# Qualitative decision making with bipolar information

**Didier Dubois, Hélène Fargier**
IRIT-CNRS
118 route de Narbonne
31062 Toulouse Cedex 9, France
{dubois,fargier}@irit.fr

### Abstract

Decisions can be evaluated by sets of positive and negative arguments — the problem is then to compare these sets. Studies in psychology have shown that in this case the scale of evaluation of decisions is generally bipolar. Moreover decisions are often made on the basis of an ordinal ranking of the arguments rather than on a genuine numerical evaluation of their degrees of attractiveness or rejection, hence the qualitative nature of the decision process in practice. In this paper, assuming bipolarity of evaluations and qualitative ratings, we present and axiomatically characterise two decision rules based on possibilistic order of magnitude reasoning that are capable of handling positive and negative affects. They are extensions of the maximin and maximax criteria to the bipolar case. A bipolar extension of possibility theory is thus obtained. In order to overcome the lack of discrimination power of the decision rules, refinements are also proposed, capturing both the efficiency principle and the idea of order of magnitude reasoning.

## Introduction

It is known from many experiments in cognitive psychology that humans evaluate alternatives or objects for the purpose of decision-making by considering positive and negative aspects separately (Osgood, Suci, & Tannenbaum 1957; Cacioppo & Berntson 1994; Slovic *et al.* 2002). In other words, they implicitly use bipolar evaluation scales. Decisions are moreover often made on the basis of an ordinal ranking of the strength of the arguments rather than on a numerical evaluation, hence the qualitative nature of the decision process (Gigerenzer, Todd, & the ABC group 1999).

The present work is part of a first attempt toward formalizing and axiomatically characterizing bipolar qualitative decision rules. We consider the simple situation where each possible decision $d$ is assessed by a finite subset of arguments $\mathcal{C}(d) \subseteq X$. $X$ is the set of all possible arguments: an argument is typically a criterion satisfied by $d$, a risk run by choosing $d$, a good, or a bad, consequence of $d$. Under this view, comparing decisions amounts to comparing sets of arguments, i.e. subsets $A$, $B$ of $2^X$. The point is that some arguments are positive, and thus attractive for the decision maker, while others are negative and should be avoided. For instance, when choosing a house, having a garden is a positive argument $g^+$, missing a garden is a negative argument $g^-$. Being close to an airport is a negative argument $a^-$– but being far is not necessarily positive, and being too far is definitively negative – hence a second argument $a'^-$. So, comparing a house with a garden but close to the airport with one without garden and very far from the airport amounts to comparing $\{g^+, a^-\}$ and $\{g^-, a'^-\}$. The final decision obviously depend on the strength of the different elements.

As said previously we focus in this paper on *qualitative* bipolar decision making, i.e. on models that rank decisions on the basis of ordinal rather than on numerical evaluations. Among other motivations is the fact that the elicitation of the information required by a quantitative decision model is often not an easy task in practice. Another motivation is the genuine quantitativeness of human reasoning.

The handling of qualitative information in decision making is not a new question. The most famous decision rule of this kind is the maximin rule of Wald (1950). It only presupposes that the arguments in $X$ can be ranked in terms of merits by means of some utility function $u$ mapping on any ordinal scale. Decisions are then ranked according to the merit of their worst arguments, following a pessimistic attitude. This approach captures the handling of negative affects. Purely positive decisions are sometimes separately handled in a symmetric way, namely on the basis of their best arguments. The case of ordinal ranking procedures using bipolar (both positive and negative) information has retained less attention. It is worth mentioning the works of Benferhat *et al.* (2006) who separately merge all positive affects into a degree of satisfaction (using the max rule) and all the negative affects into a degree of tolerance (using the min rule). Decisions are first selected according to tolerance and then the most satisfactory solution is obtained.

In the present paper, we follow a more systematic direction of research, characterising a set of procedures that are simultaneously ordinal and bipolar. Unsurprisingly, the reader will see that the corresponding decision rules are strongly related to possibility theory. The paper is structured as follows. The next section is devoted to the background on monotonic set comparison. The third section presents two basic qualitative and bipolar rules. We then show how the basic properties of bipolar reasoning can be expressed axiomatically and going further, which axioms can capture the principles of *qualitative* bipolar decision-making. Efficient criteria are then studied that are more decisive than the basic ones without giving up their qualitative nature.

## Background

Comparing sets of more or less important elements is an old issue in uncertain reasoning, logics, measurement theory, etc. Let us first recall that, for any relation $\succeq$ on a power set $2^X$, one can define its symmetric part ($A \sim B \iff A \succeq B$ and $B \succeq A$), its asymmetric part ($A \succ B \iff A \succeq B$ and $not(B \succeq A)$) and an incomparability relation: $A \asymp B \iff not(A \succeq B)$ and $not(B \succeq A)$. $\succeq$ is said to be *quasi-transitive* iff $\succ$ is transitive. $\succeq$ is a weak order iff it is complete and transitive. In the latter case, the comparison can be captured by a monotonic set function or "capacity":

**Definition 1** *A capacity on $X$ is a mapping $\sigma$ defined from $2^S$ to $[0,1]$ that is consistent ($\sigma(\emptyset) = 0$), non trivial ($\sigma(S) = 1$), and monotonic, i.e. such that:*
$\forall A, B \subseteq X, A \subseteq B \Rightarrow \sigma(A) \leq \sigma(B)$

Capacities are meaningful in argument-based decision: if $d$ is supported by a set of positive (resp. negative) arguments $A$ ($\mathcal{C}(d) = A$), then this decision can be evaluated by means of $\sigma(A)$ — i.e. capacities suit the situations where all the elements of $X$ are positive (resp. negative). The essence of capacities, i.e. the monotony principle saying that the larger the set, the higher its importance, is not tied to the use of a single numerical measure. Halpern (1997) (see also (Dubois & Fargier 2004)) has extended the monotonicity principle to the relational framework, thus defining a more general concept of *comparative capacity*:

**Definition 2** *A relation $\succeq$ on a power set $2^X$ is said to be positively monotonic (or "orderly") i.e. satisfies: $A \subseteq C, D \subseteq B$, $A \succeq B \Rightarrow C \succeq D$* [1].

*A relation $\succeq$ on a power set $2^X$ is a comparative capacity iff it is reflexive, quasi-transitive, non-trivial ($X \succ \emptyset$) and positively monotonic*

Contrary to numerical capacities, comparative capacities are not necessarily complete and transitive relations. For instance, the *discrimax order* ( (Behringer 1977); see also (Dubois, Fargier, & Prade 1996)) relies on a rank function (possibility distribution) $\pi : X \mapsto [0,1]$. It is defined by:

$$A \succeq_{Discrimax} B \text{ iff } \Pi(A \setminus B) \geq \Pi(B \setminus A),$$

where $\Pi(V) = \max_{x \in V} \pi(x)$. This definition of $\succeq_{Discrimax}$ yields a complete but not fully transitive comparative capacity (indifference is not transitive).

Another example is given by a family of probability distributions, say $\mathcal{F}$. The following relation yields a transitive but incomplete comparative capacity :

$$A \succeq_{\mathcal{F}} B \iff \forall P \in \mathcal{F}, P(A) \geq P(B)$$

In the present paper, we aim at focusing on and at characterising the *bipolar* decision making situations that are qualitative, rather than quantitative. The following section presents two basic rules along this line.

[1]Remark that the monotony of $\succeq$ implies the monotony of $\succ$, i.e. $A \subseteq C, D \subseteq B, A \succ B \Rightarrow C \succ D$

## Two basic ordinal rules for comparing bipolar sets

For each decision $d$, let $\mathcal{C}(d)$ be the set of arguments relevant for $d$, including positive and negative ones. For the sake of simplicity, we assume the positiveness (respectively the negativeness) of an argument is not a matter of degree. Hence we can suppose that $X$ is divided into two subsets: $X^+$ is the set of positive arguments, $X^-$ is the set of negative arguments. Considering a house, having a garden is a positive argument $g^+ \in X^+$ and missing a garden is a negative argument $g^- \in X^-$. Obviously, only one of the two arguments belongs to $\mathcal{C}(d)$. Having a garden shelter (argument $s^+$) is also positive, but not really an important one, so that its absence is not a negative argument (we simply don't care). Being in the historical center of the city is also a positive argument $h^+ \in X^+$. Some arguments can present both a positive and a negative aspect: for instance, being far from the main roads is a guarantee of quietness but induces longer transportation times. This kind of argument can be encompassed through a duplication process: we will have two sub-arguments $r^+$ and $r^-$, the former one in $X^+$ and the latter one in $X^-$. They are either both present in $\mathcal{C}(d)$, or both absent of it. Hence comparing a house $d$ far from the main routes, with a large garden and a shelter with a house $d'$ missing a garden but ideally located in the city amounts to comparing two sets of arguments, $\mathcal{C}(d) = \{r^+, r^-, s^+, g^+\}$ and $\mathcal{C}(d) = \{g^-, h^+\}$.

