

Modelling Uniformity and Control during Knowledge Acquisition

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Abstract

We develop a formal system dealing ‘spatially’ with certain aspects of uniformity and control during knowledge acquisition. To this end, we study an appropriate modality, which is essentially due to Baskent, against the background of dynamic epistemic logic. The new connective turns out to be rather complex, but hybridizing the source language enables us to prove some of the desired fundamental properties of the arising logic like completeness or decidability.

Introduction

Moss and Parikh’s *effort operator*, cf (Moss & Parikh 1992) and (Dabrowski, Moss, & Parikh 1996), constitutes the connecting link between two different approaches to reasoning about non-static aspects of knowledge: *topologic*, and, on the other hand, dynamic epistemic logic.¹ The connection we indicated is, however, rather loose. This will become apparent in a minute when particularly the first system is inspected to some extent, but already now the main reason for that can be stated: The effort operator of *topologic* models actions of agents *implicitly* while these are *explicitly* present in dynamic epistemic logic.

The goal of this paper is, among other things, bringing *topologic* a little closer to dynamic epistemic logic. This is done with the aid of an interesting modality describing *controlled shrinking*.

For convenience of the reader, we now recall the language underlying *topologic*, L . We lead up to the new operator afterwards, having our eye on dynamic epistemic logic at the same time.

In L , the knowledge of an agent in question is described by the space of all *knowledge states*. These are the sets of states the agent considers possible at a time. If an effort is made to acquire knowledge, then this appears as a *shrinking procedure* regarding that space of sets. The formulas of

topologic may contain both a modality K describing knowledge and an operator \Box expressing effort. The semantic domains are triples (X, \mathcal{O}, V) called *subset spaces*, which consist of a non-empty set X of states, a set \mathcal{O} of subsets of X representing the knowledge states of the agent,² and a valuation V determining the states where the atomic propositions are true. The operator K then quantifies over some knowledge state $U \in \mathcal{O}$, whereas \Box quantifies ‘downward’ over \mathcal{O} since shrinking elements of \mathcal{O} and gaining knowledge correspond to each other.

Thus \Box models *some* knowledge acquisition procedure, i.e., a sequence of actions that are not specified further unless they increase knowledge. Unlike that the actions considered in dynamic epistemic logic are quite concrete. In fact, they strongly resemble the program constructs known from propositional dynamic logic (in short, PDL); cf (van Ditmarsch, van der Hoek, & Kooi 2007), Ch. 5, and (Goldblatt 1992), § 10, respectively.

It is, therefore, fairly natural to try combining *topologic* and PDL in order to reach the aforementioned goal. Since procedures change knowledge states, subset spaces must be enriched with partial functions operating on \mathcal{O} for that. Such functions are intended to represent the ‘elementary’ actions. The more complex ones come into play when composition, alternation, iteration, etc are included.

The idea of combining those two systems, which has just been given by way of a hint, was elaborated in (Heinemann 2007). Certain features of *topologic* get lost by doing this though. For instance, a \Box -like construct appears *only* as the reflexive and transitive closure of procedures then; i.e., we do no longer have the original effort operator at our disposal since it may be independent of those closures and we do not have the interaction of the modalities under control in this case. On account of this we follow a different approach here, which is now motivated by two examples.

Our first example concerns the *computation of binary streams*. This is one of the wide-spread cases for which the process of acquiring knowledge can be modelled by a *tree*; cf (Georgatos 1997). In fact, let \mathcal{B} be the set of all infinite 0-1-sequences endowed with the initial-segment topology, \mathcal{T} .

²These sets are sometimes called *the opens* since a topological interpretation of knowledge is supported by the system under discussion; see the papers cited above. (By the way, this justifies the name *topologic*.)

¹To become acquainted with these systems the reader is referred to the following books published quite recently: the handbook (Aiello, Pratt-Hartmann, & van Benthem 2007), of which Chapter 6 contains a rather detailed exposition of *topologic*, and, respectively, the textbook (van Ditmarsch, van der Hoek, & Kooi 2007).

