# Preference-Based Default Reasoning

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### Abstract

It is well-known that default reasoning and preference-based decision making both make use of preferential relations between possible worlds resp. alternatives. In this paper, we explore this methodological relationship in more detail by considering inference as a decision making problem. A foundational approach to preference fusion is used to define a non-monotonic inference relation System ARS which turns out to be a refinement of System Z. We compare System ARS to other default reasoning strategies and prove that it satisfies an irrelevance property which is violated by System Z.

## Introduction

Many inference systems in nonmonotic reasoning make use of orderings of the defaults in conditional knowledge bases to assign a plausibility to each possible world, see (Benferhat *et al.* 1993) for a survey. The resulting preference relation on the set of possible worlds is then taken as the basis for inference. In this paper, we will use a foundational method for preference fusion to derive such a preference relation on worlds. Hence, we consider inference as a decision making problem deciding which worlds are more plausible than others on the base of explicit criteria. In order to evaluate the resulting inference relation, we compare it to well-known reasoning systems such as System Z. In System Z (Goldszmidt & Pearl 1996), where the plausibility of a world is defined by the priority of the highest default it falsifies, lower defaults do not effect possible consequences. Although this is reasonable in most cases it may have unexpected results. The inference relation is very cautious and is afflicted with the so-called *irrelevance* problem: plausible consequences may not be deduced in the presence of irrelevant information.

The notion of irrelevance applied here focuses on the conclusions which should be deduced regarding the new information. More precisely, a conditional knowledge base, which does not contain any information about two atoms $x$ and $y$, should be irrelevant for the default $(y|x)$: if any premises including $x$ are given, then $y$ should be deduced.

Consider for instance the well-known set of default rules $\Delta_{Tweety} = \{\delta_1 = (f|b), \delta_2 = (\overline{f}|p), \delta_3 = (b|p)\}$, constituting that a bird normally flies, that a penguin normally cannot fly and that it is normally a bird, expanded by any default $\delta_4 = (y|x)$ with new atoms $x, y \notin \{p, f, b\}$. Given two different possible worlds $\omega$ and $\omega'$ which both satisfy $x$ and only differ in terms of the atom $y$, an inference operator should always prefer the world which satisfies $y$, even under exceptional circumstances. Instead System Z gives equal plausibility to $\omega = pbfx\overline{y}$ and $\omega' = pbfxy$. Given a penguin-bird, which can fly, and given $x$, $y$ would not be concluded.

Our system addresses problems like these and resembles System Z in various ways. In the method proposed in this paper we adopt the ordering of defaults of System Z, but take a completely different way to obtain the plausibilities of the worlds, as we make use of a basic preference fusion operator by Andreka, Ryan and Schobbens (*ARS*) (Andreka, Ryan, & Schobbens 2002). We extend work begun in (Ritterskamp & Kern-Isberner 2007).

In the following section we introduce the preliminaries of System Z and the fundamentals of the *ARS* approach. Afterwards, we present our system and discuss the property of irrelevance consequently. Finally, we compare our method to similar approaches and conclude.

## Preliminaries

Let $\Delta$ be a set of defaults which constitutes a conditional knowledge base and $\Omega$ be the set of possible worlds. We consider default rules $(c|p)$ with the default consequence $c$ and the premise $p$ being propositional formulas on a language $\mathcal{L}$. A default rule $(c|p)$ is satisfied (falsified) by a world $\omega$, if $\omega \models (p \Rightarrow c)$ holds (does not hold). We speak of verification, if $\omega \models p \wedge c$. Given a set of defaults $\Delta$ we say that $\omega$ satisfies $\Delta$ (written $\omega \models \Delta$) if every default $\delta \in \Delta$ is satisfied, and we say it falsifies if there exists at least one default $\delta \in \Delta$ which is falsified by $\omega$.

Following Spohn's (Spohn 1987) approach of ordinal conditional functions a world can be ranked on an ordinal scale according to its degree of plausibility. The more normal a world is the smaller is its ranking value.

**Definition 1** *An ordinal conditional function (a ranking function) is a function $\kappa : \Omega \rightarrow \mathbb{N} \cup \{\infty\}$ with $\kappa(\omega) = 0$ for at least one $\omega \in \Omega$.*

Each formula $f$ can now be assigned a ranking value, which is the ranking value of a minimal world according to the ranking of the worlds, i.e. $\kappa(f) = \min\{\kappa(\omega) | \omega \models f.\}$

A conditional knowledge base entails a default $(c|p)$, if all minimal worlds satisfying $p$ satisfy $c$. This is the notion of preferential entailment, which is the basis for inference with conditional knowledge bases of this kind. In the special case of admissible ranking functions there is another way to figure out preferential entailment.

