Some Spatial and Spatio-Temporal Operators Derived from the Topological View of Knowledge

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Abstract

In this paper, we extend Moss and Parikh's approach to reasoning about topological properties of knowledge. We turn that system in a spatio-temporal direction by successively adding various modalities having on the one hand an epistemic interpretation and facilitating on the other hand spatial or spatio-temporal specifications up to a certain degree. The first of these operators is related to disjointness regarding space and ignorance regarding knowledge, and the second one to overlapping and, respectively, quantifying across all possible agents. The third one turns up along with increase of sets and no learning of agents, respectively. A fourth operator is already present in the basic system. Apart from the first case we establish the soundness, completeness and decidability of the accompanying logics. In the first case, however, we up to now could only prove that a certain naturally arising sublogic is decidable.

Introduction

The logic of knowledge and time as well as spatial logics are amongst the frequently used formal tools for coping with the appropriate modeling and reasoning tasks. However, it has hardly been recognized that both notions in a sense live under the same roof. But the bi-modal *logic of knowledge and effort* invented by Moss and Parikh in the papers (Moss & Parikh 1992; Dabrowski, Moss, & Parikh 1996) really makes possible a qualitative description of procedures gaining knowledge and offers some expressive power concerning spatial features, too. The latter is true since a certain *topological component* of knowledge is revealed by Moss and Parikh's system. In fact, as an agent's knowledge is represented by the space of all *knowledge states* of the agent, knowledge acquisition appears as a *shrinking procedure* regarding this space of sets. Thus concepts from topology like *closeness* or *neighbourhood* turn up together with knowledge in a natural way. This aspect will be reinforced in the present paper.

Moss and Parikh suggestively called their system topologic, and we adopt this naming here. In the following, we briefly recall the basics of the language underlying topologic. As it has just been indicated, formulas may contain two one-place operators: a modality K describing the knowledge of the agent and another one, $\Box$, describing (computational) effort. The domains for evaluating formulas are *subset spaces* $(X, \mathcal{O}, V)$ consisting of a non-empty set X of states, a set $\mathcal{O}$ of subsets of X representing the knowledge states of the agent, and a valuation V determining the states where the atomic propositions are true. The operator K then quantifies across any knowledge state $U \in \mathcal{O}$, whereas $\Box$ quantifies 'downward' across $\mathcal{O}$. That is, more knowledge, i.e., closer proximity to states of 'complete' knowledge can be achieved by *descending* (with respect to the set inclusion relation in $\mathcal{O}$), and just this is modeled by $\Box$.

Several classes of subset spaces, including the ordinary topological ones, could be characterized by means of topologic; cf (Georgatos 1994; 1997; Weiss & Parikh 2002). Actually, the language of topologic proved to be quite suitable for dealing with 'locally describeable' properties of points and sets. However, more expressiveness is needed to capture non-local notions like *disjointness* or *overlapping* as well.

The just mentioned spatial relations remind one of the *Region Connection Calculus, RCC,* which is ubiquitous in AI; see (Randell, Cui, & Cohn 1992). But while that system formalizes 'externally' the relations any two sets may satisfy we take the 'internal', modal point of view here. That is, starting at some situation we regard as relevant *all* sets from $\mathcal{O}$ such that the respective property is satisfied. In this way, already the effort modality $\Box$ appears as a spatio-temporal one. The temporal component of $\Box$ is rather implicit, actually, whereas the spatial one corresponds to the relation *'contains-as-a-subset'*. In the paper, we consider connectives for disjointness, overlapping and *'is-part-of'* besides.

Obviously, our approach is not in competition with RCC because the underlying idea of space is quite different. One might say at most that both frameworks are complementary to each other. Nevertheless, the present paper shows that the Moss-Parikh formalism in fact can be taken as a basis for a certain kind of spatio-temporal reasoning which may be called 'epistemic'. This alternative view may yield some new insights into the logical nature of spatial concepts.

The rest of this paper is organized as follows. In the next section we recapitulate the modal logic of subset spaces from (Dabrowski, Moss, & Parikh 1996). This gives us the necessary presuppositions for the matters developed afterwards. In Section 3, we integrate an operator describing disjointness into the language of topologic. We give some examples of valid formulas of the extended language, and we touch on the question of expressive power there. We then present a list of axioms suggesting themselves and show that the set of all derivable formulas is decidable. Unfortunately, we could not prove completeness in this case up to now. In Section 4, we report some of the results known for the overlap operator. Section 5 deals with the *'is-part-of'* relation, which temporal-epistemically corresponds to *no learning in the course of time,* but represents implicitly the past in our context. This section contains the main technical issues of the paper, aside from the second part of Section 3. The heading of all these sections is to indicate the spatial or spatio-temporal idea formalized each time. Finally, we give a brief summary and point to further aspects and future research.