The final decision obviously depends on the strength of the different arguments. In a purely qualitative, ordinal approach to decision-making, let us suppose that the importance of arguments is a transitive and complete notion, i.e. that it can be described on a totally ordered scale $L = [0_L, 1_L]$, e.g. by a function $\pi : X \mapsto L = [0_L, 1_L]$.

$\pi(x) = 0_L$ means that the decision maker is indifferent to argument $x$ ; $1_L$ is the highest level of attraction or repulsion (according to whether it applies to a positive or negative argument). $\pi$ is supposed to be non trivial, i.e. at least one $x$ receives a positive rating. Whenever $\pi(x) > \pi(y)$, the strength of $x$ is considered at least one order of magnitude higher than the one of $y$, so that $y$ is negligible in front of $x$. So the strength of $A$ shall be measured by the following qualitative possibility measure that reflects the order of magnitude of the elements in $A$ (irrespective of their signs) - hence the notation $OM$:

$$\forall A \subseteq X, OM(A) = \max_{x \in A} \pi(x)$$

### The Bipolar Qualitative Pareto Dominance Rule

Any $A \subseteq X$ can also be partitioned in two subsets: $A^+ = A \cap X^+$ and $A^- = A \cap X^-$ are respectively the positive and negative arguments in $A$. As said in the introduction, if all the arguments were negative, we could apply Wald's cautious principle, i.e. decide that (the object evaluated by) $A$ is preferable to (the object evaluated by) $B$ iff $OM(A) = OM(A^-) \leq OM(B) = OM(B^-)$. On the contrary, if all the arguments were positive, we could apply the optimistic rule, i.e. decide that $A$ is preferable to $B$ iff $OM(A) = OM(A^+) \geq OM(B) = OM(B^+)$.

In the bipolar case, one could consider that each of the two scales defines a criterion, i.e. that the ranking of decisions relies on two ordinal evaluations, namely on the pair $(OM(A^+), OM(A^-))$. This yields the following Pareto-like rule (Dubois & Fargier 2005), which does not assume commensurateness between the evaluation of positive and negative arguments:

$$A \succeq^{Pareto} B \iff \begin{array}{ll} & OM(A^+) \geq OM(B^+) \\ \text{and} & OM(A^-) \leq OM(B^-) \\ \text{where} & OM(V) = \max_{x\in v} \pi(x) \end{array}$$

This is decision on the basis of unanimous agreement between positive and negative arguments. In other terms:

- $A$ is strictly preferred to $B$ ($A \succ^{Pareto} B$) in two cases: either $OM(A^+) \geq OM(B^+)$ and $OM(A^-) < OM(B^-)$, or $OM(A^+) > OM(B^+)$ and $OM(A^-) \leq OM(B^-)$.
- $A$ and $B$ are indifferent when $OM(A^+) = OM(B^+)$ and $OM(A^-) = OM(B^-)$.
- There is a conflict when $OM(A^+) > OM(B^+)$ and $OM(A^-) > OM(B^-)$. Then $A$ is not comparable with $B$: $\succeq^{Pareto}$ is a partial relation.

**Proposition 1** *$\succeq^{Pareto}$ is reflexive, transitive but not necessarily complete.*

$\succeq^{Pareto}$ is perhaps too partial: for instance, when $OM(A^-) > OM(A^+)$, $\succeq^{Pareto}$ concludes that $A$ is incomparable with $B = \emptyset$ and this even if the positiveness of $A$ is negligible w.r.t its negativeness. In this case, one would rather say that getting $A$ is bad and that getting nothing is preferable. Another drawback is observed when $OM(A^+) > OM(B^+)$ and $OM(A^-) = OM(B^-)$: this enforces $A \succ^{Pareto} B$, and this even if $OM(A^+)$ is very weak w.r.t the order of magnitude of the negative arguments — in the latter case, a rational decider would examine the negative arguments in details before concluding.

### The Bipolar Possibility Relation

The Pareto decision rule does not account for the fact that the two evaluations that are used can *share a common scale*. The following new decision rule takes this commensurability into account. The principle at work here is simple: any argument against $A$ (resp. against $B$) is an argument pro $B$ (resp. pro $A$) and conversely, and the most supported decision is preferred:

$$A \succeq^{Biposs} B \Leftrightarrow \begin{array}{l} \max(OM(A^+), OM(B^-)) \\ \geq \max(OM(B^+), OM(A^-)) \end{array}$$

This rule focuses on the arguments of highest importance, deciding that $A$ is at least as good as $B$ iff, at that decision level, i.e. at level $OM(A \cup B)$, there are arguments in favor of $A$ or arguments attacking $B$. Thus $A \succ^{Biposs} B$ iff, at the highest level, there is at least a positive argument for $A$ or an argument against $B$, but no negative argument against $A$ and no positive argument pro $B$.

Importantly, like $\succeq^{Pareto}$, $\succeq^{Biposs}$ collapses to the $\max$ rule if $X = X^+$ and to Wald's pessimistic rule if $X = X^-$. Like $\succeq^{Pareto}$ also, $\succeq^{Biposs}$ satisfies the weak unanimity principle:

$$\begin{array}{ll} & OM(A^+) \geq OM(B^+) \\ \text{and} & OM(B^-) \geq OM(A^-) \end{array} \Rightarrow A \succeq^{Biposs} B$$

But the converse implication is not valid. It may happen that $A \sim^{Biposs} B$ while $OM(A^+) > OM(B^+)$ and $OM(B^-) = OM(A^-)$.

Finally, $\succeq^{Biposs}$ is reflexive and *only quasi-transitive*. But is it is complete, contrary to $\succeq^{Pareto}$.

**Proposition 2** *$\succeq^{Biposs}$ is complete and quasi transitive, but $\sim^{Biposs}$ is not necessarily transitive.*

However, $\succeq^{Biposs}$ and $\succeq^{Pareto}$ are very rough rules that may be not decisive enough. In particular, if may happen that $A \subsetneq B$ while none of the rules is making a strict preference between $A$ and $B$ — the usual drowning effect of possibility theory reappears here. Variants of the bipolar possibility relation will be presented in the last section of the paper that overcome this difficulty. But the two rules have the advantage to capture the essence of ordinal decision making, as shown by the axiomatic study of the next section.

## Axioms for ordinal comparison on a bipolar scale

As usual in axiomatic characterisations, an abstract relation $\succeq$ on $2^X$ is considered and the natural properties that it should obey are formalised — here, the property that should be required for qualitative bipolar comparison relations. Let us begin by the general properties any bipolar procedure should sensibly obey.

### Axioms for monotonic bipolar set relations

The basic notion of bipolar reasoning over sets of arguments is the separation of $X$ in good and bad arguments. The first axiom thus states that any argument is either positive or negative, i.e. better than nothing or worse than nothing:

**Clarity of Arguments (CA)** $\forall x \in X, \{x\} \succeq \emptyset$ or $\emptyset \succeq \{x\}$

One can then partially order the elements of $\mathbb{X} = X \cup \{\emptyset\}$ and build a partition of it. Denoting $\succeq_{\mathbb{X}}$ the restriction of $\succeq$ to $\mathbb{X}$, we have:

$$\begin{array}{ll} x \succeq_{\mathbb{X}} y \iff \{x\} \succeq \{y\} & X^+ = \{x, \{x\} \succ \emptyset\} \\ x \succeq_{\mathbb{X}} \emptyset \iff \{x\} \succeq \emptyset & X^- = \{x, \emptyset \succ \{x\}\} \\ \emptyset \succeq_{\mathbb{X}} x \iff \emptyset \succeq \{x\} & X^0 = \{x, \emptyset \sim \{x\}\} \end{array}$$

In the sequel, $\succeq_{\mathbb{X}}$ is called the *ground relation* of $\succeq$. Now, arguments that are indifferent to the decision maker should obviously not affect the preference. This is the meaning of the next axiom:

**Status Quo Consistency (SQC)**
If $\{x\} \sim \emptyset$ then $\forall A, B : A \succeq B \;\Leftrightarrow\; A \cup \{x\} \succeq B$
$\Leftrightarrow\; A \succeq B \cup \{x\}$.

The Status quo consistency axiom allows to forget about $X_0$.