**Definition 2** *Given a default base* $\Delta$ *a ranking function* $\kappa$ *is called admissible w.r.t.* $\Delta$*, iff* $\forall(c|p) \in \Delta : \kappa(p \wedge c) < \kappa(p \wedge \overline{c})$.

Due to the condition above any default $(c|p)$ is entailed, or equivalently $c$ can be inferred given $p$ (written $p \mid\!\sim c$), if the degree to which it is possible to have $p \wedge c$ is greater than the degree to which $p \wedge \overline{c}$ is possible, i.e. $p \mid\!\sim c$ iff $\kappa(p \wedge c) < \kappa(p \wedge \overline{c})$.

We use the ranking values as a comparison criterion for possible worlds. Therefore in the following and without loss of generality we only consider ranking functions where for every ranking value $k \leq \max_{\omega\in\Omega}(\kappa(\omega))$ there exists at least one world $\omega$ with $\kappa(\omega) = k$ . These ranking functions can be expressed by binary relations $\leq_R$ on the set $\Omega$. More precisely, the binary relations considered here are total preorders, i.e. reflexive and transitive relations. A total preorder $R$ on a finite set $M$ can be described as an ordered partition of $M$ into $k+1$ sets $M_0, \dots, M_k$ where each element in $M$ is contained in exactly one set. We have

$$\omega \leq_R \omega' \text{ iff } (\omega, \omega') \in R \text{ iff } (\omega \in M_i \wedge \omega' \in M_j \wedge i \leq j).$$

For instance $R = \{(\omega_1, \omega_1), (\omega_2, \omega_2), (\omega_1, \omega_2)\}$ is represented by sets $M_0 = \{\omega_1\}$ and $M_1 = \{\omega_2\}$. We have $\omega <_R \omega'$ if $\omega \leq_R \omega'$ and not $\omega' \leq_R \omega$ , and $\omega = \omega'$ if both $\omega \leq_R \omega'$ and $\omega' \leq_R \omega$.

## System Z

System Z (Goldszmidt & Pearl 1996) makes use of an ordered partition of the default rules $\Delta$ respecting a certain tolerance relationship between these rules: Every default rule in $\Delta_i$ has to be tolerated by every default rule in every layer $\Delta_j$ with $j \geq i$. Additionally, each layer $\Delta_i$ of the ordering of default rules has to be maximal. A rule $(c|p)$ is tolerated by other rules $(c_1|p_1), \dots (c_k|p_k)$ if a world $\omega$ exists with $\omega \models (c \wedge p) \wedge \bigwedge_i (p_i \Rightarrow c_i)$.

With the ranking values

$$\kappa^Z(\omega) = \begin{cases} 0 & \omega \models \Delta_j \\ \max\{j | \ \exists \delta_i \in \Delta_j : \ \omega \text{ fals. } \delta_i\} + 1 & else \end{cases}$$

the total preorder $\leq_Z$ is generated:

$$\omega \leq_Z \omega' \text{ iff } \kappa^Z(\omega) \leq_Z \kappa^Z(\omega').$$

Due to the maximality of the layers for every ranking value $k \leq \max_{\omega\in\Omega}(\kappa^Z(\omega))$ there exists at least one world $\omega$ with $\kappa^Z(\omega) = k$. Figure 1 shows a graphical representation of the relation: for each ranking value there is a layer which contains the worlds this value is assigned to, the layers are arranged in ascending order.

| |
|---|
| $\{\omega \| \kappa^Z(\omega) = i+1\}$ |
| $\dots$ |
| $\{\omega \| \kappa^Z(\omega) = 0\}$ |

Figure 1: Graphical representation of $R_Z$

## ARS-Approach

Andreka, Ryan and Schobbens (Andreka, Ryan, & Schobbens 2002) define priority operators that map a set of relations to a single relation. The combination satisfies natural conditions, which are a variant of Arrow's conditions (Arrow 1950).

A preference relation $R$ on a set $M$ is simply any relation $R \subseteq M \times M$. Let $(m_1, m_2) \in R^<$ iff $(m_1, m_2) \in R$ and $(m_2, m_1) \notin R$. A priority operator $o$ is denoted by a priority graph $(N, <, v)$, where $<$ is a strict partial order on a set $N$ and $v$ is a function from $N$ to a set of variables $V$. The operator $o$ maps a set of relations $(R_x)_{x\in V}$ to the relation $o((R_x)_{x\in V})$ defined by $(m, n) \in o((R_x)_{x\in V})$ iff

$$\forall i \in N\text{: } ((m, n) \in R_{v(i)} \vee \exists j \in N\text{: } (j < i \wedge (m, n) \in R^<_{v(j)}))$$

In this definition the lexicographic rule is used with a priority on the set $N$ and therefore indirectly on the relations $(R_x)_{x\in V}$. Due to the use of the function $v$ it is possible to use a relation $R_x$ multiple times, which increases the expressive power of priority operators.