This set can be depicted as the full infinite binary tree. Moreover, every node of this tree can be annotated with the basic neighbourhood $U \in \mathcal{T}$ containing all possible prolongations of the initial segment leading to this node. In this way, a *tree of subsets of \mathcal{B}* results. Now, computing a concrete function $g : \mathbb{N} \rightarrow \{0, 1\}$ yields, step by step, a *bisection* of the actual open, constituting the respective knowledge state of the observer. Thus shrinking of opens proceeds in a *uniform* and *controlled* way here. Furthermore, and this is also an important point, only *some* elements of the domain (viz those from the canonical basis of \mathcal{T}) are needed for describing how more and more knowledge of g is obtained.

A similar phenomenon appears whenever the *measurement precision* has to be increased repeatedly during an *experiment* (say by one decimal place each time). Again, a uniform and controlled process turns up, this time concerning the knowledge of the measured object.

More examples of controlled shrinking from as diverse fields as philosophy, belief revision, basic topology, and public announcement logic,³ are given in Ch. 5 of the paper (Baškent 2007). These examples led the author to introduce a new modality $[F]$, which, roughly speaking, captures control by addressing all (or, more generally, some of) the distinguished functions on \mathcal{O} that represent a correspondingly (eg, syntactically) given procedure.

The precise definition and a more detailed discussion of $[F]$ will be provided in the next section. The problems arising out of this operator too will immediately become apparent there. Due to the intrinsic complexity of $[F]$, the development of the relevant modal theory seems to be difficult, and almost nothing is known about it.⁴ Somewhat surprisingly, we can make considerable headway by using methods from *hybrid logic*; see (Blackburn, de Rijke, & Venema 2001), Sec. 7.3, or Ch. 14 of the handbook (Blackburn, van Benthem, & Wolter 2007). This is carried out in the technical part of this paper.

The paper is organized as follows. In the next section, we insert the operator of controlled shrinking in *topologic* and comment on the new language, in particular, with regard to systems encompassing PDL. Then we recall the concepts from hybrid logic we need later on, and we briefly revisit *hybrid topologic*; see (Heinemann 2008). The final technical section contains an account of the resulting logic and of the basic issues indicated in the abstract. Concluding the paper, we sum up and state some of the remaining questions.

Controlled Shrinking

We now define the extended language, \mathcal{L} . Throughout this paper, we confine ourselves to the single-agent case.

The syntax of \mathcal{L} is based on a denumerable set $\text{Prop} = \{p, q, \dots\}$ of symbols called *proposition letters*. The set Form of all *formulas* over Prop is given by the rule

$$\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \beta \mid K\alpha \mid \Box\alpha \mid [F]\alpha$$

³See, eg, (Baltag, Moss, & Solecki 1998), and also (van Ditmarsch, van der Hoek, & Kooi 2007), Ch. 4.

⁴Nevertheless, just abstracting the operator $[F]$ from the given examples is the unquestionable merit of the paper (Baškent 2007).

then. While K and \Box denote the well-known modalities of knowledge and effort, the new operator $[F]$ should describe controlled shrinking.⁵ The missing boolean connectives are treated as abbreviations, as needed. The symbol \Diamond designates the dual of the effort operator. The duals of the other modalities are displayed by putting the corresponding letters in angle brackets; thus $\langle K \rangle$ denotes the dual of K and $\langle F \rangle$ denotes the dual of $[F]$.

Now, we turn to the semantics of \mathcal{L} . For a start, we fix the relevant domains. We let $\mathcal{P}(X)$ designate the powerset of a given set X .

Definition 1 (Controlled structures) 1. Let be given a triple $\mathcal{S} := (X, \mathcal{O}, \mathcal{F})$ such that

- (a) X is a non-empty set,
- (b) $\mathcal{O} \subseteq \mathcal{P}(X)$ is a set of subsets of X , and
- (c) $\mathcal{F} = \{f \mid f : \mathcal{O} \rightarrow \mathcal{O}\}$ is a set of one-place partial functions satisfying, for all $U \in \mathcal{O}$, the condition $f(U) \subseteq U$ whenever $f(U)$ exists.

Then \mathcal{S} is called a *controlled subset frame*.

- 2. Let $\mathcal{S} = (X, \mathcal{O}, \mathcal{F})$ be a controlled subset frame. The elements of the set

$$\mathcal{N}_{\mathcal{S}} := \{(x, U) \mid x \in U \text{ and } U \in \mathcal{O}\}$$

are called the *neighbourhood situations* of \mathcal{S} .