Let us now discuss the property of monotonicity. Monotonicity in the sense of Definition 2 can obviously not be obeyed as such in a bipolar scaling. Indeed, if $B$ is a set of negative arguments, it generally holds that $A \succ A \cup B$. We rather need axioms of monotonicity *specific to* positive and negative arguments – basically, the one of bipolar capacities (Greco, Matarazzo, & Slowinski 2002), expressed in a comparative way.

**Positive monotonicity**
$\forall C, C' \subseteq X^+, \forall A, B : A \succeq B \Rightarrow C \cup A \succeq B \setminus C'$

**Negative monotonicity**
$\forall C, C' \subseteq X^-, \forall A, B : A \succeq B \Rightarrow C \setminus A \succeq B \cup C'$

Now, the bipolar scale encodes all the relevant information, saying that only the positiveness and the negativeness of $A$ and $B$ are to be taken into account: if $A$ is at least as good as $B$ on both the positive and the negative sides, then $A$ is at least as good as $B$. This is expressed by the already encountered axiom of weak unanimity.

**Weak unanimity**
$\forall A, B, A^+ \succeq B^+$ and $A^- \succeq B^- \Rightarrow A \succeq B$

Notice that the axiom of weak unanimity can in some cases be reinforced by a second and more restrictive axiom, strong unanimity. It claims that only indifference on both sides results in indifference. We will see in next section that it is the essence of the Bipolar Qualitative Pareto rule.

**Strong unanimity** $\forall A, B \neq \emptyset$:

$$\begin{array}{lcccl} A^+ \succeq B^+ & \text{and} & A^- \succ B^- & \Rightarrow & A \succ B \\ A^+ \succ B^+ & \text{and} & A^- \succeq B^- & \Rightarrow & A \succ B \end{array}$$

Finally, adding an axiom of non triviality,

**Non-Triviality**: $X^+ \succ X^-$

we get a generalisation of comparative capacities:

**Definition 3** *A relation on a power set* $2^X$ *is a* monotonic bipolar set relation *iff it is reflexive, quasi-transitive and satisfies the properties of Clarity of Arguments, Status Quo Consistency, Non-Triviality, Weak unanimity, Positive and Negative Monotonicity*

**Proposition 3** $\succeq^{Biposs}$*is a monotonic bipolar set relation*

**Proposition 4** $\succeq^{Pareto}$ *is a monotonic bipolar set relation*

In the present work, we are interested in relations that are entirely determined by the strength of the individual arguments $X$. In agreement with the existence of a totally ordered scale for weighting arguments, the ground relation $\succeq_{\mathbb{X}}$ is supposed to be weak order. Then a minimal condition of coherence with $\succeq_{\mathbb{X}}$ is that if an argument is replaced by a better one (resp. a worse one), the preference cannot be reversed. This can be viewed as a condition of monotonicity with respect to $\succeq_{\mathbb{X}}$:

**Monotonicity w.r.t. $\succeq_{\mathbb{X}}$ or "$\mathbb{X}$-monotonicity"**
$\forall A, B, x, x'$ such that $A \cap \{x, x'\} = \emptyset$ and $x' \succeq_{\mathbb{X}} x$

$$\begin{array}{lcl} A \cup \{x\} \succ B & \Rightarrow & A \cup \{x'\} \succ B \\ A \cup \{x\} \sim B & \Rightarrow & A \cup \{x'\} \succeq B \\ B \succ A \cup \{x'\} & \Rightarrow & B \succ A \cup \{x\} \\ B \sim A \cup \{x'\} & \Rightarrow & B \succeq A \cup \{x\} \end{array}$$

This very natural axiom is richer that it seems. For instance, it implies a property of exchangeability of equally strong arguments. This kind of property is often called "anonymity" in social choice and decision theory. A property close to anonymity should also be required when a positive argument blocks a negative argument of the same strength: this blocking effect should not depend on the arguments themselves, but on their position in the scale only. Hence the axioms of positive and negative cancellation:

**Positive Cancellation (POSC)** $\forall x, z \in X^+, y \in X^-$:
$\{x, y\} \sim \emptyset$ and $\{z, y\} \sim \emptyset \Rightarrow x \sim_{\mathbb{X}} z$.

**Negative Cancellation (NEGC)** $\forall x, z \in X^-, y \in X^+$:
$\{x, y\} \sim \emptyset$ and $\{z, y\} \sim \emptyset \Rightarrow x \sim_{\mathbb{X}} z$.

It makes sense to summarize the above requirement into a single axiom we call *Simple grounding* as follows:

**Simple grounding**: $\succeq$ is said to be simply grounded if and only if $\succeq_{\mathbb{X}}$ is a weak order, $\succeq$ is monotonic with respect to $\succeq_{\mathbb{X}}$ and satisfies positive and negative Cancellation.

**Proposition 5** $\succeq^{Biposs}$ *is simply grounded.*

**Proposition 6** $\succeq^{Pareto}$ *is simply grounded.*

## Axiomatizing qualitative bipolar relations

Definition 3 is actually very general and encompasses numerous models, be they qualitative (e.g. the two rules in the previous section) or not (e.g. cumulative prospect theory in its full generality). As we are interested in preference rules that derive from the principles of ordinal reasoning only, we now focus on axioms that account for ordinality.

The ordinal comparison of sets was extensively used, especially in Artificial Intelligence (see for instance (Dubois 1986; Lehmann 1996; Halpern 1997; Dubois & Fargier 2004)). The basic concept of ordinal reasoning is Negligibility that presupposes that each level of importance should be interpreted as an order of magnitude, much higher than the next lower level.

**NEG** $\forall A, B, C \subseteq X^+ : A \succ B$ and $A \succ C \Rightarrow A \succ B \cup C$

Axiom NEG has been around in AI, directly under this form or through more demanding versions – the "union property" of non monotonic reasoning or Halpern's (1997) "Qualitativeness" axioms (see (Dubois & Fargier 2004) for a discussion). Ordinal reasoning generally comes along with a notion of closeness preservation, that does away with the idea of counting:

**CLO** $\forall A, B, C \subseteq X^+$ $A \sim B$ and $A \sim C \Rightarrow A \sim B \cup C$
$\forall B, C \subseteq X^+ : B \succeq C \Rightarrow B \sim B \cup C$

These axioms were proposed and justified in ordinal reasoning, when only one scale (the positive one) is to be considered. But they are not sufficient when a negative scale has also to be taken into account. We need for instance to express that if there is a very bad consequence $B$, so bad that $A \succ B$ and $C \succ B$, then whatever the negative arguments in $A$ and $C$, $B$ is still worse than $A \cup C$:

$$\forall A, B, C : A \succ B \text{ and } C \succ B \Rightarrow A \cup C \succ B$$

Other cases should be encompassed, that compare sets with both negative and positive elements. For instance, if $A$ is so good that it can cope with globally negative $B$ and also win the comparison with $C$, then $A \cup B$ is still better than $C$:

$$\forall A, B, C : A \cup B \succ \emptyset \text{ and } A \succ C \Rightarrow A \cup B \succ C$$

And similarly, if a globally negative $A$ ($A \prec \emptyset$) is so bad that it is outperformed by $C$ ($C \succ A$) and cannot be enhanced by $B$ ($\emptyset \succ A \cup B$), then $C \succ A \cup B$, i.e.:

$$\forall A, B, C : \emptyset \succ A \cup B \text{ and } C \succ A \Rightarrow C \succ A \cup B.$$

All these properties can be expressed by the following axiom of global negligibility:

**GNEG** $\forall A, B, C, D : A \succ B$ and $C \succ D \Rightarrow A \cup C \succ B \cup D$

This property is classical in a purely positive scale – in this case, it is a consequence of NEG and positive monotony. That is why it is usually not explicitly required in positive frameworks. But when a framework with two scales is to be taken into account, the NEG condition is no longer sufficient for getting GNEG. So, in order to keep the property that it is as the foundation of a pure order of magnitude reasoning, bipolar qualitative frameworks have to explicitly require GNEG.