In (Andreka, Ryan, & Schobbens 2002) it is shown that every priority operator can be expressed by combinations of two basic operators called *but* and *on the other hand*.

In our approach we only use the operator *but*, which we will call $\oplus$ in this paper. Given two relations $R_1$ and $R_2$ on the same set $M$, then $(m, n) \in R_1 \oplus R_2$ iff $(m, n) \in (R_1 \cap R_2) \cup R_2^<$.

# System ARS

In System Z a stratification of defaults is an arrangement in specificity order. Defaults being tolerated by only a few other defaults are considered more specific. The more specific the knowledge, the more it should be emphasized. System Z achieves this emphasis by ranking the worlds according to the numbers of the highest falsified layer, which fortifies the influence of more specific defaults. In our approach we use the same stratification but take a different method to obtain the ordering of the worlds. First of all we let the sets of defaults induce input relations for preference fusion. We then use the operator $\oplus$, which fortifies the influence of the second operand, inductively on the input relations. Due to the character of the operator its application leads to a lexicographic definition of preference in our system, which we describe in detail in this section.

Our approach is possible due to different reasons. Firstly, the operator $\oplus$ preserves the property of being total preorders, secondly we get admissible ranking functions, which can be used as the basis for inference. These important properties are going to be proved later using similarities of the ranking of the worlds in System Z and in our approach. This

| $R(i)$ | $K_{i+1}$ | $R(i+1)$ |
|---|---|---|
| $<$ | $=$ | $<$ |
| $>$ | $=$ | $>$ |
| $=$ | $=$ | $=$ |
| $<$ | $<$ | $<$ |
| $>$ | $<$ | $<$ |
| $=$ | $<$ | $<$ |
| $<$ | $>$ | $>$ |
| $>$ | $>$ | $>$ |
| $=$ | $>$ | $>$ |

Table 1: Operator table for $\oplus$

similarity to System Z is another reason for regarding our approach promising.

## The New Ranking $R_\oplus$

We let the layers $\Delta_i$ used in System Z, i.e. sets of default rules, induce preorders $K_i$ on the set of possible worlds.

**Definition 3** *Given two possible worlds $\omega$ and $\omega'$, then $\omega \leq_i \omega'$(iff $(\omega, \omega') \in K_i$) iff $\omega' \models \Delta_i \Rightarrow \omega \models \Delta_i$.*

Equivalently we can use ranking functions $\kappa_i : \Omega \rightarrow \{0,1\}$ to describe the input relations, $\kappa_i(\omega) = 0$ iff $\omega \models \Delta_i$.

As suggested above, the operator $\oplus$ should emphasize more specific knowledge. This can be achieved by using higher input relations as second operands and the former results as first operands. Respecting only the first layer of defaults $\Delta_0$ the resulting ranking of the worlds should be the same as $K_0$, so $R(0) = K_0$ in this case. Recursively applied

$$R(i+1) = R(i) \oplus K_{i+1} = (R(i) \cap K_{i+1}) \cup K_{i+1}^{<}, \quad i \geq 1,$$

leads to ranking of worlds $R_\oplus = R(k)$, with $k+1$ being the number of layers. Equivalently,

$$\omega \leq_{R(i+1)} \omega' \text{iff } (\omega <_{i+1} \omega') \vee ((\omega =_{i+1} \omega') \wedge (\omega \leq_{R(i)} \omega')).$$

Only in the case of equivalence of two worlds in $K_{i+1}$ the former result $R(i)$ has an impact on their relationship in $R(i+1)$. Otherwise $K_{i+1}$ prevails. Table 1 shows the application of the operator $\oplus$.

We will now show that $R_\oplus$ is a total preorder. This important feature makes it possible to use ranking functions to describe it.

**Lemma 1** *Let $R_1$ and $R_2$ be two total preorders. $R_1 \oplus R_2$ is also a total preorder.*

*Proof*. $R = R_1 \oplus R_2 = (R_1 \cap R_2) \cup R_2^{<}$. The operator $\oplus$ is part of the ARS framework, hence it preserves transitivity. As both relations are reflexive it is easy to see that $R$ remains reflexive: every tuple $(\omega, \omega)$ will remain even in the intersection set of $R_1$ and $R_2$. The result $R$ is also total, because we get at least $(\omega_1, \omega_2) \in R$ or $(\omega_2, \omega_1) \in R$: If we have equality of two worlds $\omega_1$ and $\omega_2$ in $R_2$ none of the tuples $(\omega_1, \omega_2)$ or $(\omega_2, \omega_1)$ of $R_1$ will be lost when constructing $R_1 \cap R_2$. Otherwise the union with $R_2^{<}$ will definitely contain one of the two tuples. □