- 3. Let $\mathcal{S} = (X, \mathcal{O}, \mathcal{F})$ be a controlled subset frame and $V : \text{PROP} \rightarrow \mathcal{P}(X)$ be a mapping. Then V is called an \mathcal{S} -valuation.
- 4. Let $\mathcal{S} = (X, \mathcal{O}, \mathcal{F})$ be a controlled subset frame and V be an \mathcal{S} -valuation. Then,

$$\mathcal{M} := (X, \mathcal{O}, \mathcal{F}, V)$$

is called a *controlled subset space*, or, in short, a *CSS* (based on \mathcal{S}).

Three facts are worth mentioning for item 1 of this definition: First, control is represented by *functions on \mathcal{O}* , which is quite natural because of the aforementioned examples. Second, *partiality* should obviously be admitted in order to be general enough. And third, the requirement ' $f(U) \subseteq U$ ' in item 1 (c) shows shrinking, i.e., knowledge acquisition.

The neighbourhood situations introduced in item 2 of Definition 1 make up the atomic semantic entities of our language. They are used for evaluating formulas; see the next definition. In a sense, the set component of a neighbourhood situation measures the uncertainty about the associated state component at any one time.

We are mainly interested in *interpreted systems*, which are here formalized by the use of subset spaces (Definition 1.4). The assignment of sets of states to proposition letters by means of valuations (see item 3 above) is in accordance with the usual logic of knowledge; cf (Fagin *et al.* 1995) or (Meyer & van der Hoek 1995).

The final remark on Definition 1 concerns uniformity. Consider once again the tree example from the introduction.

⁵We adopt the name, which does not reflect the aspect of uniformity, from (Baškent 2007). However, the definitions in the present paper are a little different from those there.

In this example, proceeding in a controlled mode means going down the subset tree either one step left or one step right. Consequently, the set \mathcal{F} there consists of two functions which model the respective direction. Note that making progress in this way is uniform for all stages of the whole process (i.e., for all levels of the tree). Therefore, control and uniformity are captured at the same time.

Our next task is defining the relation of satisfaction. This is done with respect to a CSS \mathcal{M} . Thus satisfaction, which should hold between neighbourhood situations of the underlying frame and formulas from Form , is designated $\models_{\mathcal{M}}$. In the following, neighbourhood situations are written without brackets.

Definition 2 (Satisfaction; validity) *Let be given a CSS $\mathcal{M} = (X, \mathcal{O}, \mathcal{F}, V)$ based on $S = (X, \mathcal{O}, \mathcal{F})$, and let $x, U \in \mathcal{N}_S$ be a neighbourhood situation. Then*

$$\begin{aligned} x, U \models_{\mathcal{M}} p & : \iff x \in V(p) \\ x, U \models_{\mathcal{M}} \neg \alpha & : \iff x, U \not\models_{\mathcal{M}} \alpha \\ x, U \models_{\mathcal{M}} \alpha \wedge \beta & : \iff x, U \models_{\mathcal{M}} \alpha \text{ and } x, U \models_{\mathcal{M}} \beta \\ x, U \models_{\mathcal{M}} K\alpha & : \iff \text{for all } y \in U : y, U \models_{\mathcal{M}} \alpha \\ x, U \models_{\mathcal{M}} \Box \alpha & : \iff \left\{ \begin{array}{l} \forall U' \in \mathcal{O} : (x \in U' \subseteq U \\ \Rightarrow x, U' \models_{\mathcal{M}} \alpha) \end{array} \right. \\ x, U \models_{\mathcal{M}} [F]\alpha & : \iff \left\{ \begin{array}{l} \forall f \in \mathcal{F} : (x \in f(U) \\ \Rightarrow x, f(U) \models_{\mathcal{M}} \alpha), \end{array} \right. \end{aligned}$$

for all $p \in \text{Prop}$ and $\alpha, \beta \in \text{Form}$. In case $x, U \models_{\mathcal{M}} \alpha$ is true we say that α holds in \mathcal{M} at the neighbourhood situation x, U . Furthermore, a formula α is called valid in \mathcal{M} iff it holds in \mathcal{M} at every neighbourhood situation. (Manner of writing: $\mathcal{M} \models \alpha$.)

Note that the meaning of proposition letters is independent of the opens by definition, hence ‘stable’ with respect to \Box and $[F]$. This fact will also find expression in the logical system considered later on; see footnote 9 below.