A similar reasoning applies to axiom CLO. A more general property than the one usual for unipolar qualitative scales must be explicitly required for bipolar scales:

**GCLO** $\forall A, B, C, D\ A \succeq B$ and $C \succeq D \Rightarrow A \cup C \succeq B \cup D$

**Proposition 7** *$\succeq^{Biposs}$ satisfies GNEG and GCLO*

**Proposition 8** *$\succeq^{Pareto}$ satisfies GNEG and GCLO*

Conversely, consider a weak order $\succeq_{\mathbb{X}}$ on $\mathbb{X}$, encoding the signed (positive or negative) order of magnitude of the different arguments. Applying the principles of qualitative bipolar reasoning described by the previous axioms can lead to several different rules. But $\succeq^{Pareto}$ is the unique decision induced by $\succeq_{\mathbb{X}}$, understood as a bipolar ordinal scale, by applying only the principle of (i) bipolar qualitative decision making and (ii) weak and strong unanimity. Indeed, any other preference relation in the family refines it, i.e. obeys the preference $A \succ^{Pareto} B$ when the Bipolar Qualitative Pareto Dominance rule allows to conclude. This straightforward result can be formalized as follows:

**Definition 4** *$\succeq'$ refines $\succeq$ iff $\forall A, B : A \succ B \Rightarrow A \succ' B$*

**Theorem 1** *For any weak order $\succeq_{\mathbb{X}}$ on $\mathbb{X}$, let $\mathcal{F}(\succeq_{\mathbb{X}}) = \{\trianglerighteq, \trianglerighteq_{\mathbb{X}} \equiv \succeq_{\mathbb{X}}\}$ be the set of monotonic bipolar rules with, relation $\geq_{\mathbb{X}}$.*
*$\succeq^{Pareto}$ is the least refined* transitive *monotonic bipolar set relation in $\mathcal{F}(\succeq_{\mathbb{X}}^{Pareto})$ that satisfies axioms GNEG, GCLO and obeys the principles of weak and strong unanimity.*

Theorem 1 shows that the Bipolar Qualitative Pareto Dominance rule is the least committed qualitative monotonic bipolar set relation that follows from strong unanimity. However, we pointed out that this decision rule is too demanding, as it can induce a counterintuitive strict preference. So, let us relax the unanimity postulate, requiring *weak unanimity* only and assume completeness. The following characterisation of the bipolar possibility relation is obtained.

**Theorem 2** *The following propositions are equivalent:*
*- $\succeq$ is a simply grounded* complete *monotonic bipolar set relation on $2^X$, that satisfies GNEG and GCLO.*
*- there exists a mapping $\pi : X \mapsto [0_L, 1_L]$ such that $\succeq \equiv \succeq^{Biposs}$.*

In summary $\succeq^{Biposs}$ is a natural model of bipolar order of magnitude. In particular, any rule in accordance with GNEG has to follow the strict preference prescribed by $\succ^{Biposs}$

But $\succeq^{Biposs}$ (and it is also the case of $\succeq^{Pareto}$) suffers from a drowning effect, as usual in standard possibility theory. For instance, when $B$ is included in $A$ and even if all its elements are positive, then $A$ is not necessarily strictly preferred to $B$. In other worlds, the closeness axioms concludes indifference even in some cases where we would like apply the principle of efficiency to make the decision.

## The efficiency principle

The proper extension of the principle of efficiency to the bipolar framework, that should be at work here, has one positive and one negative side:

**Positive efficiency** $B \subseteq A$ and $A \setminus B \succ \emptyset \Rightarrow A \succ B$
**Negative efficiency** $B \subseteq A$ and $A \setminus B \prec \emptyset \Rightarrow A \prec B$

$\succeq^{Biposs}$ and $\succeq^{Pareto}$ also fail the classical condition of preferential independence, also called the principle of preadditivity, that implies the above conditions of efficiency. This condition simply says that arguments present in both $A$ and $B$ should not influence the decision:

**Preferential Independence**:
$\forall A, B, C, (A \cup B) \cap C = \emptyset : A \succeq B \iff A \cup C \succeq B \cup C$

Except in very special cases where all the arguments are of different levels of importance ($\succeq_{\mathbb{X}}$ is a linear order), these axioms are incompatible with axioms of ordinality when completeness or transitivity is enforced. It is already true in the pure positive case, i.e. when $X^-$ is empty (Fargier & Sabbadin 2003). But this impossibility result is not damning: the solution is to build relations that are in agreement with $\succeq^{Biposs}$ (we shall give up $\succeq^{Pareto}$ because of its important drawbacks) and satisfy Preferential Independence. Such rules are presented in the next section.

## Refining the bipolar possibility relation

To overcome the drowning effect, we can indeed propose comparison principles that refine $\succeq^{Biposs}$, that is, relations $\succ$ compatible with $\succeq^{Biposs}$ but more decisive.

In a preliminary work (Dubois & Fargier 2005), we have investigated a kind of dominance rule that is close to the Bipolar Possibility Relation. Like $\succeq^{Biposs}$, it focuses on arguments of maximal strength, but applies a more restrictive dominance principle: $A$ is at least as good as $B$ iff, at level $OM(A \cup B)$ the existence of arguments in favor of $B$ is counterbalanced by the existence of arguments in favor of $A$ and the existence of arguments against $A$ is cancelled by the existence of arguments against $B$. Formally:

**Definition 5** $A \succeq^{DPoss}$ *iff:*

$$OM(B^+) = OM(A \cup B) \quad \Rightarrow \quad OM(A^+) = OM(B^+)$$
$$OM(A^-) = OM(A \cup B) \quad \Rightarrow \quad OM(B^-) = OM(A^-)$$

$\succeq^{DPoss}$ refines $\succeq^{Biposs}$: when $OM(A^+) = OM(B-) = OM(A^-) > OM(B^+)$, $\succeq^{Biposs}$ will conclude to indifference but $\succeq^{DPoss}$ will strictly prefer $A$, because at the highest level there are pros and cons $A$, cons $B$ but no pro $B$. $\succeq^{DPoss}$ is a reflexive and transitive rule but it is very often incomplete. For instance, any set containing a positive and a negative argument of the highest level, i.e. a conflicting set, is incomparable to any set of arguments of a lower level ($\succeq^{Biposs}$ would rather conclude to an indifference). The rule is thus very interesting from a theoretic descriptive point of view. On the other hand, even if more decisive than $\succeq^{Biposs}$, it still suffers from the drowning effect and does not satisfy preferential independence. It can even violate positive and negative efficiency.

There is also a limited, but natural, lexicographic refinement of $\succeq^{Biposs}$, namely the one consisting in the leximax comparison of the vectors $(OM(A^+), OM(B^-))$ and $(OM(B^+), OM(A^-))$. Unfortunaly, it also fail to satisfy the principle of efficiency. This is due to the fact that, like $\succeq^{Biposs}$, $\succeq^{DPoss}$ and $\succeq^{Pareto}$, it collapse with classical comparative possibilities when $X = X^+$.

So, let us now focus of a set of relations that all do satisfy Preferential Independence, and are thus both positively and negatively efficient, and refine $\succeq^{Biposs}$.

The "discri" rule just adds the principle of preferential independence to the ones proposed by $\succeq^{Biposs}$, simply cancelling the elements that appear in both sets before applying the rule:

**Definition 6 (Discri)**
$A \succeq^{Discri} B \iff A \setminus B \succeq^{Biposs} B \setminus A$

$\succeq^{Discri}$ is complete but not transitive; its strict part, $\succ^{Discri}$ is obviously transitive. When $X = X^+$ (resp. $X = X^-$), sets of positive (resp. negative) arguments are to be compared; unsurprisingly, it is easy to check that in this case, $\succeq^{Discri}$ collapses to the discrimax (resp. discrimin) procedure.

$\succeq^{Discri}$ simply cancels any argument appearing in both $A$ and $B$. One could moreover accept the cancellation of any positive (resp. negative) argument in $A$ by another positive (resp. negative) argument in $B$ that shares the same order of magnitude. This yields the following two rules based on a levelwise comparison by cardinality. The arguments in $A$ and $B$ are scanned top down, until a level is reached such that the numbers of positive and negative arguments presented by the two alternatives are different; then, the set with the least number of negative arguments and the greatest number of positive ones is preferred: a Pareto comparison of the two cardinality-based criteria is performed by $\succeq^{Bilexi}$.

**Definition 7 ($i$-section)** *For any level $i \in L$:*
$A_i = \{x \in A, \pi(x) = i\}$ *is the i-section of A.*
$A_i^+ = A_i \cap X^+$ *(resp. $A_i^- = A_i \cap X^-$ ) is its positive (resp. negative) i-section.*

**Definition 8 (BiLexi)**
$A \succeq^{Bilexi} B \iff |A_{lb}^+| \geq |B_{lb}^+|$ *and* $|A_{lb}^-| \leq |B_{lb}^-|)$
*where* $lb = Argmax\{i : |A_i^+| \neq |B_i^+|$ *or* $|A_i^-| \neq |B_-^\pm|\}$

It easy to show that $\succeq^{Bilexi}$ is reflexive, transitive, but not complete. Indeed, if at the decisive level ($lb$) one of the set wins on the positive side, and the other on the negative side, a conflict is revealed and the procedure concludes to an incomparability. This information is particulary interesting, and should not be confused with indifference: in case of incomparability, the decision maker is perplex as no alternative seems better than the other: for any choice, there is a reason to regret it. In case of indifference, both alternatives are equally satisfactory, and the choice can be made without any regret. So, $\succeq^{Bilexi}$ concludes to an incomparability if and only if there is a conflict between the positive view and the negative view *at the decisive level*. From a descriptive point of view, this range of incomparability is a good point in favor of $\succeq^{Bilexi}$.

