \int_{\bar{B}_i} F(x)\, dV_{\bar{B}_i}(x)$$

where $\bar{B}_i = (b_1,\ldots,b_{i-1},b_{i+1},\ldots,b_k)$, is a belief distribution over the b_i-axis. Such a distribution will be referred to as a local distribution.

The rationale behind the local distributions is that the resulting belief in, e.g., the point 0.1 in a sense is the sum of all beliefs over the vectors where 0.1 is the first component, i.e., the totality of belief in this point.

Distributions over Expected Utility Spaces

Let $B_V = [0,1]^k$ be a unit cube with an associated belief distribution F. Then the local distributions $f_i(x_i)$ can be calculated through the concept of S-projections. Then the following rule for the distribution over the sum $z = \sum x_i$ has well-defined semantics.

Definition 9 *The belief distribution $h(z)$ on a sum $z = \sum_{i=1}^k x_i$ of a set of (independent) variables $\{x_i\}_{i=1}^k$, associated with belief distributions $f_i(x_i)$, is given by evaluating the integral*

$$h(z) = \int_{\mathcal{S}_z} \prod_{i=1}^k f_i(x_i)\, dS_z$$

where $\mathcal{S}_z = \{(x_1,\ldots,x_k)z = \sum_{i=1}^k x_i\} \subset [0,1]^k$, $0 \le z \le k$, and dS_z is the surface area element.

Theorem 1 *The function $h(z)$ given in Definition 9 is a belief distribution.*

Proof.

$$\int_0^1 h(z)\,dz = \int_0^1 \Big(\int_{\mathcal{S}_z} \prod_{i=1}^k f_i(x_i)\,dS_z\Big)dz =$$

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^k f_i(x_i)\,dx_1 \ldots dx_k =$$

$$\Big(\int_0^1 f_1(x_1)\,dx_1\Big) \cdot \ldots \cdot \Big(\int_0^1 f_k(x_k)\,dx_k\Big) = 1^k = 1$$

□

Definition 10 *Given a belief distribution F over a cube B, the centroid F_c of F is*

$$F_c = \int_B xF(x)\,dV_B(x)$$

where V_B is some k-dimensional Lebesgue measure on B.

Centroids are preserved under projections, in the sense that the centroid of an S-projection of F on $A \subset \operatorname{im} F$ share coordinates with the centroid of F (Ekenberg, Danielson, and Thorbiörnson 2005). Thus, a local distribution of a belief distribution preserves the centroid in that dimension. From (Ekenberg, Danielson, and Thorbiörnson 2005), it is clear that the centroid is multiplicative, i.e., for two local belief distributions f and g on independent variables x and y with centroids f_c and g_c, the centroid of the distribution on their product $x \cdot y$ is given by $f_c \cdot g_c$. An equally important property in decision analysis is that the centroid is additive as well.

Theorem 2 *The horizontal centroid h_c of h, where h is defined as in Definition 9, is the sum of the horizontal centroids of each f_i, i.e., the horizontal centroid is additive.*

Proof.

$$h_c = \int_0^1 z \cdot h(z)\,dz = \int_0^1 z\Big(\int_{\mathcal{S}_z} \prod_{i=1}^k f_i(x_i)\,dS_z\Big)\,dz =$$

$$\int_0^1 \Big(\int_{\mathcal{S}_z} (\sum_{i=1}^k x_i) \prod_{i=1}^k f_i(x_i)\,dS_z\Big)\,dz =$$

$$\int \cdots \int_{[0,1]^k} (\sum_{i=1}^k x_i) \prod_{i=1}^k f_i(x_i)\,dx_1 \ldots dx_k =$$

$$\int \cdots \int_{[0,1]^k} x_1 \cdot \Big(\prod_{i=1}^k f_i(x_i)\Big) + \ldots +$$

$$+ x_k \cdot \Big(\prod_{i=1}^k f_i(x_i)\Big)\,dx_1 \ldots dx_k =$$

$$\int \cdots \int_{[0,1]^k} x_1 \cdot \Big(\prod_{i=1}^k f_i(x_i)\Big)dx_1 \ldots dx_k + \ldots +$$

$$+ \int \cdots \int_{[0,1]^k} x_k \cdot \Big(\prod_{i=1}^k f_i(x_i)\Big)dx_1 \ldots dx_k =$$

$$\Big(\int_0^1 x_1 \cdot f_1(x_1)\,dx_1\Big) \cdot \ldots \cdot \Big(\int_0^1 f_k(x_k)\,dx_k\Big) + \ldots +$$

$$+ \Big(\int_0^1 f_1(x_1)\,dx_1\Big) \cdot \ldots \cdot \Big(\int_0^1 x_k \cdot f_k(x_k)\,dx_k\Big) =$$

$$f_{1_c} \cdot 1^{k-1} + \ldots + f_{k_c} \cdot 1^{k-1} = \sum_{i=1}^k f_{i_c}$$