For the rest of this section, we deal with the operator $[F]$. This is the key construct of our language. Definition 2 lights up how it is expressed that $\alpha \in \text{Form}$ is valid due to controlled shrinking, viz by quantifying over all functions that are responsible for that. At this point, the position of the present approach between *topologic* and PDL-like epistemic logics becomes quite clear: While we do not have any explicit control in *topologic*, and even syntactically presented control in dynamic epistemic logic, control (in terms of the functions from \mathcal{F}) is at least present in the semantics here; the modality $[F]$ then allows to speak about it in the formal language.

Definition 2 also indicates that by no means all the results that can be obtained by some effort must be yielded in a controlled manner. This property is, in fact, a validity for controlled subset spaces, as the first item of the subsequent proposition shows.

Proposition 3 *Let \mathcal{M} be a CSS. Then, for all $\alpha \in \text{Form}$,*

1. $\mathcal{M} \models \Box \alpha \rightarrow [F]\alpha$
2. $\mathcal{M} \models K[F]\alpha \rightarrow [F]K\alpha$.

Replacing $[F]$ with \Box in the second item yields the *Cross Axiom of topologic*. This axiom is the crucial schema of that

system; cf (Dabrowski, Moss, & Parikh 1996). – The proof of Proposition 3 is straightforward from Definition 2 and, therefore, omitted.

By integrating controlled shrinking into *topologic* we have really won more expressive power. This is demonstrated by the next example.

Example 4 *We consider the tree example from the introduction one last time. For all binary strings $w \in \{0, 1\}^*$, we let $w\mathcal{B} := \{h \in \mathcal{B} \mid h \text{ is prefixed by } w\}$. Then, for every $a \in \{0, 1\}$, we define $f_a : \mathcal{B} \rightarrow \mathcal{B}$ as follows: $f_a(w\mathcal{B}) := wa\mathcal{B}$ for all $w \in \{0, 1\}^*$, and $f_a(U) := \text{undefined}$ for all other elements $U \in \mathcal{T}$. (Thus these functions are the ones representing uniformity and control on the topological space $(\mathcal{B}, \mathcal{T})$; see the final remark on Definition 1 above.) Now, consider the controlled subset frame $S = (\mathcal{B}, \mathcal{T}, \{f_0, f_1\})$. Using this frame we are able to specify a certain liveness property of a program computing a concrete function $g : \mathbb{N} \rightarrow \{0, 1\}$:*

$$g, \mathcal{B} \models_{\mathcal{M}} \Box \Diamond \langle F \rangle \top$$

holds for all CSSs \mathcal{M} based on S . – Note that we are free to choose the functions we are interested in, as long as these are in accordance with the demands for uniformity or control. Thus if we restrict the frame S to $\{f_0\}$ in the third component, then the same formula says that a machine computing g time after time outputs the bit ‘0’.

Concluding this section, we look at Definition 2 in view of a logical theory of $[F]$. For the development of such a theory, the following question must be answered: How can we come to grips with the functions from \mathcal{F} ? Note that the operator $[F]$ provides only ‘indirect’ access to these functions. But for a model to be constructed, we must be able to handle them extensionally, and the semantics of $[F]$ must be respected at the same time. – In the next section, we supply the prerequisites for a possible answer.

Hybridization

In this section, the just defined language \mathcal{L} is hybridized. The logic invented in (Heinemann 2008) is sketched in addition. First, however, we recall the basic features from hybrid logic we need for that.⁶

The key idea of the hybrid approach is adding a set Nom of *nominals*, i.e., a set of *names of states*, to a modal language. One may take nominals as special proposition letters having a unique denotation. Already this simple concept is very powerful since a good deal more properties of frames can be expressed than before. However, hybrid logic develops its abilities in full only after integrating particular *binders*. For the purposes of this paper, the so-called *satisfaction operators* play a part. These operators enable one to evaluate formulas at the denotation of a nominal under discussion: For an arbitrary $i \in \text{Nom}$, the formula $@_i \alpha$ holds at some state x iff α holds at the state y denoted by i . Thus

⁶(Blackburn 2000) gives a nice overview (including history), in particular, of the various areas of application of hybrid logic; see also the references cited above.

the information stored in named states can be retrieved by means of satisfaction operators.