Shrinking

At the beginning of the technical part of this paper we briefly recall the language and the logic of subset spaces from (Dabrowski, Moss, & Parikh 1996). Let $\mathrm{PROP} = \{A, B, \ldots\}$ be a denumerable set of symbols. The elements of PROP are called *proposition letters*. We define the set WFF of *well-formed formulas over* PROP by the rule $\alpha ::= A \mid \neg\alpha \mid \alpha \wedge \beta \mid \mathsf{K}\alpha \mid \Box\alpha$. The operators K and $\Box$ represent *knowledge* and *effort,* respectively, as it is common for topologic. The missing boolean connectives $\top, \bot, \vee, \rightarrow, \leftrightarrow$ are treated as abbreviations, as needed. The duals of K and $\Box$ are denoted L and $\Diamond$, respectively.

We now turn to the relevant semantics. First, we define the domains for interpreting formulas. Given a set X, let $\mathcal{P}(X)$ be the powerset of X.

Definition 1 (Subset frames; subset spaces) *1. Let a set $X \neq \emptyset$ be given, and suppose that $\mathcal{O} \subseteq \mathcal{P}(X)$ is a set of subsets of X. Then, $\mathcal{F} := (X, \mathcal{O})$ is called a* subset frame.

2. *Let $\mathcal{F} = (X, \mathcal{O})$ be a subset frame. The* set $\mathcal{N}_{\mathcal{F}}$ of neighbourhood situations *of $\mathcal{F}$ then is defined by $\mathcal{N}_{\mathcal{F}} := \{(x, U) \mid x \in U \text{ and } U \in \mathcal{O}\}$. (Neighbourhood situations are often written without brackets later on.)*
3. *Let $\mathcal{F}$ be as above. A mapping $V : \mathrm{PROP} \longrightarrow \mathcal{P}(X)$ is called an $\mathcal{F}$–*valuation.
4. *A* subset space *is a triple $\mathcal{M} := (X, \mathcal{O}, V)$, where $\mathcal{F} := (X, \mathcal{O})$ is a subset frame and V an $\mathcal{F}$–valuation; $\mathcal{M}$ is called* based on $\mathcal{F}$.

The relation of satisfaction, which is now defined with regard to subset spaces, holds between neighbourhood situations and formulas. The obvious boolean cases are omitted.

Definition 2 (Satisfaction; validity) *Let $\mathcal{M} = (X, \mathcal{O}, V)$ be a subset space.*

1. *Let x, U be a neighbourhood situation of $\mathcal{F} = (X, \mathcal{O})$. Then*

$$\begin{array}{lcl} x, U \models_{\mathcal{M}} A & :\Longleftrightarrow & x \in V(A) \\ x, U \models_{\mathcal{M}} \mathsf{K}\alpha & :\Longleftrightarrow & \forall\, y \in U : y, U \models_{\mathcal{M}} \alpha \\ x, U \models_{\mathcal{M}} \Box\alpha & :\Longleftrightarrow & \begin{cases} \forall\, U' \in \mathcal{O} : x \in U' \subseteq U \\ \quad \textit{implies } x, U' \models_{\mathcal{M}} \alpha, \end{cases} \end{array}$$

where $A \in \mathrm{PROP}$ and $\alpha, \beta \in \mathrm{WFF}$. In case $x, U \models_{\mathcal{M}} \alpha$ is true we say that α holds in $\mathcal{M}$ at *the neighbourhood situation x, U.*

2. *A formula α is called* valid in $\mathcal{M}$ *(written '$\mathcal{M} \models \alpha$'), iff it holds in $\mathcal{M}$ at every neighbourhood situation of the frame $\mathcal{M}$ is based on.*

Note that the meaning of proposition letters is independent of neighbourhoods by definition, thus 'stable' with respect to $\Box$. This fact is reflected by the axiom schema $(A \rightarrow \Box A) \wedge (\Diamond A \rightarrow A)$, where $A \in \mathrm{PROP}$.

We now look for further validities in subset spaces. Actually, the formula schema $\mathsf{K}\Box\alpha \rightarrow \Box\mathsf{K}\alpha$ is typical of topologic. This schema was called the *Cross Axiom* in the paper (Dabrowski, Moss, & Parikh 1996) and plays a key role in the proofs of the completeness and the decidability of that logic. The Cross Axiom describes the basic interaction between knowledge and effort. The temporal-epistemic notion of *perfect recall,* cf (Fagin *et al.* 1995), Sec. 4.4.4, is closely related to this axiom, which spatially describes *shrinking*.