Now, if one looks for an even more decisive procedure, one could accept to drop this information about the conflict. A complete and transitive refinement of $\succeq^{Bilexi}$ will be obtained:

**Definition 9 (Lexi)**
$A \succeq^{Lexi} B \iff \exists i \in L$ *such that*

$$\begin{cases} (\forall j > i, & |A_j^+| - |A_j^-| & = & |B_j^+| - |B_j^-|) \\ \textit{and} & (|A_i^+| - |A_i^-| & > & |B_i^+| - |B_i^-|) \end{cases}$$

Finally, the projection of both $\succeq^{Bilexi}$ and $\succeq^{Lexi}$ on $X^+$ (resp. $X^-$) is complete and transitive and amounts to the leximax (resp. leximin) preference relation (Deschamps & Gevers 1978). Each projection is thus representable by a qualitative capacity (see e.g. (Moulin 1988)). Thus:

**Proposition 9** *There exist two capacities $\sigma^+$ and $\sigma^-$ such that:*
$A \succeq^{Lexi} B \Leftrightarrow \sigma^+(A^+) - \sigma^-(A^-) \geq \sigma^+(B^+) - \sigma^-(B^-)$

$$A \succeq^{Bilexi} B \Leftrightarrow \textit{and} \begin{cases} \sigma^+(A^+) & \succeq^{Bilexi} & \sigma^-(B^+) \\ \sigma^+(B^-) & \succeq^{Bilexi} & \sigma^-(A^-) \end{cases}$$

The proposition is obvious using the classical encoding of the leximax (unipolar) procedure by a capacity, e.g. $\sigma^+(V) = \sigma^-(V) = \sum_{i \in L} |V_i| \cdot |X|^i$.

The three rules obviously define monotonic bipolar relations. Each of them refines $\succeq^{Biposs}$ and satisfies Preferential Independence. They can be ranked from the least to the most decisive ($\succ^{Lexi}$), which is moreover complete and transitive.

**Proposition 10** $A \succ^{Biposs} B \Rightarrow A \succ^{Discri} B \Rightarrow A \succ^{Bilexi} B \Rightarrow A \succ^{Lexi} B$

## Related works

Results in cognitive psychology did point out the importance of bipolar reasoning in human decisions. Psychologists have shown that the simultaneous presence of positive and negative affects prevents decisions from being simple to make (Osgood, Suci, & Tannenbaum 1957) (see also (Slovic *et al.* 2002)), except when all the arguments have different orders of magnitude.

In this case, one can indeed apply the "take the best" approach advocated by (Gigerenzer & Goldstein 1996). In this approach, each criterion $a$ is supposed to have a positive side (it generates a positive argument $a^+$) and a negative side (it generates a negative argument $a^-$): fulfilling the criterion is a pro, missing it is a cons and is as bad as the pro is good. This is the typical example of having or not a garden. The criteria are then supposed to be of very different order of magnitude, so that they can be ranked lexicographically: there does not exist $a$, $b$ such that $\pi(a) = \pi(b)$. So, we can rank criteria from the strongest to the weakest when comparing alternatives. As soon as there is an $a$ that is in favor of $d$ and in disfavor of $d'$, $d$ is preferred to $d'$ (hence the name "take the best"). Applied to such linearly ranked criteria, the discri and lexi bipolar rules have this very behavior. But these rules are able to account for more decision situations — e.g. several criteria can share the same degree of importance. In this sense, they are a natural extension of the "Take the best" qualitative rule advocated by psychologists.

Cumulative Prospect Theory (Tversky & Kahneman 1992) is another attempt to account for positive and negative arguments. Contrary to the "Take the best" approach, it is often oriented toward quantitative evaluations of decisions. Cumulative Prospect Theory assumes that reasons supporting a decision and reasons against it can be measured by means of two capacities $\sigma^+$ and $\sigma^-$, $\sigma^+$ reflecting the importance of the group of positive affects, $\sigma^-$ the importance of the group of negative affects. The higher $\sigma^+$, the more convincing the set of arguments and conversely the higher $\sigma^-$, the more deterring $A$. This approach moreover admits that it is possible to map these evaluations to a so-called "net predisposition" score expressed on a single scale:

$$\forall A \subseteq X, NP(A) = \sigma^+(A^+) - \sigma^-(A^-)$$

where $A^+ = A \cap X^+, A^- = A \cap X^-$. Variants exist that measure utility by some other function of $\sigma^+$ and $\sigma^-$.

On the other hand, since the comparison of net predispositions systematically provides a complete and transitive preference, it can fail to capture a large range of decision-making attitudes: the point is that, contrasting affects make decision difficult, so that the comparison of objects characterised by bipolar evaluations does not systematically yields a complete and transitive relation. It can imply some incomparabilities. That is why bicapacities were generalized by means of bipolar capacities (Greco, Matarazzo, & Slowinski 2002). The idea is to use two measures, a measure of positiveness (that increases with the addition of positive arguments and the deletion of negative arguments) and a measure of negativeness (that increases with the addition of negative arguments and the deletion of positive arguments), but without combining then. Then $A$ is preferred to $B$ iff it is the case with respect to both measures – i.e. according to the sole Pareto principle. This allows the representation of conflicting evaluations and can lead to a partial order.

Our approach is clearly a qualitative counterpart to the above works. In the $\succeq^{Lexi}$ relation, the positive and negative sets of affects are evaluated separately by capacities $\sigma^+$ and a $\sigma^-$ and the aggregated in agreement with net predisposition. The bilexi-rule does not merge the positive and the negative, thus allowing the expression of conflicts. This concludes our argumentation in favor of $\succeq^{Lexi}$ and $\succeq^{Bilexi}$: they comply with the spirit of CTP as well at its practical advantages (transitivity and representability by a pair of functions), they are efficient and in accordance with but more decisive than pure order-of-magnitude reasoning.

## Conclusion

This paper has focused on a particular class of bipolar decision making situations, namely those that are qualitative rather than quantitative in essence. The proposed work is an extension of possibility theory to the handling of sets containing two-sorted elements considered as positive or negative. The results were couched in a terminology borrowing to argumentation and decision theories, and indeed we consider they can be relevant for both. Our framework is a qualitative counterpart to Cumulative Prospect Theory and more recent proposals like bipolar capacities. The paper is also relevant to argumentative reasoning for the evaluation of sets of arguments in inference processes (Cayrol & M-C.Lagasquie-Schiex 2002), and to argument-based decisions (Amgoud & Prade 2004). The next step is naturally the extension to (qualitative) bipolar criteria whose satisfaction is a matter of degree (Grabisch & Labreuche 2005). In the future, comparison between our decision rules and those adopted in the above works as well as aggregation processes in finite bipolar scales (Grabisch 2004) is in order.

This paper has adopted a prescriptive point of view in the sense that the rules were studied with respect to the properties that a qualitative theory of bipolar decision making should obey. In the meantime, we are currently testing their descriptive power, i.e. their ability to represent the behavior of human decision makers. Our first results confirm $\succ^{Biposs}$ as being the basis of ordinal decision making and suggest $\succeq^{Lexi}$ as a decision rule for the decision makers that satisfy preferential independence. $\succeq^{Bilexi}$ seems to be the best model for that subset of decision makers for whom conflicts lead to incomparability.

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## Proofs

**Proof of Proposition 1.** *transitivity of* $\succeq^{Pareto}$: suppose $A \succeq^{Pareto} B$ and $B \succeq^{Pareto} C$. By definition, we get $OM(A^+) \geq OM(B^+)$, $OM(A^-) \leq OM(A^-)$, $OM(B^+) \geq OM(C^+)$, $OM(B^-) \leq OM(C^-)$. Then obviously $OM(A^+) \geq OM(C^+)$ and $OM(A^-) \geq OM(C^-)$: $A \succeq^{Pareto} C$.

The reflexivity of $\succeq^{Pareto}$ is obvious.

$\succeq^{Pareto}$ *is not complete*: It is enough to consider the following counter example: $A = \{a^+, a^-\}$, with $a^+ \in X^+$, $a^- \in X^-$, $\pi(a^+) = \pi(a^-) > 0$. It holds that $OM(\emptyset^+) = OM(\emptyset^-) = 0$. So $OM(A^+) > OM(\emptyset^+)$ and $OM(A^-) > OM(\emptyset^-)$. $A$ and $\emptyset$ are not comparable.