There is another hybrid peculiarity worth mentioning here: unorthodox proof rules. Such rules facilitate a proper treatment of names in the framework of modal logic. We shall list most of the unorthodox proof rules that will be utilized for \mathcal{L} below.

Now, due to the two-component semantics of \mathcal{L} we enrich this language with *two disjoint* sets of nominals called *names of states* and *names of sets*, respectively. Let $N_{stat} = \{i, j, \dots\}$ and $N_{sets} = \{I, J, \dots\}$ be the corresponding sets of symbols. Moreover, let A designate the *global modality* and E its dual; see (Blackburn, de Rijke, & Venema 2001), Sec. 7.1. The global modality is mainly used for simulating satisfaction operators in our bi-modal setting, as the reader will see in a moment.

But first we have to make the necessary adjustments concerning the semantics. To begin with, we introduce hybrid CSSs.

Definition 5 (Hybrid CSSs) *Let be given a controlled subset frame $S = (X, \mathcal{O}, \mathcal{F})$.*

1. *Assume that, for every $U \in \mathcal{O}$, any two distinct elements of $\{f(U) \mid f \in \mathcal{F}\}$ are incomparable with respect to inclusion. Then S is called regular.*
2. *A mapping $V : \text{Prop} \cup N_{stat} \cup N_{sets} \rightarrow \mathcal{P}(X)$ is called a hybrid S -valuation iff the following two conditions are satisfied:*
 - (a) *$V(i)$ is either \emptyset or a singleton subset of X for every $i \in N_{stat}$, and*
 - (b) *$V(I) \in \mathcal{O}$ for every $I \in N_{sets}$.*
3. *Let S be regular and V be a hybrid S -valuation. Then, $\mathcal{M} := (X, \mathcal{O}, V)$ is called a hybrid controlled subset space, or, in short, an HCSS.*

The regularity requirement from item 1 of Definition 5 is quite reasonable: If one is able to proceed to both U and U' by controlled shrinking, and if $U \subseteq U'$ is true additionally, then U may be excluded without worsening the process in question as a whole.⁷ Note that the above examples too are in accordance with this requirement.

Item 2 of Definition 5 takes into account that nominals may have an empty denotation. This is appropriate to us for technical reasons, but not usual for standard hybrid logic; see (Heinemann 2008) for a discussion regarding this. – We now extend Definition 2 accordingly.

Definition 6 (Hybrid satisfaction and validity) *Let $\mathcal{M} = (X, \mathcal{O}, \mathcal{F}, V)$ be an HCSS based on $S = (X, \mathcal{O}, \mathcal{F})$, and let $x, U \in \mathcal{N}_S$ be a neighbourhood situation. Then*

$$\begin{aligned} x, U \models_{\mathcal{M}} i & : \iff x \in V(i) \\ x, U \models_{\mathcal{M}} I & : \iff V(I) = U \\ x, U \models_{\mathcal{M}} A\alpha & : \iff \begin{cases} \text{for all } x', U' \in \mathcal{N}_S : \\ x', U' \models_{\mathcal{M}} \alpha, \end{cases} \end{aligned}$$

⁷Formally, if $W, U, U' \in \mathcal{O}$, $f(W) = U$, $f'(W) = U'$, and $U \subseteq U'$, then redefine $f(W) := \text{undefined}$. (f must not be left out completely since the regularity condition may be satisfied at some elements of \mathcal{O} different from W .)

where $i \in N_{stat}$, $I \in N_{sets}$ and $\alpha \in \text{Form}$.⁸

Thus the intended meaning is truly assigned to nominals. Furthermore, the modality A turns out to be ‘global’ with respect to neighbourhood situations here.

The formulas of the form $i \wedge I$, where $i \in N_{stat}$ and $I \in N_{sets}$, play an important part below since they can be taken as names of neighbourhood situations. We are able to associate a satisfaction operator with such a name by virtue of $@_{(i \wedge I)} \alpha \equiv E(i \wedge I \wedge \alpha)$, where $\alpha \in \text{Form}$. Thus formulas of the above form function as ‘proper’ nominals for controlled subset spaces.

The operators $@_{(i \wedge I)}$ are needed for axiomatizing the logic of HCSSs. This logic is naturally based on *hybrid topologic*. Hence we should briefly review the latter system.