The complete list of axioms for topologic guarantees that, for every Kripke model validating all those axioms,

- the accessibility relation $\xrightarrow{\mathsf{K}}$ belonging to the knowledge operator is an equivalence,
- the accessibility relation $\xrightarrow{\Box}$ belonging to the effort operator is reflexive and transitive, and
- the composite relation $\xrightarrow{\Box} \circ \xrightarrow{\mathsf{K}}$ is contained in the composite relation $\xrightarrow{\mathsf{K}} \circ \xrightarrow{\Box}$.

We obtain a logical system by adding the standard proof rules of modal logic, i.e., *modus ponens* and *necessitation with respect to each modality*. We call this system T, indicating topologic.

Theorem 3 (Soundness and completeness I) *A formula α is valid in all subset spaces, iff α is* T*–derivable.*

In (Dabrowski, Moss, & Parikh 1996), Sec. 2.2, completeness is proved by a non-trivial step-by-step construction. Unfortunately, topologic does not satisfy the finite model property with respect to the class of all subset spaces; cf (Dabrowski, Moss, & Parikh 1996), Sec. 1.3. Thus one has to make a little detour via a suitable class of auxiliary Kripke models. In this way, decidability is obtained nevertheless; cf (Dabrowski, Moss, & Parikh 1996), Sec. 2.3.

Theorem 4 (Decidability I) *The logic* T *is decidable.*

The just indicated proof methods will be appropriately modified and supplemented in the subsequent sections of this paper.

Disjointness

In this section, we add a unary operator D, describing disjointness from the actual neighbourhood, to the language of topologic. We first define the semantics of D precisely and then give some examples and comments related to expressiveness. After that we ask for an axiomatization of the accompanying logic, TD. The main part of this section is devoted to the proof of the decidability of a natural sublogic of TD. Among other things, a scarce filtration is used for that.

Let $\mathrm{WFF}_{\mathsf{D}}$ be the set of formulas which results when we take $\alpha ::= \mathsf{D}\alpha$ as an additional generative rule. Furthermore, let $\widehat{\mathsf{D}}$ denote the dual of D. The semantics of D is defined as follows.

Definition 5 (Semantics of D) *Let $\mathcal{F} = (X, \mathcal{O})$ be a subset frame, $\mathcal{M} = (X, \mathcal{O}, V)$ a subset space based on $\mathcal{F}$, and $(x, U) \in \mathcal{N}_{\mathcal{F}}$ a neighbourhood situation. Then, for all $\alpha \in \mathrm{WFF}_{\mathsf{D}}$,*

$$x, U \models_{\mathcal{M}} \mathsf{D}\alpha : \Longleftrightarrow \begin{cases} \forall\,(y, U') \in \mathcal{N}_{\mathcal{F}} : \textit{if } U' \cap U = \emptyset, \\ \qquad \textit{then } y, U' \models_{\mathcal{M}} \alpha. \end{cases}$$

It is natural to ask in which way D, K and $\Box$ interact. Some answers are given by the following proposition.

Proposition 6 *Let $\mathcal{M}$ be any subset space and $\alpha \in \mathrm{WFF}_{\mathsf{D}}$ a formula. Then*

$$\mathcal{M} \models (\mathsf{D}\alpha \to \mathsf{K}\mathsf{D}\alpha) \wedge (\mathsf{D}\alpha \to \mathsf{D}\mathsf{K}\alpha) \wedge (\Diamond\mathsf{D}\alpha \to \mathsf{D}\alpha)\,.$$

The proof of Proposition 6 is straightforward from Definition 2 and Definition 5 and, therefore, omitted.

Now we search for interesting properties of spaces which can be formalized with the aid of the new operator. As it turns out, D develops its abilities only in connection with a certain additional means of expression, viz basic *hybrid logic*. The special feature of hybrid logic is the use of names of worlds (called *nominals*) in formulas; cf (Blackburn, de Rijke, & Venema 2001), Sec. 7.3. Due to the two-component semantics of topologic we have nominals for both states and subsets here. (A corresponding system, called *hybrid* topologic, was initially developed in (Heinemann 2003).) With that, we now show as an example that a certain higher separation condition related to *regularity* can be captured; cf (Bourbaki 1966), § 8.4, Proposition 11.

Definition 7 *Let $\mathcal{F} = (X, \mathcal{O})$ be a subset frame. $\mathcal{F}$ is called* pseudo (T 3), *iff for all $x, y \in X$ and $U, U' \in \mathcal{O}$ such that $x \in U' \subset U$ and $y \in U \setminus U'$ there exists some $\widetilde{U} \in \mathcal{O}$ satisfying $y \in \widetilde{U}$ and $U' \cap \widetilde{U} = \emptyset$.*

Note that hybrid topologic without D is able to cope with lower separation axioms only; cf (Heinemann 2003).