**Proof of Proposition 2.**

The completeness of $\succeq^{Biposs}$ follows from the simple fact that it is based on the comparison of two measures.

*Transitivity of* $\succ^{Biposs}$: suppose $A \succ^{Biposs} B$ and $B \succ^{Biposs} C$. Four cases are possible:

- $\exists a^+ \in A^+, \forall x \in A^- \cup B^+, \pi(a^+) > \pi(x)$ and $\exists b^+ \in B^+, \forall x \in B^- \cup C^+, \pi(b^+) > \pi(x)$. Then $\pi(a^+) > \pi(b^+)$ and thus $\forall x \in C^+, \pi(a^+) > \pi(x)$: $A \succ^{Biposs} C$.
- $\exists b^- \in B^-, \forall x \in A^- \cup B^+, \pi(b^-) > \pi(x)$ and $\exists c^- \in C^-, \forall x \in B^- \cup C^+, \pi(c^-) > \pi(x)$. Then $\pi(c^-) > \pi(b^-)$ and thus $\forall x \cup A^-, \pi(c^-) > \pi(x)$. So, $\exists c^- \in C^-, \forall x \in A^- \cup C^+, \pi(c^-) > \pi(x)$: $A \succ^{Biposs} C$.
- $\exists a^+ \in A^+, \forall x \in A^- \cup B^+, \pi(a^+) > \pi(x)$ and $\exists c^- \in C^-, \forall x \in B^- \cup C^+, \pi(c^-) > \pi(x)$. So, the maximum of $a^+$ and $c^-$ has a possibility degree strictly better than any $x \in A^-$ and than any $x \in C^+$: $OM(A^+ \cup C^-) > OM(A^- \cup C^+)$.
- $\exists b^- \in B^-, \forall x \in A^- \cup B^+, \pi(b^-) > \pi(x)$ and $\exists b^+ \in B^+, \forall x \in B^- \cup C^+, \pi(b^+) > \pi(x)$. Hence the contradiction $\exists b^- \in B^-, \forall b^+ \in B^+, \pi(b^-) > \pi(b^+)$ and $\exists b^+ \in B^+, \forall b^- \in B^-, \pi(b^+) > \pi(b^-)$. This case never occurs.

So, $A \succ^{Biposs} B$ and $B \succ^{Biposs} C$ implies $A \succ^{Biposs} C$: $\succ^{Biposs}$ is transitive, i.e. $\succeq^{Biposs}$ quasi transitive.

$\sim^{Biposs}$ is not transitive. Indeed choose $a^+, a^-, b^+, b^-, c^+, c^-$ such that $\pi(b^+) = \pi(b^-) = \alpha > \pi(a^+) = \pi(c^-) = \beta > \pi(a^-) = \pi(c^+) = \gamma$, and build $A = \{a^-, a^+\}, B = \{b^-, b^+\}, C = \{c^-, c^+\}$. So $OM(A^+ \cup B^-) = OM(C^+ \cup B^-) = OM(A^- \cup B^+) = OM(B^- \cup B^-) = \alpha$. We thus have $A \sim^{Biposs} B$ and $B \sim^{Biposs} C$, but $OM(A^+ \cup C^-) = \beta > \gamma = OM(A^- \cup C^+)$, hence $A \succ^{Biposs} C$.

Note that the transitivity of $A \succeq B, B \succeq C \implies A \succeq C$ is preserved as soon as $OM(B^+) \neq OM(B^-)$, i.e. as soon as $B$ is not suffering from an internal conflict.

**Proof of Proposition 3.**

Quasi transitivity is proved in Proposition 2.

Positive and negative monotony, as well as SQC follow from the monotony of $OM$, that is a possibility measure (i.e. $OM(U \cup V) \geq OM(V) \forall U, V$).

Non triviality of $\succeq^{Biposs}$ is obtained from the non triviality of $\pi$ (there exists $x$ such as $\pi(x) > 0$), that implies $OM((X^+)^+ \cup (X^-)^-) = OM(X^+ \cup X^-) > 0$ while $OM((X^+)^- \cup (X^-)^+) = OM(\emptyset) = 0$.

Clarity of argument is also trivial: if $x$ is positive, then $OM(\{x\}^+ \cup (\emptyset)^-) = \pi(x) \geq OM(\{x\}^- \cup (\emptyset)^+) = OM(\emptyset) = 0$: $\{x\} \succeq^{Biposs} \emptyset$. If $x$ is negative, we get in the same way $\{x\} \preceq^{Biposs} \emptyset$. If $x$ has a degree 0, we get $OM(\{x\}^+ \cup (\emptyset)^-) = \pi(x) \geq OM(\{x\}^- \cup (\emptyset)^+) = OM(\emptyset) = 0$: $\{x\} \sim^{Biposs} \emptyset$.

For proving weak unanimity, recall that $A^+ \succeq^{Biposs} B^+$ implies $\exists a^+ \in A^+, \forall x \in A^+ \cup B^+, \pi(a^+) \geq \pi(x)$ and that $A^- \succeq^{Biposs} B^-$ implies $\exists b^- \in B-, \forall x \in B^- \cup A^-, \pi(b^-) \geq \pi(x)$. So, $OM(A^+ \cup B^-) = \max(\pi(a^+), \pi(b^-)) \geq OM(A^- \cup B^+)$: $A \succeq^{Biposs} B$.

**Proof of Proposition 4.**

The proof is very similar to the previous one.

The quasi transitivity of $\succeq^{Pareto}$ is follows from its transitivity (Proposition 1).

Positive and negative monotony, as well as SQC follow from the monotony of $OM$, that is a possibility measure.

Non triviality of $\succeq^{Pareto}$ is obtained from the non triviality of $\pi$, in a similar way as in the proof drawn for $\succeq^{Biposs}$

Clarity of argument is also trivial: for a positive $x$, then $OM(\{x\}^+ = \pi(x) \geq OM((\emptyset)^+)$ and since $\{x\}^- = \emptyset$, $OM(\{x\}^- = 0 = OM((\emptyset)^-)$: $\{x\} \succeq^{Pareto} \emptyset$. The proof is symmetric if $x$ is in $X-$.

Weak (and strong unanimity) are obvious for Pareto: they immediately follow from the definition.

**Proof of Proposition 5.** When restricted to singletons, $\succeq^{Biposs}$ ranks the positive arguments by decreasing order of $\pi$, then the null arguments ($\pi = 0$), then the negative arguments by increasing value of $\pi$. This ranking defines a complete and transitive relation. This proves the $\succeq_{\mathbb{X}}$ induced by $\succeq^{Biposs}$ is a complete pre-order.

Axioms POSC is easy to check, since $\{x+, y-\} \sim \emptyset$ (resp. $\{z+, y-\} \sim \emptyset$) iff $\pi(x^+) = \pi(y^-)$ (resp. $\pi(z^+) = \pi(y^-)$) and $x^+ \sim_{\mathbb{X}} z^+$ iff $\pi(x^+) = \pi(z^+)$. The proof of NEGC is similar.

The $\mathbb{X}$-monotonicity of $\succeq^{Biposs}$ is shown as follows. Let $A, x, x'$ be such that $A \cap \{x, x'\} = \emptyset$ and $x' \succeq_{\mathbb{X}} x$. Three cases are possible:

Case A: $x \in X^+$. Then $x' \in X^+$ and $\pi(x') \geq \pi(x)$.

- if $A \cup \{x\} \succ^{Biposs} B$: then $OM(A^+ \cup \{x\} \cup B^-) > OM(A^- \cup B^+)$. Since $\pi(x') \geq \pi(x)$, we get $OM(A^+ \cup \{x'\} \cup B^-) > OM(A^- \cup B^+)$: $A \cup \{x'\} \succ^{Biposs} B$.
- if $A \cup \{x\} \sim^{Biposs} B$: then $OM(A^+ \cup \{x\} \cup B^-) = OM(A^- \cup B^+)$. Replacing $x$ by $x'$, i.e. $\pi(x)$ by $\pi(x')$, the first OM level increases, so we get $OM(A^+ \cup \{x'\} \cup B^-) \geq OM(A^- \cup B^+)$: $A \cup \{x'\} \succeq^{Biposs} B$.
- if $B \succ A \cup \{x'\}$, then $OM(B^+ \cup A^-) > OM(A^+ \cup \{x'\} \cup B^-)$. Replacing $x'$ by $x$, i.e. $\pi(x')$ by $\pi(x)$, the second OM level decreases, so we get $OM(B^+ \cup A^-) > OM(A^+ \cup \{x\} \cup B^-)$, i.e. $B \succ^{Biposs} A \cup \{x\}$.
- if $B \sim A \cup \{x'\}$, then $OM(B^+ \cup A^-) = OM(A^+ \cup \{x'\} \cup B^-)$. Replacing $x'$ by $x$, i.e. $\pi(x')$ by $\pi(x)$, the second OM level decreases, so $OM(B^+ \cup A^-) \geq OM(A^+ \cup \{x\} \cup B^-)$, i.e. $B \succeq^{Biposs} A \cup \{x\}$.