□

When calculating expected utilities, each term is a product of a probability and a utility. Since we handle probabilities and utilities independently, the belief in each term is a product of beliefs. From this reasoning, the following rule of for the distribution over the line segment of possible expected values follows.

Definition 11 *Given one probability unit cube $B_P = (p_1, \ldots, p_k)$ and a utility unit cube $B_V = (u_1, \ldots, u_k)$, an expected utility unit cube, denoted B_{EU}, is the cross product $B_{EU} = B_P \times B_V$.*

Thus, given any point $e = (p_1, u_1, p_2, u_2, \ldots, p_k, u_k) \in B_{EU}$, there is an expected utility $z \in [0, 1]$ such that $z = p_1 \cdot u_1 + p_2 \cdot u_2 + \ldots + p_k \cdot u_k$ whenever $\sum_{i=1}^k p_i = 1$.

Definition 12 *The belief distribution $h(z)$ on a sum $z = \sum_{i=1}^k p_i u_i$ of a set of products of two variables $\{p_i \cdot u_i\}_{i=1}^k$, associated with global belief distributions $F(p_1, \ldots, p_k)$ and $G(u_1, \ldots, u_k)$ respectively, is given by evaluating the integral*

$$h(z) = \int_{\mathcal{S}_z} F(p)G(u)\,dS_z$$

where $\mathcal{S}_z = \{(p_1, u_1, \ldots, p_k, u_k) z = \sum_{i=1}^k p_i \cdot u_i\} \subset [0, 1]^{2k}$, $0 \le z \le 1$, and dS_z is the surface area element.[2]

Theorem 3 *The function $h(z)$ in Definition 12 is a belief distribution.*

Proof.

$$\int_0^1 h(z)\,dz = \int_0^1 \Big(\int_{\mathcal{S}_z} F(p)G(u)\,dS_z\Big)\,dz =$$

$$\int_{B_{EU}} F(p)G(u)\,dV_{B_{EU}} =$$

$$\Big(\int_{B_P} F(p)\,dV_{B_P}\Big)\Big(\int_{B_V} G(u)\,dV_{B_V}\Big) = 1 \cdot 1$$

□

Theorem 4 *The horizontal centroid h_c of h, where h is defined as in Definition 12, is the sum of the centroid products $f_i \cdot g_i$, i.e., the horizontal centroid is additive and multiplicative.*

[2]In this definition we use p to denote the vector components $(p_1, \ldots, p_k)$ which is a subset of the vector components included in $e = (p_1, u_1, \ldots, p_k, u_k)$. The same meaning applies to the vector u.

Proof. From Theorem 2, the following can be derived.

$$h_c = \int_0^1 z \cdot h(z)\, dz = \int_0^1 z\Big(\int_{\mathcal{S}_z} F(p)\cdot G(u)\, dS_z\Big)\, dz = \int_0^1 (\sum_{i=1}^{k} p_i u_i)\Big(\int_{\mathcal{S}_z} F(p)\cdot G(u)\, dS_z\Big)\, dz =$$

$$\Bigg(\Big(\int_0^1 p_1 \cdot f_1(p_1)\, dp_1\Big)\Big(\int_0^1 u_1 \cdot g_1(u_1)\, du_1\Big)\cdot \ldots \cdot \Big(\int_0^1 f_k(p_k)\, dp_k\Big)\Big(\int_0^1 g_k(u_k)\, du_k\Big)\Bigg) + \ldots +$$

$$+\Bigg(\Big(\int_0^1 f_1(p_1)\, dp_1\Big)\Big(\int_0^1 g_1(u_1)\, du_1\Big)\cdot \ldots \cdot \Big(\int_0^1 p_k \cdot f_k(p_k)\, dp_k\Big)\Big(\int_0^1 u_k \cdot g_k(u_k)\, du_k\Big)\Bigg) = \sum_{i=1}^{k} f_{i_c} g_{i_c}$$

□

Belief on Expected Utilities

When making pure interval assignments, there is still a belief in all intermediate points, even if it is sometimes low (zero in the extreme case). For presentational purposes, we assume below that the beliefs in the feasible values are at least *uniformly distributed* and we show effects of this assumption when exposing variables for binary operators employed when calculating expected utilities. This does not mean that the warp effects discussed are present only in these cases.