Proposition 8 *Let $\mathcal{F} = (X, \mathcal{O})$ be a subset frame. Then, $\mathcal{F}$ is pseudo (T 3) iff for all (hybrid) subset spaces $\mathcal{M}$ based on $\mathcal{F}$ the formula schema*

$$i \wedge \Diamond\,(I \wedge \mathsf{K}\neg j) \wedge \mathsf{L}j \to \Box\left(i \wedge I \to \widehat{\mathsf{D}}\mathsf{L}j\right)$$

is valid in $\mathcal{M}$, where i, j are names of states and I is any name of a subset.

The easy proof of Proposition 8 is omitted here. – Our final remark on expressiveness concerns the relevance of the disjointness operator to the logic of knowledge. First let us stress once again that D quantifies across subsets which do not intersect with any future knowledge state of the agent. Thus certain *states of ignorance* are accessed by this operator. Since the language of knowledge describes 'external' knowledge of the agent anyway,[1] it is legitimate and useful also from the epistemic point of view to extend the formalism as suggested.

We now propose several axioms for D. Apart from the first and the third formula schema listed in Proposition 6[2] we take, for all $\alpha, \beta \in \mathrm{WFF}_{\mathsf{D}}$:

1. $\mathsf{D}(\alpha \to \beta) \to (\mathsf{D}\alpha \to \mathsf{D}\beta)$
2. $\alpha \to \mathsf{D}\widehat{\mathsf{D}}\alpha$.

Note that the symmetry of the accessibility relation, $\xrightarrow{\mathsf{D}}$, belonging to the disjointness operator is expressed by Axiom 2.

In the remaining part of this section we prove that the set TD of all formulas that are derivable from the axioms of topologic and the just stated schemata by means of the standard modal proof rules is decidable. Note that TD is a subset of the set of all validities, TD.

The reader should connect R and K, S and $\Box$, and T and D, respectively, in the next definition.

Definition 9 (TD–model) *Let $M := (W, \{R, S, T\}, V)$ be a trimodal Kripke model. Then M is called an* TD–model, *iff the following conditions are satisfied:*

1. *R is an equivalence relations, S is reflexive and transitive, and T is symmetric,*
2. *$R \circ T \subseteq T$ and $S \circ R \subseteq R \circ S$,*
3. *for all $w, w', w'' \in W$ such that $w\,T\,w'$ and $w\,S\,w''$ it holds that $w'\,T\,w''$,*
4. *for all $w, w' \in W$ and $A \in \mathrm{PROP}$: if $w\,S\,w'$, then $(w \in V(A) \iff w' \in V(A))$.*

It is easy to see that all the above mentioned axioms of TD are sound for TD–models. Furthermore, the canonical model of TD itself is in fact a TD–model. This gives us the next theorem.

Theorem 10 (Kripke completeness) *The sublogic* TD *of* TD *is sound and complete with respect to the class of all TD–models.*

We now use the method of *filtration* in order to establish the finite model property of TD with respect to TD–models. Given a TD–consistent formula $\alpha \in \mathrm{WFF}_{\mathsf{D}}$, we first have to define an appropriate filter set $\Sigma \subseteq \mathrm{WFF}_{\mathsf{D}}$ for that. To this end, we let

$$\Sigma_0 := \mathrm{sf}(\alpha) \cup \{\neg\beta \mid \beta \in \mathrm{sf}(\alpha)\} \cup \{\Box\widehat{\mathsf{D}}\mathsf{D}\gamma \mid \mathsf{D}\gamma \in \mathrm{sf}(\alpha)\},$$

where $\mathrm{sf}(\alpha)$ denotes the set of all subformulas of α. We then form the closure of Σ_0 under finite conjunctions of pairwise distinct elements of Σ_0, and we close under single applications of the operator L afterwards. Finally, we form the set of all subformulas of elements of the set obtained last. Let Σ denote the resulting set of formulas.

Secondly, we consider the respective *smallest* filtrations of the accessibility relations $\xrightarrow{\mathsf{K}}$ and $\xrightarrow{\Box}$ of the canonical model, and the *symmetric* filtration of the relation $\xrightarrow{\mathsf{D}}$; cf (Goldblatt 1992), p 33. Let $M := (W, \{R, S, T\}, V)$ be the corresponding filtration of a suitably generated submodel of

[1] For a discussion of this point cf (Fagin *et al.* 1995), in particular, Sec. 4.2.