Case B: If $x' \in X^-$, then $x \in X^-$ and $\pi(x) \geq \pi(x')$. The same kind of proof in four cases can be drawn.

Case C: $x' \in X^+$ and $x \in X^-$:

- if $A \cup \{x\} \succ^{Biposs} B$: A negative argument on the left is replaced by a positive one, so $OM(A^+ \cup B^- \cup \{x'\}) \geq OM(A^+ \cup B^-) > OM(A^- \cup B^+ \cup \{x\}) \geq OM(A^- \cup B^+)$: $A \cup \{x'\} \succ^{Biposs} B$.
- Similarly, if $A \cup \{x\} \sim^{Biposs} B$: since $x$ is negative we have $OM(A^+ \cup B^-) = OM(A^- \cup B^+ \cup \{x\})$. Moreover $OM(A^+ \cup B^- \cup \{x'\}) \geq OM(A^+ \cup B^-)$ and $OM(A^- \cup B^+ \cup \{x\}) \geq OM(A^- \cup B^+)$. Hence $OM(A^+ \cup B^- \cup \{x'\}) \geq OM(A^- \cup B^+$: $A \cup \{x'\} \succeq^{Biposs} B$.
- if $B \succ A \cup \{x'\}$, a positive argument on the right is replaced by a negative one, $OM(B^+ \cup A^- \cup \{x\}) \geq OM(B^+ \cup A^-) > OM(A^+ \cup \{x'\} \cup B^-) \geq OM(A^+ \cup B^-)$: $B \succ^{Biposs} A \cup \{x\}$.
- Similarly, if $B \succ A \cup \{x'\}$ we have: $OM(B^+ \cup A^- \cup \{x\}) \geq OM(B^+ \cup A^-) = OM(A^+ \cup \{x'\} \cup B^-) \geq OM(A^+ \cup B^-)$: $B \succeq^{Biposs} A \cup \{x\}$.

**Proof of Proposition 6.** Like $\succeq^{Biposs}$, when restricted to singletons, $\succeq^{Pareto}$ ranks the positive arguments by decreasing order of $\pi$, then the null arguments, then the negative arguments by increasing value of $\pi$. This ranking defines a complete and transitive relation: the $\succeq_{\mathbb{X}}$ induced by $\succeq^{Pareto}$ is a complete pre-order. Axioms POSC and NEGC are trivial, since, if $x^+ \in X^+$ and $y- \in X^-$, it cannot happen that $\{x+, y-\} \sim^{Pareto} \emptyset$. The proof of the $\mathbb{X}$-monotonicity of $\succeq^{Pareto}$ is as easy as for $\succeq^{Biposs}$ and very similar. It is omitted for the sake of space.

**Proof of Propositions 7 and 8.**

$\succeq^{Biposs}$ *satisfies* $GCLO$. Recall that $A \succeq^{Biposs} B \iff OM(A^+ \cup B^-) \geq OM(B^+ \cup A^-)$ and $C \succeq^{Biposs} D \iff OM(C^+ \cup D^-) \geq OM(D^+ \cup C^-)$. Since the GCLO property holds for possibility measures and since $OM$ is a possibility measure, we obtain: $OM(A^+ \cup B^- \cup C^+ \cup D^-) \geq OM(B^+ \cup A^- \cup D^+ \cup C^-)$, hence $A \cup C \succeq^{Biposs} B \cup D$.

$\succeq^{Biposs}$ *satisfies* $GNEG$. Recall that $A \succ^{Biposs} B \iff OM(A^+ \cup B^-) > OM(B^+ \cup A^-)$ and $C \succ^{Biposs} D \iff OM(C^+ \cup D^-) > OM(D^+ \cup C^-)$. Since the GNEG property holds for possibility measures and since $OM$ is a possibility measure, we obtain: $OM(A^+ \cup B^- \cup C^+ \cup D^-) > OM(B^+ \cup A^- \cup D^+ \cup C^-)$, hence $A \cup C \succ^{Biposs} B \cup D$.

$\succeq^{Pareto}$ *satisfies GCLO.* Choose $A, B, C, D$ such that $A \sim^{Pareto} B$ and $C \sim^{Pareto} D$. By definition of $\succeq^{Pareto}$: $OM(A^+) = OM(B^+), OM(A^-) = OM(B^-)$, $OM(A^-) = OM(B^-)$, $OM(C^+) = OM(D^+)$, $OM(C^-) = OM(D^-)$. Since OM is a possibility measure (and thus satisfies GCLO) this implies $OM(A^+ \cup C^+) = OM(B^+ \cup D^+)$ and $OM(A^- \cup C^-) = OM(B^- \cup D^-)$: thus $A \cup C \sim^{Pareto} B \cup D$.

$\succeq^{Pareto}$ *satisfies GNEG.* Choose $A, B, C, D$ such that $A \succ^{Pareto} B$ and $C \succ^{Pareto} D$. By definition of $\succeq^{Pareto}$, four cases are possible:

(i) $OM(A^+) \geq OM(B^+)$, $OM(A^-) < OM(B^-)$ (thus $A \succ^{Pareto} B$) and $OM(C^+) \geq OM(D^+), OM(C^-) < OM(D^-)$ (thus $C \succ^{Pareto} D$): Since OM is a possibility measure we get obviously, $OM(A^+ \cup C^+) \geq OM(B^+ \cup D^+)$ and $OM(A^- \cup C^-) < OM(B^- \cup D^-)$, thus $A \cup C \succ^{Pareto} B \cup D$.

(ii) $OM(A^+) > OM(B^+)$, $OM(A^-) \leq OM(B^-)$ (thus $A \succ^{Pareto} B$) and $OM(C^+) > OM(D^+), OM(C^-) \leq OM(D^-)$ (thus $C \succ^{Pareto} D$). Since OM is a possibility measure we get obviously, $OM(A^- \cup C^-) \leq OM(B^- \cup D^+)$ and $OM(A^+ \cup C^+) > OM(B^+ \cup D^+)$, thus $A \cup C \succ^{Pareto} B \cup D$.

(iii) $OM(A^+) > OM(B^+)$, $OM(A^-) \leq OM(B^-)$ and $OM(C^+) \geq OM(D^+)$, $OM(C^-) < OM(D^-)$: once again, since OM is a possibility measure we get obviously, $OM(A^+ \cup C^+) \leq OM(B^+ \cup D^+)$ and $OM(A^- \cup C^-) < OM(B^- \cup D^-)$, thus $A \cup C \succ^{Pareto} B \cup D$.

(iv) $OM(A^+) \geq OM(B^+)$, $OM(A^-) < OM(B^-)$ and $OM(C^+) > OM(D^+)$, $OM(C^-) \leq OM(D^-)$: once again, since OM is a possibility measure we get obviously, $OM(A^+ \cup C^+) \leq OM(B^+ \cup D^+)$ and $OM(A^- \cup C^-) < OM(B^- \cup D^-)$, thus $A \cup C \succ^{Pareto} B \cup D$.

**Proof of Theorem 1.** We first build a distribution $\pi^+$ that ranks the positive arguments. This is made possible because $\succeq_{\mathbb{X}}$ is a complete pre-order. Any $\pi^+$ such that $\forall x, y \in X^+ : \pi^+(x) \geq \pi^+(y) \iff \{x\} \succeq \{y\}$ and $\pi^+(x) = 0 \iff \{x\} \sim \emptyset$ is suitable. Similarly, build a distribution that ranks the negative arguments: any $\pi^-$ such that $\forall x, y \in X^- : \pi^-(x) \geq \pi^+(y) \iff \{x\} \preceq \{y\}$ and $\pi^-(x) = 0 \iff \{x\} \sim \emptyset$ is suitable. Define $A \succeq^{Pareto} B$ as $OM^+(A^+) > OM^+(B^+)$ and $OM^-(A^-) \leq OM^-(B^-)$ where $OM^+(A^+)$ (resp. $OM^-(A^-)$) is based on $\pi^+$ (resp. $\pi^-$).