$$\begin{array}{c} K_0 \\ | \\ K_1 \\ | \\ \vdots \\ | \\ K_k \end{array}$$

Figure 2: Priority operator of $R_\oplus$

**Theorem 1** *$R_\oplus$ is a total preorder.*

*Proof*. This Theorem follows directly from Lemma 1 by induction. □

Due to its lexicographic origin the relation $R_\oplus$ can be expressed the following way:

$$\omega \leq_\oplus \omega' \Leftrightarrow$$
$$\forall j : \omega =_j \omega' \vee \exists i : (\omega <_i \omega' \wedge \forall j > i : \omega =_j \omega')$$

Given equivalence of two worlds $\omega_1$ and $\omega_2$ in every input relation, we have equivalence in $R_\oplus$.

**Lemma 2** $\omega_1 =_\oplus \omega_2$ *iff* $\forall i \in \{0, \ldots, k\} : \omega_1 =_i \omega_2$

*Proof*. Let us consider an arbitrary layer $\Delta_i$ and the application of $\oplus$ with an input relation $K_{i+1}$ on the corresponding relation $R(i)$:
$R(i+1) = R(i) \oplus K_{i+1} = (R(i) \cap K_{i+1}) \cup K_{i+1}^{<}$.
The table for $\oplus$ shows that if and only if $\omega_1 =_{R(i)} \omega_2$ and $\omega_1 =_{i+1} \omega_2$ we get $\omega_1 =_{R(i+1)} \omega_2$. We get the same for $R^\oplus$ requiring equivalence in $R(k-1)$ and $K_k$, which requires equivalence in $R(k-2)$ and $K_{k-1}$ and so on, leading to equivalence in every input relation $K_i$. □

In the ARS-framework priority operators have a graphic representation. As only the operator $\oplus$ of the framework is used, we get a linear arrangement, see figure 2.

In the following example the operator $\oplus$ is applied once, as there are only two input relations. In this case it leads to the preorder, which would also be constructed by System Z.

**Example [Tweety]** In the simple case of Tweety we have a set of three default rules $\Delta_{Tweety} = \{\delta_1 = (f|b), \delta_2 = (\overline{f}|p), \delta_3 = (b|p)\}$, constituting that a bird normally flies, that a penguin normally cannot fly and that it is normally a bird. We get two tolerance layers $\Delta_0 = \{(f|b)\}$ and $\Delta_1 = \{(\overline{f}|p), (b|p)\}$, because $(f|b)$ is being tolerated by all of the default rules while the other two rules only tolerate each other. These two sets of default rules imply the preorders $K_0$ and $K_1$, see figure 3.

| | |
|---|---|
| $\overline{p}\,\overline{f}b, p\overline{f}b$ | $K_0$ |
| $\overline{p}\,\overline{f}\,\overline{b}, p\overline{f}\,\overline{b}, \overline{p}f\overline{b}, pf\overline{b}, \overline{p}fb, pfb$ | $(f\|b)$ |
| $p\overline{f}\,\overline{b}, pf\overline{b}, pfb$ | $K_1$ |
| $\overline{p}\,\overline{f}\,\overline{b}, \overline{p}f\overline{b}, \overline{p}\,\overline{f}b, p\overline{f}b, \overline{p}fb$ | $(\overline{f}\|p)(b\|p)$ |

Figure 3: Input relations for the Tweety example

Figure 4 shows both the relations $R_\oplus$ and $R_Z$ in the Tweety example consisting of three layers. It becomes apparent, that the second input relation has a stronger influence on the plausibility ranking.

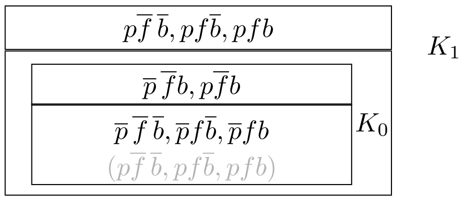


Figure 4: Resulting relation in the Tweety example

## The Inference Relation

$R_\oplus$ is a total preorder, so we can use a ranking function $\kappa_\oplus$ to describe the relation $R_\oplus$, with $\kappa_\oplus(\omega) < \kappa_\oplus(\omega')$ iff $\omega <_\oplus \omega'$. The ranking function is admissible w.r.t.$\Delta$ – we will prove this later when we get into more details of $R_\oplus$ – so it is possible to define the inference relation $\mid\!\sim_\oplus \subseteq \mathcal{L} \times \mathcal{L}$.

$$p \mid\!\sim_\oplus c \text{ iff } \kappa(p \wedge c) < \kappa(p \wedge \overline{c}).$$

Due to the ranking being a total preorder and due to the admissibility of the ranking function the inference relation $\mid\!\sim_\oplus$ satisfies several of the desired properties of commonsense reasoning. It satisfies the core of these properties, which are the rationality postulates of System P (Kraus, Lehmann, & Magidor 1990). The core represents a complete and sound axiom system for inference relations. Furthermore, inference relations of this kind are closed under rational monotony, for a proof see (Pearl 1990). The rational monotony ensures, that given $p \mid\!\sim_\oplus c$ and $p \mid\!\sim_\oplus \neg q$ we can conclude $p \wedge q \mid\!\sim_\oplus c$.