[2] The second one turns out to be a logical consequence, actually.

the canonical model; the valuation V is to assign the empty set to all proposition letters not in Σ there. Then we have the following lemma.

Lemma 11 *The structure M is a finite TD–model of which the size depends computably on the length of α.*

Proof. The items of Definition 9 which do not concern the relation T are clear from the proof of (Dabrowski, Moss, & Parikh 1996), Theorem 2.11. Since the symmetry of T follows from (Goldblatt 1992), Exercise 4.5, and the proof of the first part of item 2 is similar to the one of item 3, only the latter will be given here.

Let Γ, Θ, Ξ be points of the canonical model such that $[\Gamma]\ T\ [\Theta]$ and $[\Gamma]\ S\ [\Xi]$, where the brackets $[\ldots]$ indicate the respective classes. Take any $\mathsf{D}\gamma \in \mathrm{sf}(\alpha)$, and assume that $M, [\Theta] \models \mathsf{D}\gamma$. We must show that $M, [\Xi] \models \widehat{\mathsf{D}}\mathsf{D}\gamma$. Since $[\Gamma]\ T\ [\Theta]$ is valid, we first obtain that $M, [\Gamma] \models \widehat{\mathsf{D}}\mathsf{D}\gamma$. As the latter formula is contained in Σ, we infer $\Box\widehat{\mathsf{D}}\mathsf{D}\gamma \in \Gamma$ from that with the aid of the (dual of the) third formula schema from Proposition 6. But $\Box\widehat{\mathsf{D}}\mathsf{D}\gamma$ is contained in Σ, too. Thus $M, [\Gamma] \models \Box\widehat{\mathsf{D}}\mathsf{D}\gamma$. Consequently, $M, [\Xi] \models \widehat{\mathsf{D}}\mathsf{D}\gamma$, as desired. □

Since the model M from Lemma 11 realizes α, the claimed decidability result follows readily.

Theorem 12 (Decidability II) *The set of all* TD*–theorems is decidable.*

Overlapping

Instead of D we now add an *overlap operator* to the language of topologic. Such a modality, O, was investigated in the paper (Heinemann 2006) for the first time. To complete the picture, we state the relevant definitions and results here. Mentioning a certain relation between D and O merely goes beyond that.

This is the semantics of the overlap operator in subset spaces $\mathcal{M} = (X, \mathcal{O}, V)$ at neighbourhood situations:

$$x, U \models_{\mathcal{M}} \mathsf{O}\alpha :\Longleftrightarrow \left\{ \begin{array}{c} \forall\, U' \in \mathcal{O} : x \in U' \\ \text{implies } x, U' \models_{\mathcal{M}} \alpha. \end{array} \right.$$

Compared to the effort operator, the condition $U' \subseteq U$ obviously was left out; cf Definition 2. Thus it can be seen from the formal definition as well that shrinking is a special case of overlapping.

As to knowledge, note that O quantifies across all sets that are fixed in the same point. Since those sets may represent knowledge states of agents we have, therefore, an operator to hand by means of which the actual knowledge state of every possible agent is addressed.

The overlap modality turned out to be quite nice. In fact, O satisfies all the S5 laws. The following two theorems belong to the main results of the paper (Heinemann 2006).

Theorem 13 (Soundness and completeness II) *The set of all theorems that are derivable from the axioms for overlapping is sound and complete for subset spaces.*

Theorem 14 (Decidability III) *The logic of overlapping is decidable.*

Like D the overlap operator is particularly expressive within the framework of hybrid topologic; cf (Heinemann 2006), Sec. 5. Even more, the (dual of the) disjointness operator then will be definable in a sense: $\widehat{\mathsf{D}}\alpha$ can be 'expressed' by the schema

$$I \rightarrow \widehat{\mathsf{O}}\mathsf{L}\Diamond\left(\mathsf{K}\mathsf{O}\neg I \wedge \alpha\right),$$

where I is any set name and $\widehat{\mathsf{O}}$ denotes the dual of O. But this is only true with regard to so-called *named* models (cf (Blackburn, de Rijke, & Venema 2001), Sec. 7.3) based on subset frames $(X, \mathcal{O})$ for which $X \in \mathcal{O}$ is valid. – This is all that we wanted to say about the overlap operator here.

Increase

The remaining spatio-temporal modality to be discussed, P, captures the *'is-part-of'* relation. Thus P is complementary to the effort operator. Again, we first define the semantics and then comment on expressive power. For the arising logic, TP, we propose a corresponding axiomatization. The completeness proof for TP following after that makes up the core of this section. Finally, we obtain that TP too is decidable.