Example

The following example considers the rudimentary expected utility calculation where $\Theta = \{\theta_1, \theta_2\}$, i.e., there are two uncertain outcomes. Assume that the belief is uniformly distributed over all possible probability and utility distributions. In this case we have two 2-dimensional unit cubes, $B_V = (u_1, u_2)$ and $B_P = (p_1, p_2)$. Over B_V the belief distribution is $G(u) = 1$, since the belief is uniformly distributed. Regarding B_P, the belief distribution over the surface $p_1 + p_2 = 1$ is $F(p) = \frac{1}{\sqrt{2}}$ (this surface is a line with length $\sqrt{2}$). Through the transformation $(p_1, p_2) = P(p) = (\frac{p}{\sqrt{2}}, \frac{\sqrt{2}-p}{\sqrt{2}}), 0 \le p \le \sqrt{2}$, B_P may be replaced with the line segment $[0, \sqrt{2}]$. Furthermore, from the concept of S-projections, the local distributions over the axis is $f_1(p_1) = f_2(p_2) = g_1(u_1) = g_2(u_2) = 1$.

Consider the 3-dimensional space obtained by $B_V \times [0, \sqrt{2}]$, in this space, each vector of points (p, u_1, u_2) now represents a possible expected utility, i.e., given any point in this space there is an expected utility $z \in [0, 1]$ such that $\frac{p}{\sqrt{2}}u_1 + \frac{\sqrt{2}-p}{\sqrt{2}}u_2 = z$. The belief in a given z is then obtained by summing up all beliefs of the vectors in the space $[0, 1] \times [0, \sqrt{2}]$ which fulfill $z = \frac{p}{\sqrt{2}}u_1 + \frac{\sqrt{2}-p}{\sqrt{2}}u_2$, i.e., we have a surface integral.

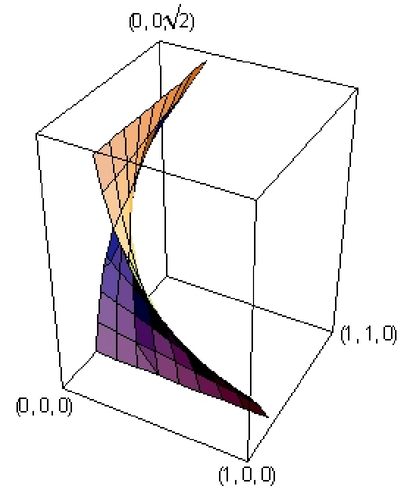

Figure 1 Surface of points fulfilling $z = 0.25$

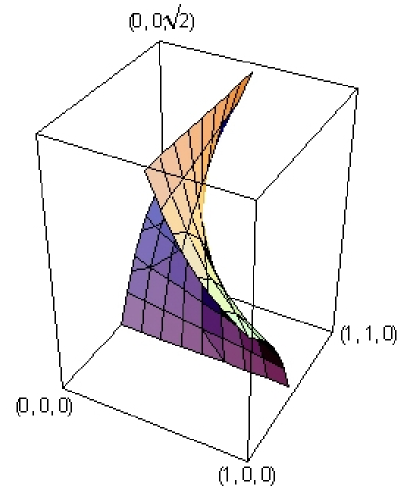

Figure 2 Surface of points fulfilling $z = 0.5$

In general, finding such areas through calculus is performed through parameterization. In this case, we wish to find the area of the surface $\mathcal{S} \in [0, 1] \times [0, \sqrt{2}]$ given by $\mathbf{F}(u_1, u_2, p) = \sqrt{2}(z - u_2) - z(u_1 - u_2) = 0$, where $z \in [0, 1]$ represents a possible expected utility.