**Difference Between $R_\oplus$ and $R_Z$** For each layer $\Delta_i$ of the defaults we have an input relation $K_i$.

While the recursive application of the operator $\oplus$ very often leads to $R_Z$, see figure $5$ a) which covers a lot of cases, the set-up in figure $5$ b) causes another ranking. The figure shows the $\kappa_i$-values of $j+1$, $j+2$ respectively, input-relations for two possible worlds. A $0$ $(1)$ means that the world satisfies (falsifies) the layer: $\kappa_i(\omega) = 0$ iff $\omega \models \Delta_i$, and $\kappa_i(\omega) = 1$ else.

$a)$

| | $i=0$ | $i=1$ | $\dots$ | $i=j$ |
|---|---|---|---|---|
| $\kappa_i(\omega_1)$ | $0/1$ | $0/1$ | $\dots$ | $1$ |
| $\kappa_i(\omega_2)$ | $0/1$ | $0/1$ | $\dots$ | $0$ |

$b)$

| | $i=0$ | $i=1$ | $\dots$ | $i=j$ | $i=j+1$ |
|---|---|---|---|---|---|
| $\kappa_i(\omega_1)$ | $0/1$ | $0/1$ | $\dots$ | $1$ | $1$ |
| $\kappa_i(\omega_2)$ | $0/1$ | $0/1$ | $\dots$ | $0$ | $1$ |

Figure 5: Example for the difference between $R_Z$ and $R_\oplus$

Crucial for this setting is the relation $K_j$ $(K_{j+1})$. Let us first look at the situation a) in figure $5$ without the existence of $K_{j+1}$, which means that $K_j$ is the relation with the highest priority. The world $\omega_1$ falsifies one of the defaults in layer $\Delta_j$ while $\omega_2$ does not falsify any of them. In System Z $\kappa^Z(\omega_2) < \kappa^Z(\omega_1)$ holds. We get the same relationship in System ARS. This is independent of the input relations $K_i$ with $i < j$, which means that the relationships of many pairs of world is the same for both Systems.

With $K_{j+1}$ as the highest relation the situation is different: As both worlds are falsifying one of the default rules in the last layer $\Delta_{j+1}$, System Z leads to $\kappa^Z(\omega_1) = \kappa^Z(\omega_2)$. This means that both tuples $(\omega_1, \omega_2)$ and $(\omega_2, \omega_1)$ have to be contained in the resulting relation. But since $(\omega_1, \omega_2)$ is not contained in $K_j$ or in $K^<_{j+1}$ and $R(j+1) = R(j) \oplus K_{j+1}$, the tuple $(\omega_2, \omega_1)$ is not in the resulting relation of System ARS, which means that System Z and System ARS may have a different behaviour.

**Connection Between $R_\oplus$ and $R_Z$** A relation $R$ is a refinement of a relation $S$ iff $R \subseteq S$.

**Theorem 2** *$R_\oplus$ is a refinement of $R_Z$.*

*Proof* . Both $R_\oplus$ and $R_Z$ are total relations. Therefore for every pair of worlds $\omega$ and $\omega\prime$ at least one of the tuples $(\omega, \omega')$ or $(\omega', \omega)$ is contained in each relation.

Assuming $R_\oplus$ is not a refinement of $R_Z$, there has to be a tuple $(\omega_1, \omega_2)$ for which we either have $\omega_1 =_\oplus \omega_2$ and $\omega_1 <_Z \omega_2$ or we have $\omega_1 >_\oplus \omega_2$ and $\omega_1 <_Z \omega_2$.

Following the assumptions and considering System Z there has to be a layer $\Delta_i$ where $\omega_2$ falsifies a default in this layer while $\omega_1$ does not (which means $\omega_1 <_i \omega_2$), and for every $\Delta_j$ with $j \geq i$ we neither have the opposite case nor both of them falsify a default in $\Delta_j$.