Like in Section 3 we introduce the relevant set of formulas, $\mathrm{WFF}_{\mathsf{P}}$. And we let analogously $\widehat{\mathsf{P}}$ denote the dual of P. The semantics of P in subset spaces $\mathcal{M} = (X, \mathcal{O}, V)$ at neighbourhood situations is then defined as follows:

$$x, U \models_{\mathcal{M}} \mathsf{P}\alpha :\Longleftrightarrow \left\{ \begin{array}{c} \forall\, U' \in \mathcal{O} : U \subseteq U' \\ \text{implies } x, U' \models_{\mathcal{M}} \alpha. \end{array} \right.$$

As it turns out, the increase operator P is the 'strongest' one among the spatial modalities considered in this paper. In fact, even the (dual of the) overlap operator is definable, at least in models based on subset frames $(X, \mathcal{O})$ satisfying $X \in \mathcal{O}$; the defining term actually reads

$$\widehat{\mathsf{O}}\alpha :\equiv \widehat{\mathsf{P}}\Diamond\alpha,$$

where $\alpha \in \mathrm{WFF}_{\mathsf{P}}$.

As an example of a spatial property which can be expressed using P we mention the *closure of $\mathcal{O}$ under supersets*. However, we need nominals for that as well so that we do not carry out this here.

We have chosen the denotation 'P' for the increase operator since it temporally corresponds to the past. This is true simply because the effort operator concerns the future.[3]

The epistemic relevance of the increase operator (in case this operator is considered independent of $\Box$) is worth mentioning. While the effort operator is associated with *no forgetting,* the increase operator comes along with *no learning;* cf (Halpern, van der Meyden, & Vardi 2004).

The following list contains the axioms involving P :

1. $\mathsf{P}(\alpha \rightarrow \beta) \rightarrow (\mathsf{P}\alpha \rightarrow \mathsf{P}\beta)$
2. $\alpha \rightarrow \Box\widehat{\mathsf{P}}\alpha$

[3]For certain systems of linear time, a related modality was examined in (Heinemann 2002). However, the motivation and the technical details for the more general case considered in the present paper are completely different from those there.

3. $\alpha \rightarrow \mathsf{P}\Diamond\alpha$
4. $\mathsf{PK}\alpha \rightarrow \mathsf{KP}\alpha$,

where $A \in \mathrm{PROP}$ and $\alpha, \beta \in \mathrm{WFF}_{\mathsf{P}}$. – Some comments on these axioms seem to be appropriate. The schemata 2 and 3 express that the accessibility relations belonging to $\Box$ and P, respectively, are inverse to each other. Note that the latter relation is reflexive and transitive, which follows logically from the axioms (including those of $\mathsf{topologic}$). Finally, the schema 4 is obviously 'dual' to the Cross Axiom and will be called the *Reverse Cross Axiom* therefore.[4]

By adding the P–necessitation rule we obtain the logical system TP, which is sound and complete for subset spaces.

Theorem 15 (Soundness and completeness III) *Let $\alpha \in \mathrm{WFF}_{\mathsf{P}}$. Then α is valid in all subset spaces, iff α is TP–derivable.*

Proof. (Sketch.) The soundness part of Theorem 15 is easy to prove. Thus we need not go into that in more detail. The proof of completeness is both an adaption to the new system and a suitable extension of the corresponding proof for $\mathsf{topologic}$; cf (Dabrowski, Moss, & Parikh 1996), Sec. 2.2. Only the part concerning the modifications is given more ore less detailedly here.

Let $\alpha \in \mathrm{WFF}_{\mathsf{P}}$ be not TP–derivable. We attain a subset space falsifying α by an infinite 'three-dimensional' step-by-step construction. In each step, an approximation to the final model is defined. In order to ensure that this 'limit structure' behaves as desired, several requirements on the intermediate models have to be kept under control.

Let $\mathcal{C}$ be the set of all maximal TP–consistent sets of formulas and $\xrightarrow{\mathsf{K}}$, $\xrightarrow{\Box}$, and $\xrightarrow{\mathsf{P}}$ the accessibility relations on $\mathcal{C}$ induced by the modalities K, $\Box$ and P, respectively. Suppose that $\Gamma_0 \in \mathcal{C}$ contains $\neg\alpha$ and is to be realized thus. We choose a denumerably infinite set of points, Y, fix an element $x_0 \in Y$, and construct inductively a sequence of quadruples (X_n, P_n, i_n, t_n) such that, for every $n \in \mathbb{N}$,