Doing so, it is not necessary to list the corresponding axioms here. The reader is referred to the paper (Heinemann 2008) regarding this. We start off with the unorthodox proof rules of *hybrid topologic* instead.

Definition 7 (Unorthodox proof rules) *The following hybrid schemata have to be added to the usual modal ones:*

$$\begin{aligned} (\text{NAME}_{stat}) \quad & \frac{j \rightarrow \beta}{\beta} \quad (\text{NAME}_{sets}) \quad \frac{J \rightarrow \beta}{\beta} \\ (\nabla\text{-ENRICHMENT}) \quad & \frac{E(i \wedge I \wedge \nabla(j \wedge J \wedge \alpha)) \rightarrow \beta}{E(i \wedge I \wedge \nabla \alpha) \rightarrow \beta}, \end{aligned}$$

where $\alpha, \beta \in \text{Form}$, $i, j \in N_{stat}$, $I, J \in N_{sets}$, $\nabla \in \{\langle K \rangle, \diamond, E\}$, and j, J are new each time (i.e., do not occur in any other syntactic building block of the respective rule).

A ‘contrapository’ reading is suggested for the reader not familiar with this kind of proof rule. For example, the rule (NAME_{stat}) is to be read ‘if β is satisfiable, then $j \wedge \beta$ is satisfiable, too’ (provided that the nominal j does not occur in β). From that, the soundness of the unorthodox rules should be obvious.

Just the ENRICHMENT-schema for $\langle F \rangle$ must be added later on in order to obtain the proof rules for the logic accompanying \mathcal{L} .

Technically, the NAME and ENRICHMENT rules are used for proving an appropriate *Lindenbaum Lemma* and an *Existence Lemma*, respectively; cf (Heinemann 2008), Lemmata 3.3 and 3.6. Both auxiliary results constitute the first steps towards the completeness theorem for *hybrid topologic*. In fact, these lemmata make possible a hybridization of the canonical model of the logical system under discussion. A model \mathcal{M} falsifying a given non-derivable formula can then be obtained via a certain space, X , of partial functions over the carrier set D of that hybridized structure. We emphasize the individual components of \mathcal{M} :

- The domain $\text{dom}(h)$ of every function $h \in X$ is a maximal subset of the set $\mathcal{Q} := \{[\Sigma] \mid \Sigma \in D\}$ of all equivalence classes $[\Sigma] := \{\Sigma' \in D \mid \Sigma \xrightarrow{K} \Sigma'\}$ of the accessibility relation \xrightarrow{K} induced by the knowledge modality K . In this connection, maximality is meant with regard to the two subsequent conditions:

⁸From now on, Form denotes the set of formulas of the enriched language.

1. $h([\Sigma]) \in [\Sigma]$ for all $[\Sigma] \in \text{dom}(h)$, and
2. $h([\Sigma]) \xrightarrow{\square} h([\Theta])$ for all $[\Sigma], [\Theta] \in \text{dom}(h)$ such that $[\Sigma]$ *precedes* $[\Theta]$ in the following sense:

$$\exists \Sigma' \in [\Sigma], \Theta' \in [\Theta] : \Sigma' \xrightarrow{\square} \Theta',$$

where $\Sigma, \Theta \in D$ and $\xrightarrow{\square}$ denotes the accessibility relation belonging to the effort modality \square .

We write $h_\Sigma := h([\Sigma])$ in case $h([\Sigma])$ exists. Furthermore, we define

- $U_\Sigma := \{h \in X \mid h_\Sigma \text{ exists}\}$, for all $\Sigma \in D$,
- $\mathcal{O} := \{U_\Sigma \mid \Sigma \in D\} \cup \{\emptyset\}$, and
- $V : \text{Prop} \cup N_{\text{stat}} \cup N_{\text{sets}} \longrightarrow \mathcal{P}(X)$ by

$$h \in V(c) : \iff \begin{cases} c \in h_\Sigma \text{ for some } \Sigma \in D \\ \text{such that } h_\Sigma \text{ exists,} \end{cases}$$

for all $c \in \text{Prop} \cup N_{\text{stat}} \cup N_{\text{sets}}$.

With that, we obtain the relevant *Truth Lemma*, cf (Heinemann 2008), Lemma 3.11, from which the completeness of *hybrid topologic* can easily be concluded. – We shall come back to the above construction in the next section.