The first case $\omega_1 =_\oplus \omega_2$ conflicts with $\omega_1 <_i \omega_2$ because of Lemma 2. In the other case $\omega_1 >_\oplus \omega_2$ leads to another conflict: considering the highest layer $\omega_1 >_k \omega_2$ is not possible because of the existence of an $i$ with $\omega_1 <_i \omega_2$ (see above) and the only other possibility to get $>$ is $\omega_1 >_{R(k-1)} \omega_2$, leading to the same problem. □

The similarity of the structure of both world rankings $R_Z$ and $R_\oplus$ is a result of the refinement relationship. Two worlds $\omega_1$ and $\omega_2$ with $\kappa^Z(\omega_1) = \kappa^Z(\omega_2)$ may get separated, resulting in a segmentation of a layer.

**Inference** The segmentation of the layers may lead to additional inferences that System Z does not conclude. The inferences of System Z remain.

**Theorem 3** *For every $p, c \in \mathcal{L}$ we have:*

$$\textit{If } p \mid\!\sim_Z c \textit{ then } p \mid\!\sim_\oplus c.$$

*Proof* . Due to the refinement relation between $R_Z$ and $R_\oplus$ we have $\kappa^Z(\omega_1) < \kappa^Z(\omega_2) \;\Rightarrow\; \kappa_\oplus(\omega_1) < \kappa_\oplus(\omega_2)$. □

Using a ranking of worlds as a foundation for inference is reasonable, if its ranking function is admissible. It has been proved in (Lehmann 1989) that this is the basis for reasonable inference. Now we show the important result, that the ranking function of $R_\oplus$ is admissible.

**Theorem 4** *The ranking function of $R_\oplus$ is admissible with respect to $\Delta$.*

| | $\Delta_0$ | | $\Delta_1$ | |
|---|---|---|---|---|
| | $(f\|b)$ | $(y\|x)$ | $(b\|p)$ | $(\overline{f}\|p)$ |
| $\omega_1 = p\,b\,f\,x\,y$ | | | | false |
| $\omega_2 = p\,b\,f\,x\,\overline{y}$ | | false | | false |

Figure 6: Irrelevance

*Proof* . Let $(c|p)$ be an arbitrary default of $\Delta$. As the ranking function of System Z is admissible we look at $\omega_f$ being a minimal world of $R_Z$ with $\omega_f \models p \wedge \overline{c}$ and $\omega_v$ a minimal world with $\omega_v \models p \wedge c$. Because of the admissibility of the ranking function in System Z we have $\omega_v <_Z \omega_f$. As $R_\oplus$ is a refinement of $R_Z$ we know that $\omega_v <_Z \omega_f \Rightarrow \omega_v <_\oplus \omega_f$. □

## Irrelevance

In the following we analyze the introductory example of irrelevance in more detail. The example demonstrates that new independent defaults possibly do not have any effect on the conclusions, although they should have. Afterwards we present a general property of irrelevance and show that our system satisfies this property.

**Example [Irrelevance]** Consider the set of defaults $\Delta_{Tweety}$ expanded by any default $\delta_4 = (y|x)$ with two new atoms $x, y \notin \{p, f, b\}$. The default $\delta_4$ is independent of the other defaults and is therefore contained in $\Delta_0$. System Z gives equal plausibility to every world which falsifies $\Delta_1$, because $\delta_4$ in the lower layer does not have an influence. So given a flying penguin-bird and $x$, $y$ is not inferred. Figure 6 shows the falsification behaviour of the two crucial worlds $\omega_2 = pbfx\overline{y}$ and $\omega_1 = pbfxy$. In System ARS we get $\omega_1 < \omega_2$, so $pbfx \mathrel{|\!\sim}_\oplus y$. This conclusion is reasonable: the premise $x$ is given and $\Delta$ does not contain any other default using $x$ or $y$, so $y$ should be inferred.

Our notion of irrelevance considers the conclusions which should be drawn with a new default $(y|x)$ when it is added to a knowledge base $\Delta$: $\Delta' = \Delta \cup \{(y|x)\}$. Given any premise $\psi$, the modified inference relation $\mathrel{|\!\sim}_{\Delta'}$ should conclude $\psi \wedge x \mathrel{|\!\sim}_{\Delta'} y$ if neither $\psi$ nor any of the defaults in $\Delta$ mention $x, y$. We show that under the constraint that $\psi$ is compatible with all most normal rules, our system satisfies the following property.

**(Irr)** Let $x, y$ be atoms that are not mentioned in the defaults in $\Delta$, and let $\Delta' = \Delta \cup \{(y|x)\}$.
If $\psi$ is a formula in which $x, y$ do not occur, then

$$\psi \wedge x \mathrel{|\!\sim}_{\Delta'} y.$$

At first we analyze the effect of the new default on the minimal worlds satisfying $\psi \wedge x$. Let $R_\oplus$ resp. $\widehat{R}_\oplus$ denote the world ranking of our system for $\Delta$ resp. $\Delta'$. Due to the syntactical independence of $(y|x)$ from the other defaults we get $(y|x) \in \Delta'_0$. Let $\min_R\{\Phi\}$ be the set of minimal worlds satisfying a formula $\Phi$ according to a world ranking $R$.