1. $X_n \subseteq Y$ is a finite set containing x_0,
2. $(P_n, \leq)$ is a finite partial order containing p_0,
3. $i_n : P_n \longrightarrow \mathcal{P}(X_n)$ is an injective function from P_n into the set of all non-empty subsets of X_n such that $p \leq q \iff i_n(p) \supseteq i_n(q)$ holds for all $p, q \in P_n$,
4. $t_n : X_n \times P_n \longrightarrow \mathcal{C}$ is a *partial* function assigning a maximal TP–consistent set to certain pairs contained in $X_n \times P_n$ such that, for all $x, y \in X_n$ and $p, q \in P_n$,
   (a) $t_n(x, p)$ exists iff $x \in i_n(p)$; in this case it holds that
      i. if $y \in i_n(p)$, then $t_n(x,p) \xrightarrow{\mathsf{K}} t_n(y,p)$,
      ii. if $p \leq q$, then $t_n(x,q) \xrightarrow{\Box} t_n(x,p)$,
      iii. if $q \leq p$, then $t_n(x,p) \xrightarrow{\mathsf{P}} t_n(x,q)$,
   (b) $t_n(x_0, p_0) = \Gamma_0$.

We now explain to what extent the intermediate structures (X_n, P_n, i_n, t_n) approximate the desired model. Actually, it can be guaranteed that, for all $n \in \mathbb{N}$,

5. $X_n \subseteq X_{n+1}$,
6. P_{n+1} is a *faithful* extension of P_n, i.e., a substructure of P_n such that no element of $P_{n+1} \setminus P_n$ lies between any two elements of P_n,
7. $i_{n+1}(p) \cap X_n = i_n(p)$ for all $p \in P_n$,
8. $t_{n+1} \mid_{X_n \times P_n} = t_n$.

Furthermore, the construction complies with the following requirements on existential formulas. For all $n \in \mathbb{N}$,

9. if $\mathsf{L}\beta \in t_n(x,p)$, then there are $k \in \mathbb{N}$, $n < k$, and $y \in i_k(p)$ such that $\beta \in t_k(y,p)$,
10. if $\Diamond\beta \in t_n(x,p)$, then there are $k \in \mathbb{N}$, $n < k$, and $q \in P_k$ such that $p \leq q$ and $\beta \in t_k(x,q)$,
11. if $\widehat{\mathsf{P}}\beta \in t_n(x,p)$, then there are $k \in \mathbb{N}$, $n < k$, and $q \in P_k$ such that $q \leq p$ and $\beta \in t_k(x,q)$.

Let us assume for the moment that the construction has been carried out successfully so that all these requirements are met. Let (X, P, i, t) be the union of the structures (X_n, P_n, i_n, t_n), i.e.,

- $X = \bigcup\limits_{n \in \mathbb{N}} X_n$ and $P = \bigcup\limits_{n \in \mathbb{N}} P_n$,
- i is given by $i(p) = \bigcup\limits_{m \geq n} i_m(p)$, where n is the smallest number l such that $p \in P_l$, and
- t is given by $t(x,p) := t_n(x,p)$, where n is the smallest number l such that $t_l(x,p)$ is defined.

Moreover, let $\mathcal{O} := \mathrm{Im}(i)$ and $\mathcal{F} := (X, \mathcal{O})$. Finally, we define a suitable $\mathcal{F}$–valuation. This definition is a bit involved. Let $V(A)$ be the set of all $x \in X$ such that there exist $y \in X$ and $q \in P$ satisfying

$$(t(y,p_0), t(x,q)) \in (\xrightarrow{\Box} \circ \xrightarrow{\mathsf{P}})^* \text{ and } A \in t(y,p_0),$$

for all $A \in \mathrm{PROP}$. Then $\mathcal{M} := (X, \mathcal{O}, V)$ is a subset space, for which the subsequent *Truth Lemma* can be proved by structural induction.

Lemma 16 (Truth Lemma) *For all formulas $\beta \in \mathrm{WFF}_{\mathsf{P}}$ and neighbourhood situations $(x, i(p)) \in \mathcal{N}_{\mathcal{F}}$, we have that $x, i(p) \models_{\mathcal{M}} \beta$ iff $\beta \in t(x,p)$.*

Letting now $x := x_0$, $p := p_0$ and $\beta := \neg\alpha$ (where x_0, p_0 and α are as above), then Theorem 15 follows immediately from Lemma 16.

It remains to define (X_n, P_n, i_n, t_n), for all $n \in \mathbb{N}$. The case $n = 0$ is straightforward. In the induction step, some existential formula contained in some maximal TP–consistent set $t_n(x,p)$, where $x \in X_n$ and $p \in P_n$, must be realized in a way meeting all the above requirements. We confine ourselves to the case of the operator P.