It is true that *hybrid topologic* is decidable as well; cf (Heinemann 2008), Theorem 4.4. This is proved by establishing a certain finite model property of this logic. The suitable method of proof for that is *filtration*; cf (Goldblatt 1992), § 4. For later purposes, we now go into that proof a little.

First of all, the filter set must be arranged in a way which facilitates the validation of all the axioms of the system for the filtrated model, called ad hoc $\overline{\mathcal{M}}$. In particular, the nominals occurring in the (consistent) formula γ for which we want to find a finite model have to be taken into account for that. Now, the point comes which is crucial to hybrid decidability: It is sufficient for the verification of the axioms involving nominals to check only the instances in which nominals from the filter set occur. This reduction can be achieved by modifying the valuation of $\overline{\mathcal{M}}$ appropriately. Therefore, it is no longer necessary to show that the first-order frame property associated with the respective axiom is valid for $\overline{\mathcal{M}}$. – This technique too will be applied in the next section.

The hybrid logic of controlled shrinking

The main outcome of this paper is a completeness and, respectively, a decidability result for the hybrid logic arising from \mathcal{L} , which is designated LC (‘Logic of Control’). In this section, both theorems are stated and the respective proofs are outlined.

LC contains three additional axiom schemata: those two from Proposition 3, and the subsequent purely hybrid one corresponding to regularity:

$$(*) \quad \langle F \rangle (i \wedge I) \wedge \langle F \rangle (i \wedge J) \wedge @_{(i \wedge I)} \neg J \rightarrow @_{(i \wedge I)} K \square \neg J \wedge @_{(i \wedge J)} K \square \neg I,$$

where $i \in N_{\text{stat}}$ and $I, J \in N_{\text{sets}}$. The effect of the latter schema will be discussed in a minute.⁹

⁹The expected schema $(p \rightarrow [F]p) \wedge (\langle F \rangle p \rightarrow p)$ is derivable from the analogous one for \square , which is an axiom of *topologic* ($p \in \text{Prop}$); cf the first remark right after Definition 2.

Since the proof rules of LC were mentioned in the previous section already we can directly strive for completeness. To this end, we extend the structure (X, \mathcal{O}, V) defined above by a fitting set of functions. The following relation between opens from \mathcal{O} will prove to be useful for that:

$$U_\Sigma \prec U_\Theta : \iff \exists h \in U_\Sigma \cap U_\Theta : h_\Sigma \xrightarrow{[F]} h_\Theta,$$

where $\xrightarrow{[F]}$ denotes the accessibility relation induced by the operator of controlled shrinking and the other notations are taken from above. Now, we call a subset $\mathcal{K} \subseteq \mathcal{O}$ a \prec -chain iff the reflexive and transitive closure of \prec restricted to \mathcal{K} is a linear order. Let \mathfrak{K} be the set of all \prec -chains contained in \mathcal{O} that are *maximal* (with respect to inclusion). For every $\mathcal{K} \in \mathfrak{K}$, we would like to define a set-valued partial function $f_{\mathcal{K}}$ as follows:

$$f_{\mathcal{K}}(U) := \begin{cases} U' & \text{if } U' \in \mathcal{K} \text{ and } U \prec U' \\ \text{undefined} & \text{otherwise,} \end{cases}$$

for all $U \in \mathcal{O}$. It turns out that this is in fact possible.

Lemma 8 *For every $\mathcal{K} \in \mathfrak{K}$, the relation $f_{\mathcal{K}}$ is a partial function, $f_{\mathcal{K}} : \mathcal{O} \rightarrow \mathcal{O}$.*

Note that this is a subtle point since it cannot be excluded from the outset that \mathcal{K} contains two or more \prec -successors of the same open (induced by different elements of \mathcal{K}):

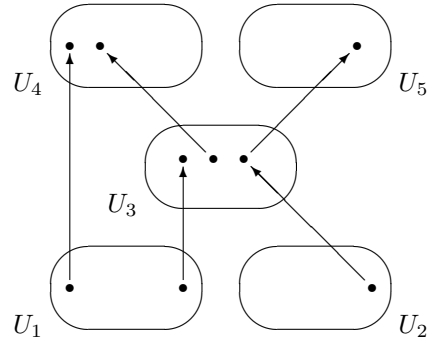


Figure 1: A non-regular situation

In this picture, arrows should indicate the accessibility relation belonging to $[F]$. Thus any \prec -chain $\mathcal{K} \in \mathfrak{K}$ containing $U_1 \prec U_3 \prec U_4$ as a subchain violates the regularity condition (whereas, eg, $U_2 \prec U_3 \prec U_5$ is an admissible subchain of some $\mathcal{K}' \in \mathfrak{K}$).