**Lemma 3** *Suppose that* $\omega \in \min_{R_\oplus}\{\psi\}$ *do not falsify any* $\delta \in \Delta_0$. *Then* $\min_{\widehat{R}_\oplus}\{\psi \wedge x\} \subseteq \min_{R_\oplus}\{\psi\}$ *and* $\omega \models xy$ *for all* $\omega \in \min_{\widehat{R}_\oplus}\{\psi \wedge x\}$.

*Proof* . Firstly, there exists at least one world $\omega \in \min_{R_\oplus}\{\psi\}$ with $\omega \models \psi xy$, as $x$ and $y$ do not occur in $\Delta$. In particular $\omega \models \psi x$. This world $\omega$ does not falsify any $\delta \in \Delta_0$. Assume that there exists a world $\omega_{pot} \models \psi$ with $\omega_{pot} \in \min_{\widehat{R}_\oplus}\{\psi \wedge x\}$ and $\omega_{pot} \notin \min_{R_\oplus}\{\psi\}$. Due to $\omega <_{R_\oplus} \omega_{pot}$ there has to exist a maximal index $i$ with $\omega \models \Delta_i$ and $\omega_{pot} \not\models \Delta_i$. As $\omega$ does not falsify $(y|x)$, adding the default does not have an influence on the falsification behaviour of $\omega$. Therefore $\omega <_{\widehat{R}_\oplus} \omega_{pot}$ holds regardless of the index $i$. This contradicts to $\omega_{pot} \in \min_{\widehat{R}_\oplus}\{\psi \wedge x\}$.

Secondly, due to Lemma 2 we know that we have equivalence in all input relations for any two worlds $\omega_1, \omega_2 \in \min_{\widehat{R}_\oplus}\{\psi \wedge x\}$, which means that these worlds have equal falsification behaviour. Additionally, due to $\min_{\widehat{R}_\oplus}\{\psi \wedge x\} \subseteq \min_{R_\oplus}\{\psi\}$ these worlds do not falsify any $\delta \in \Delta_0$. Therefore the default $(y|x)$ is decisive, which means that given $\omega_1 \models xy \wedge \omega_2 \models x\overline{y}$, this would contradict to $\omega_2 \in \min_{\widehat{R}_\oplus}\{\psi \wedge x\}$, as $\omega_1 <_{\widehat{R}_\oplus} \omega_2$. □

We are now in the position to show that the inference relation $\mathrel{|\!\sim}^{\Delta}_{\oplus}$ satisfies the irrelevance property in important cases.

**Theorem 5** *Let the worlds* $\omega \in \min_{R_\oplus}\{\psi\}$ *do not falsify any* $\delta \in \Delta_0$. *The following holds:*

$$\psi \wedge x \mathrel{|\!\sim}^{\Delta'}_{\oplus} y.$$

*Proof* . Due to Lemma 3 we know that $\omega_1 < \omega_2$ for all $\omega_1 \in \min_{\widehat{R}_\oplus}\{\psi \wedge xy\}$, $\omega_2 \in \min_{\widehat{R}_\oplus}\{\psi \wedge x\overline{y}\}$. □

## Comparison with Similar Approaches

In this section, we compare our method to similar approaches. For another approach to overcome System Z's problems see e.g. (Delgrande & Schaub 1994).

### Lexicographic Entailment

The lexicographic entailment goes back to (Benferhat *et al.* 1993) and (Lehmann 1995). Let $\omega$ and $\omega'$ be two worlds and let $|\Delta_j^{sat}(\omega)|$ be the number of defaults in $\Delta_j$ which are satisfied (not falsified) by $\omega$.

$$\omega <_{lex} \omega' \text{ iff}$$
$$1. \exists 1 \le i \le k : |\Delta_i^{sat}(\omega)| > |\Delta_i^{sat}(\omega')|, \text{ and}$$
$$2. \forall j > i, j \le k : \; |\Delta_j^{sat}(\omega)| = |\Delta_j^{sat}(\omega')|$$

Although the lexicographic system looks similar to the System ARS, the rankings of the worlds $R_{lex}$ and $R_\oplus$ differ to a great extent. $R_{lex}$ takes into account the number of defaults which are not falsified, which means that worlds with the same falsification behaviour (concerning System ARS) may get different ranking values. At first glance it looks like the number of not falsified defaults is an additional discrimination criteria which results in $R_{lex}$ being a refinement of $R_\oplus$. The following example shows that this is not the case.