Let $\widehat{\mathsf{P}}\beta \in t_n(x,p)$. We choose some element $y \in Y$ not yet processed, and we take some $q \notin P_n$. Then we let $X_{n+1} := X_n \cup \{y\}$, $P_{n+1} := P_n \cup \{q\}$, and

$$i_{n+1}(p') := \begin{cases} i_n(p') & \text{if } p' \in P_n \\ i_n(p) \cup \{y\} & \text{if } p' = q, \end{cases}$$

for all $p' \in P_{n+1}$; furthermore, we let $q \leq p$, but be incomparable with any other element of P_n. Finally, we define t_{n+1} as follows. Let Γ be a maximal TP–consistent

[4] One can prove that the schema 4 is derivable, too. However, since this schema is used below we keep it in the above list.

set such that $t_n(x,p) \xrightarrow{\mathsf{P}} \Gamma$ and $\beta \in \Gamma$. Due to the Reverse Cross Axiom, there are elements $\Gamma_q^z \in \mathcal{C}$ such that $t_n(z,p) \xrightarrow{\mathsf{P}} \Gamma_q^z \xrightarrow{\mathsf{K}} \Gamma$, for all $z \in i_n(x,p)$. Now, let

$$t_{n+1}(x',p') := \begin{cases} t_n(x',p') & \text{if } x' \in X_n \text{ and } p' \in P_n \\ \Gamma_q^z & \text{if } x' \in i_n(p) \text{ and } p' = q \\ \Gamma & \text{if } x' = y \text{ and } p' = q \\ \text{undefined} & \text{otherwise,} \end{cases}$$

for all $x' \in X_{n+1}$ and $p' \in P_{n+1}$. This completes the definition of the approximating structure in the induction step, and we must now check that the properties 1 – 8 and 11 remain valid (9 and 10 are irrelevant to the present case). Apart from 4 all items are more or less obvious from the construction. Thus only the verification of property 4 is left. As 4 (b) too is obvious we concentrate on 4 (a). First, it follows from the definition of t_{n+1} that the requirement on the domain of this function is satisfied. Then, (i) is clear from the definition of t_{n+1} and the validity of this condition for n. (Note that $\xrightarrow{\mathsf{K}}$ is an equivalence relation.) For (iii) we argue as follows. Assume that $x' \in i_{n+1}(p')$, and let $q' \in P_{n+1}$ be such that $q' \leq p'$. We must show that $t_{n+1}(x',p') \xrightarrow{\mathsf{P}} t_{n+1}(x',q')$ is valid then. If $p', q' \in P_n$, then this follows from the induction hypothesis. If $p', q' \notin P_n$, then (iii) is true since $\xrightarrow{\mathsf{P}}$ is reflexive. The case $q' \in P_n$ and $p' \notin P_n$ is not possible. It remains to consider the case $p' \in P_n$ and $q' \notin P_n$. But then we have that $p' = p$, $q' = q$ and $x' \neq y$, where p, q and y are from the construction in the induction step. Thus $t_{n+1}(x',p') \xrightarrow{\mathsf{P}} t_{n+1}(x',q')$ follows from the definition of t_{n+1}. Finally, (ii) follows in a similar manner, but we must additionally use in this case that $\xrightarrow{\mathsf{P}}$ and $\xrightarrow{\Box}$ are inverse to each other. In this way, property 4 is proved for t_{n+1}.

In order to ensure that *all* possible cases are eventually exhausted, processing has to be suitably scheduled with regard to each of the modalities involved. This can be done by means of appropriate enumerations. Concerning this and the construction in case of a modality of topologic, the reader is referred to the paper (Dabrowski, Moss, & Parikh 1996) for further details. – With that the proof sketch of Theorem 15 is finished. $\Box$

By applying similar techniques as in the case of the disjointness operator (see the second part of Section 3), we obtain that TP is decidable.[5]

Theorem 17 (Decidability IV) *The logic* TP *is decidable.*

Concluding remarks

Several spatial modalities derived from the topo-logical view of knowledge were presented in this paper. In a sense, these operators are hierarchically ordered with regard to expressiveness. Except for inequality, the given list seems to be complete for the framework of topologic. Apart from the case of the disjointness operator we obtained the desired meta-theorems for the resulting logics, viz soundness, completeness, and decidability.

[5]Unfortunately, we cannot go into details regarding this here. – The paper (Heinemann 2007) contains more about the operator P.

It remains to solve the completeness problem for TD. Moreover, the complexity of the logics must be determined each time. Afterwards, the approach should be revisited from an overall point of view, encompassing all those operators simultaneously.

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