The axiom schema $(*)$ preserves us from such critical situations and is decisive to the proof of Lemma 8 thus.

Now, we let

- $\mathcal{F} := \{f_{\mathcal{K}} : \mathcal{O} \rightarrow \mathcal{O} \mid \mathcal{K} \in \mathfrak{K}\}$.

Then we obtain the subsequent *Truth Lemma*.

Lemma 9 *The just constructed model $\mathcal{M} = (X, \mathcal{O}, \mathcal{F}, V)$ constitutes an HCSS. Moreover, for all formulas $\alpha \in \text{Form}$, elements $h \in X$, and points $\Sigma \in D$ such that $h \in U_\Sigma$, we have that*

$$h, U_\Sigma \models_{\mathcal{M}} \alpha \iff \alpha \in h_\Sigma.$$

The proof of the second part of this lemma is done by induction on the structure of formulas. It should be mentioned that both schemata from Proposition 3 are needed for the case $\alpha = [F]\beta$ of the induction step. Actually, it is suitable to replace $[F]\beta$ with $\langle F \rangle \beta$ in this case. Note that $h, U_{[\Sigma]} \models_{\mathcal{M}} \langle F \rangle \beta$ is valid iff there exists some $\mathcal{K} \in \mathcal{R}$ such that $\beta \in h_{\Theta}$, where $\Theta \in D$ satisfies $U_{[\Theta]} = f_{\mathcal{K}}(U_{[\Sigma]})$.

The first of our main results now follows immediately.

Theorem 10 *The system LC is sound and complete with respect to the class of all HCSSs.*

Concerning decidability, we follow the strategy which was briefly described at the end of the previous section. Thus we prove that the additional axioms are valid for the filtrated model. This is quite easy in case of the ‘Inclusion Axiom’, $\Box\alpha \rightarrow [F]\alpha$. Only the defining properties of the so-called *minimal* filtration, cf (Goldblatt 1992), 4.5, are used for that. A more involved argument is necessary for the ‘Second Cross Axiom’, $K[F]\alpha \rightarrow [F]K\alpha$. But, fortunately, one can proceed as for the usual Cross Axiom in this case; cf (Dabrowski, Moss, & Parikh 1996), Sec. 2.3. The case of the ‘Regularity Axiom’, $(*)$, is the most complicated one, as expected. Here the basic filter set has to be modified. As a first approximation, all subformulas of $(*)$ that contain only nominals occurring in γ must be added (where γ is as above). Note that the filter set remains finite in doing so. Now, the filtration machinery has to be applied carefully. As a consequence, the desired finite model property of the logic results. This implies the following theorem.

Theorem 11 *The set of all LC-derivable formulas is decidable.*

Concluding remarks

In this paper, we took up a spatial view of knowledge to the effect that subset spaces appeared as the appropriate models. We developed a logical system being located between *topologic* and dynamic epistemic logic. The intermediary element was a modality describing uniformity and control during knowledge acquisition. Because of the immanent complexity of this operator we hybridized the source language in order to obtain a sound and complete axiomatization as well as the decidability of the arising logic. The functional structure of the hybridized canonical model was decisively utilized for that.

Generality, simplicity, and the trans-sectoral position between two areas of epistemic reasoning, make up the merits of the new system. Nevertheless, it remains a lot to be done with regard to both theory and practicality. One problem concerning the first field was already mentioned above: What about the purely modal theory of controlled shrinking? Another one, for which most systems related to *topologic* are notorious, is computational complexity.

As to extensions, one might think of a combination of temporal systems (eg, those from (Heinemann 1999)) and the one presented in this paper. (In this connection, see (Henriksen & Thiagarajan 1999) for an example of dynamic temporal logic.) Moreover, multi-agent versions of such systems would further reduce the gap between *topologic* and dynamic epistemic logic.

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