We may regard the graph $p = g(u_1, u_2)$ as a parametric surface $\mathcal{S}$ with parameterization

$$u_1 = x, u_2 = y, p = \frac{\sqrt{2}(z - u_2)}{u_1 - u_2}$$

The domain D of $g(u_1, u_2)$ in the u_1u_2-plane where $0 \le g(u_1, u_2) \le \sqrt{2}$ is $D = \{(u_1, u_2) 0 \le u_1 \le z \Leftrightarrow z \le u_1 \le 1\} \cup \{(u_1, u_2) z \le u_1 \le 1 \Leftrightarrow 0 \le u_2 \le z\}$. The area of the surface over the two subsets of the domain D is equal due to symmetry, and the parametric region coincides with D, so the surface integral over $\mathcal{S}$ can be expressed as a double integral over D,

$$h(z) = \iint_{\mathcal{S}} dS =$$

$$2\int_0^z \int_z^1 \sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\, dydx =$$

$$2\int_0^z \int_z^1 \sqrt{1+\left(\frac{\sqrt{2}(z-u_2)}{(u_1-u_2)^2}\right)^2+\left(\frac{\sqrt{2}(z-u_2)}{(u_1-u_2)^2}-\frac{\sqrt{2}}{u_1-u_2}\right)^2}\, dydx=$$

$$2\int_0^z \int_z^1 \sqrt{1+2\left(\frac{(z-u_2)}{(u_1-u_2)^2}\right)^2+\left(\frac{(z-u_1)}{(u_1-u_2)^2}\right)^2}\, dydx$$

Monte Carlo-integration of this expression leads to the graph in Figure 3 below. As can be seen, the belief is more concentrated towards the centroid than in uniform distributions, i.e., the distribution is concave.

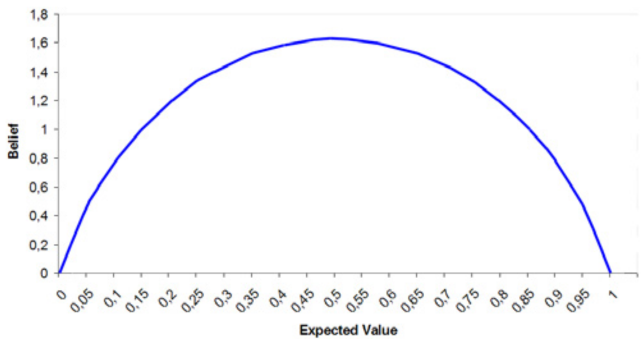

Figure 3 Shape of belief distribution $h(z)$ in Example 1 obtained from Monte Carlo-integration.

According to Definition 12, the belief in a certain z is derived from the area of each such surface when the component distributions are uniform. For $z = 0$ and $z = 1$, the surface will have an area of zero.

Decision Rule

To illustrate the need for a new decision rule within this framework, we will consider the following decision situation under risk. In the decision matrix below, p_{ij} corresponds to $P(\theta_j|A_i)$ and u_{ij} corresponds to $u(\theta_j|A_i)$

	θ_1	θ_2
A_1	$p_{11} \in [0.05; 1], u_{11} \in [70; 80]$	$p_{12} \in [0; 0.95], u_{12} \in [10; 90]$
A_2	$p_{21} \in [0.3; 0.4], u_{21} \in [70; 80]$	$p_{22} \in [0.6; 0.7], u_{22} \in [10; 90]$
A_3	$p_{31} = 0.4, u_{31} \in [70; 80]$	$p_{32} = 0.6, u_{32} \in [10; 90]$

Figure 4 Decision matrix with three alternatives and two uncertain states.

To receive a $weak$ preference order on this three alternatives, one proposed candidate is the Γ-maximin decision rule. Employing Γ-maximin will give us the order $A_3 \succ A_2 \succ A_1$, thus the decision maker is obliged to choose A_3 if accepting Γ-maximin.

However, for presentational purposes in this paper we assume that if the decision maker states his belief through intervals, then she has uniform belief on each interval, i.e., the belief in all feasible points is equal. Let z_1, z_2, z_3 be variables representing possible expected utilities of A_1, A_2, A_3 respectively, and let $h_1(z_1), h_2(z_2), h_3(z_3)$ denote the belief distributions on the possible expected utilities of each alternative. Through Monte Carlo simulations[3] we obtain the shapes of the belief distributions on the possible expected values.