The set of defaults is $\Delta_{\text{diff}} = \Delta_{Tweety} \cup \{\delta_4 = (l|b)\}$, which is the normal Tweety example with the additional default that birds normally have legs. If we compare the two worlds $\omega_1 = pbf\overline{l}$ and $\omega_2 = p\overline{b}f\overline{l}$ we get $\omega_1 <_{lex} \omega_2$ but $\omega_2 <_\oplus \omega_1$, see figure 7 for the falsification behaviour of $\omega_1$ and $\omega_2$. This is a proof that there can be no refinement relationship.

| | $\Delta_0$ | | $\Delta_1$ | |
|---|---|---|---|---|
| | $(f\|b)$ | $(l\|b)$ | $(b\|p)$ | $(\overline{f}\|p)$ |
| $\omega_1 = p\,b\,f\,\bar{l}$ | | false | | false |
| $\omega_2 = p\,\bar{b}\,f\,\bar{l}$ | | | false | false |

Figure 7: Example for the comparison with $R_{lex}$

## Basic Preference Descriptions

Now we compare the ranking of the worlds in our approach with the preference relations, which are defined by Brewka's basic preference descriptions (Brewka 2004). A basic preference description consists of a ranked knowledge base (RKB) $K = \{(f_i, v_i)\}$, assigning priorities $v_i \geq 1$ to goals which are represented by formulas $f_i$, and one out of four basic preference strategies. Given a RKB the four strategies imply orderings on the set of possible worlds. Let $K^n(m)$ denote the set of the satisfied goals with rank $n$ for a world $m$. The subset-strategy is defined as follows:

**Definition 4** *Given two possible worlds $m_1$ and $m_2$ and a RKB $K$, then $m_1 R_{\subseteq} m_2$ iff $K^n(m_1) = K^n(m_2)$ for all $n$ or there is an $n$ such that $K^n(m_1) \supset K^n(m_2)$, and for all $j > n:\ K^j(m_1) = K^j(m_2)$.*

The relation $R_{\subseteq}$ describes the ordering used in Brewka's preferred subtheories approach (Brewka 1989).

The input relations in our approach take into account the layers $\Delta_i = \{(c_0|p_0), \ldots, (c_{k_i}|p_{k_i})\}$ without considering each single default separately: we use ranking functions $\kappa_i : \Omega \to \{0, 1\}$ to describe the input relations, $\kappa_i(\omega) = 0$ iff $\omega \models \Delta_i$. This is why it is sufficient to specify a single goal for each layer $\Delta_n$ given by $f^n = \bigwedge_{j=1..k_n}(p_j \Rightarrow c_j)$. This formula is satisfied, if and only if none of the defaults in the layer is falsified. We can replicate the prioritization of higher layers by assigning each layer $\Delta_j$ the priority $j+1$. The basic preference description $K^{\subseteq}$ constituted by the resulting RKB $K = \{(f^n, n+1)\}$ and the subset-strategy $\subseteq$ leads to same ordering of models as in our system.

**Theorem 6** *Let $\Delta = \{\Delta_1, \ldots, \Delta_k\}$ with $\Delta_i = \{(c_0|p_0), \ldots, (c_{k_i}|p_{k_i})\}$ be a ranking of defaults. The ranking of worlds $R_{\oplus}$ is equivalent to the ranking induced by the basic preference relation $K^{\subseteq}$ with $K = \{(f^n, n+1)\}$, $0 \leq n \leq k$, where $f^n = \bigwedge_{j=1..k_n}(p_j \Rightarrow c_j)$.*

*Proof .* The structure of the defined RKB is special, as there is only one goal $f^n$ for each rank $n$. Therefore for all $n$ we have $|K^n(m)| \in \{0, 1\}$ and $K^n(m_1) \supset K^n(m_2)$ if and only if $m_1 \models f^n \wedge m_2 \not\models f^n$. If both worlds falsify or do not falsify $f^n$, we get $K^n(m_1) = K^n(m_2)$. Using these connections in the definition of the subset-strategy leads to our ranking of the worlds $R^{\oplus}$. □

## Summary and Outlook

In this paper, we presented an approach to default reasoning based on techniques from preference fusion. Given a knowledge base of default rules, the basic idea is to consider the plausibility relation induced by each rule on the set of possible worlds, preferring the worlds that satisfy the rule to those that falsify it. Fusing these simple preference relations gives rise to a complex preference structure on possible worlds that can be used for further conditional inferences. We proved that the resulting inference operation refines the well-known System Z, but shows improved inference properties with respect to the addition of irrelevant information. Moreover, we compare our System ARS to related approaches.

As part of ongoing work, we will elaborate on this connection between preference fusion and default reasoning in more detail. In particular, we will study relations between commonly accepted postulates from nonmonotonic reasoning, on one hand side, and preference fusion, on the other.

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