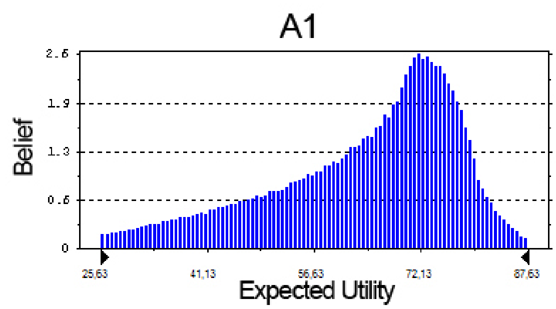

Figure 5 Belief distribution on z_1

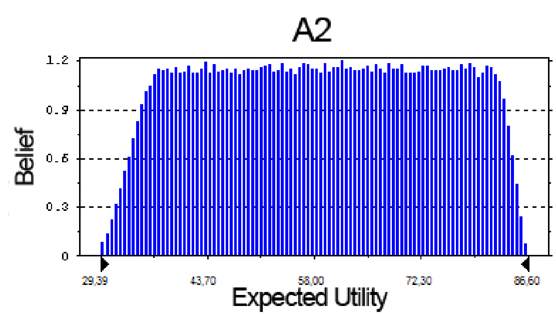

Figure 6 Belief distribution on z_2

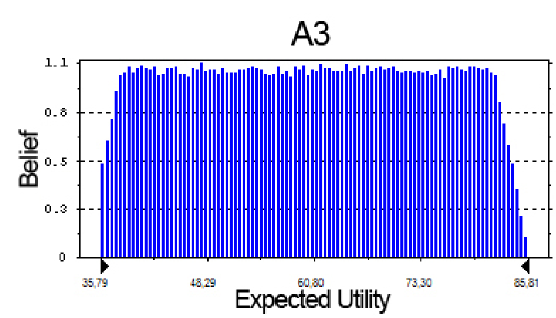

Figure 7 Belief distribution on z_3

From Theorem 4, it is clear that the horizontal centroid is additive, meaning that

$$h_{1_c} = 0.525 \cdot 75 + 0.475 \cdot 50 = 63.125$$
$$h_{2_c} = 0.35 \cdot 75 + 0.65 \cdot 50 = 58.75$$
$$h_{3_c} = 0.4 \cdot 75 + 0.6 \cdot 50 = 60$$

[3]Simulations performed with the software Crystal Ball, 300 000 trials.

As can be seen in Figures 5-7 above, when basing a decision on lower bounds (which is the case in the Γ-maximin decision rule), the decision is based on points which do not represent the decision maker's implicit belief in an adequate manner. Furthermore, since the belief distributions over the expected value intervals are not uniform, even in cases where the decision maker states the initial belief in terms of uniform belief, a decision rule should take account of the fact that belief tend to focus on sub-intervals containing the centroid.

In the light of this, basing a decision only on extreme points such as lower bounds seems to be too conservative and unnecessarily pessimistic. If a weak preference order is desired (avoiding situations like, e.g., indecision or incomparability) the following decision rule based on a *generalized expected utility* is suggested, which basically can be described as the expected utility vector at the centroid of a belief distribution h as defined in Definition 12 (Ekenberg and Thorbiörnson 2001).

Definition 13 *The principle of maximizing the generalized expected utility is accepted if a decision making agent chooses the alternative A^*, whenever*

$$A^* = \arg\max_A(\mathbf{G}(A))$$

for all $A \in \mathcal{A}$, where $\mathbf{G}(A) = h_c$ as defined in Definition 12.

This decision rule is a good candidate when the belief distributions have a major part of the belief mass concentrated to some, relatively small, neighborhood of each centroid. Furthermore, it seems very attractive from a computational viewpoint due to the properties of the centroid. Note that employing this rule yields the preference order $A_1 \succ A_3 \succ A_2$ in the example given in this section.

Concluding Remarks

When a decision maker is modeling uncertain and imprecise information in terms of first-order information such as intervals or convex sets of probability and utility distributions, we assume, only for presentational purposes, that the initial belief is uniformly distributed on the input variables if not stated otherwise. However, uniform distributions on possible first-order probability and utility distributions lead to severely non-uniform distributions on the possible expected utilities. Regardless of whether the decision maker's belief is uniform or not, expressed or implied, this warp effect presents additional information of value for making a well-informed decision. The use of uniform belief in the paper is for presentational clarity is does not constitute a delimiting property. In fact, having non-uniform centre-weighted belief adds even more to the warp effects.

This calls for that decision rules based on interval extreme points do not account for a decision maker's belief, in the sense that the second-order belief in neighborhoods of expected utility points near interval boundaries is significantly lower than neighborhoods containing the centroid or lying closer to the center of the expected utility interval. As the information given in this warp effect of second-order belief is derived purely from the properties of the expected utility formula, and how this formula combines our beliefs and desires into real numbers representing a preference order, the implication is that the interior of the intervals should be accounted for to at least the same extent as the interval boundaries.

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