Let us now prove that any relation in the family refines the Pareto ordering, i.e. let us prove that $A \succ^{Pareto} B$ implies $A \succ B$ for any relation $\succeq$ that satisfies the axioms and has the same ground relation $\succeq_{\mathbb{X}}$

The proof is very easy: suppose that $OM^+(A^+) > OM^+(B^+)$ and $OM^-(A^-) \leq OM^-(B^-)$ (the proof of $OM(B^-) > OM(A^-)$ and $OM(A^+) \geq OM(B^+)$ implies $A \succ B$ can then be made by symmetrically).; then there is an $a^+ >\in A^+$ such that $\forall x \in B^+, \{a^+\} \succ \{x\}$. Then by GNEG $\{a^+\} \succ B^+$ and by positive monotony $A^+ \succ B^+$. If $OM^-(A^-) \leq OM^-(B^-)$ then there is a $b^- \in B^-$ such that $\forall x \in A^-, x \succeq \{b^-\}$. Then by GCLO $A^- \succeq \{b^-\}$ and by positive monotony $A^- \succeq B^-$. Strong unanimity then implies $A \succ B$. QED

Note that we could prove similarly that $A \sim^{Pareto} B$ implies $A \sim B$ for any relation that satisfy the axioms and is build on the same $\succeq_{\mathbb{X}}$: using GCLO, positive monotony and negative monotony we get $A^- \sim B^-$ and $A^+ \sim B^+$. Then weak unanimity implies $A \sim B$.

Moreover, from $OM^+(A^+) > OM^+(B^+)$ and $OM^-(A^-) < OM^-(B^-)$ we conclude that for any $\succeq$ in the family, $A^+ \succ B^+$ and $A^- \prec B^-$ . In this case, Pareto concludes to an incomparability, but other rules could make other decisions.

**Proof of Theorem 2.**

Let us first build $\pi$. By distinguishability, any singleton $\{x\}$ is comparable to $\emptyset$. So $X^+ = \{x, \{x\} \succ \emptyset\}$, $X^- = \{x, \{x\} \prec \emptyset\}$ and $X^0 = \{x, \{x\} \sim \emptyset\}$ are soundly defined.

Let us define a relation $\geq$ on $X$ as follows:

- $x \in X^0 : \forall y, y \geq x$.
- $x, y \in X^+ : x \geq y \iff \{x\} \succeq \{y\}$
- $x, y \in X^- : x \geq y \iff \{y\} \succeq \{x\}$
- $x \in X^+, y \in X^- : x \geq y \iff not(\emptyset \succ \{x, y\})$
- $x \in X^+, y \in X^- : y \geq x \iff not(\{x, y\} \succ \emptyset)$

We can prove that $\geq$ is complete: since $\succeq_{\mathbb{X}}$ is complete, so is $\geq$ within $X^+$ and within $X^-$. $\geq$ is complete within $X^0$ by definition. Now, consider any $x^+ \in X^+$, $y^- \in X^-$ and $z^0 \in X^0$. By SQC, $x^+ > z^0$ and $z^0 > y^-$. Let us see how $\geq$ compares $x^+$ and $y^-$. Since $\succeq$ is complete, it holds that $\{x^+, y^-\} \succeq \emptyset$ or $\{x^+, y^-\} \preceq \emptyset$. By definition of $\geq$, this means $x^+ \geq y^-$ or $y^- \geq x^+$. So, $\geq$ is complete.

We prove now that $\geq$ is transitive. Suppose that $x \geq y$ and $y \geq z$. $x \geq z$ is trivial when $x, y, z \in X^+$ ((because $\succeq_{\mathbb{X}}$ is a transitive and can be identified with $\geq$ within $X^+$). For the same reason, $x \geq z$ when $x, y, z \in X^-$.

If $x \in X^0$, then $x \geq y$ means by SQC that $y$ is also in $X^0$, which in turn implies $z \in X^0$. So, by SQC again, $x \equiv z$.

If $y \in X^0$, $y \geq z$ implies by SQC that $z$ is also in $X^0$. So, by SQC again, $x \geq z$.

If $z \in X^0$, $x \geq z$ is always true (by status quo consistency again).

Suppose now $x \in X^+$, $y \in X^-$ and $z \in X^-$. Then by definition, $x \geq y$ means $\{x, y\} \succeq \emptyset$ and $y \geq z$ means $\{z\} \succeq \{y\}$. By $\mathbb{X}$-monotonicity we can replace $y$ by $z$ without reversing the preference: $\{x, z\} \succeq \emptyset$, i.e. $x \geq z$.

Transitivity can be proved in a similar way when $x \in X^+$, $y \in X^+$ and $z \in X^-$.

The next case occurs when $y \in X^+$ $x \in X^-$ and $z \in X^-$. Suppose that $\{x, y\} \preceq \emptyset$ ($x \geq y$), $\{z, y\} \succeq \emptyset$ ($y \geq z$) and $\{x\} > \{z\}$ ($z > x$ for negative arguments). If $\{z, y\} \succ \emptyset$, then $\mathbb{X} - monotony$ implies $\{x, y\} \succ \emptyset$ (thus a contradiction). If $\emptyset \succ \{x, y\}$ then $\mathbb{X} - monotony$ implies $\emptyset \succ \{z, y\}$ (second contradiction). Last case, if $\{x, y\} \sim \emptyset$ and $\{z, y\} \sim \emptyset$, then POSC implies $\{x\} \sim \{z\}$ (last contradiction). So, $z > x$ does not hold and thus, by completeness of $\geq$, $x \geq z$.

The last case, that occurs when $y \in X^-$ $x \in X^+$ and $z \in X^+$ is proved in a symmetrical way, using $\mathbb{X} - monotony$ and NEGC.

So, $\geq$ is a complete pre-order. It can be encoded by a distribution $\pi : X \mapsto [0_L, 1_L]$, $[0_L, 1_L]$ being a totally ordered scale. Level $0_L$ is mapped to elements of $X_0$. Now, we have to show the equivalence between $\succeq$ and the relation $\succeq^{BiPoss}$ induced from $\pi$, namely, defined by $A \succeq^{BiPoss} B \iff OM(A^+ \cup B^-) \geq OM(A^- \cup B^+)$. Since the relations are complete, this amounts to showing that $OM(A^+ \cup B^-) = OM(A^- \cup B^+)$ implies $A \sim B$ and that $OM(A^+ \cup B^-) > OM(A^- \cup B^+)$ implies $A \succ B$.

Let us first prove that $OM(A^+ \cup B^-) > OM(A^- \cup B^+)$ implies $A \succ B$. Suppose that the element of highest $\pi$ value in $A^+ \cup B^-$ is $a^+ \in A^+$. So, for any $a^- \in A^-$, $\{a^+, a^-\} \succ \emptyset$ and $b^+ \in B^+$, $\{a^+\} \succ \{b^+\}$. $GNEG$ then implies that $\{a^+\} \cup A^- \succ \emptyset$ and $NEG$ implies that $\{a^+\} \succ B^+$. By $GNEG$ again we get $\{a^+\} \cup A^- \succ B^+$ and positive and negative monotony then imply $A^+ \cup A^- \succ B^+ \cup B^-$. If the element of highest $\pi$ value in $A^+ \cup B^-$ were some $b^- \in B^-$ then for any $b^+ \in B^+$, $\{b^+, b^-\} \prec \emptyset$ and for any $a^- \in B^+$, $\{a^-\} \succ \{b^-\}$. $GNEG$ then implies that $\{b^-\} \cup B^+ \prec \emptyset$ and that $b^- \prec A^-$. By $GNEG$ again we get $\{b^-\} \cup B^+ \prec A^-$ and positive and negative monotony then implies $A^+ \cup A^- \succ B^+ \cup B^-$.

Let us now prove that $OM(A^+ \cup B^-) = OM(A^- \cup B^+)$ implies $A \sim B$.

- If $Max_{a \in A}\pi(a) > Max_{b \in B}\pi(b)$, then $A$ contains at least two elements $a^+ \in A^+$ and $a^- \in A^-$ at its highest level (otherwise we cannot have $OM(A^+ \cup B^-) = OM(A^- \cup B^+)$).
  Using GNEG we get $A^+ \succ B^+$ and $A^- \prec B^-$.
  Then for all $a_i^- \in A^-$, $\{a^+, a_i^-\} \succeq \emptyset$. By GCLO we get $\{a^+\} \cup A^- \succeq \emptyset$. Positive monotony then implies that $A^+ \cup A^- \succeq \emptyset$. By reflexivity, $A^+ \succeq A^+$. So GCLO implies $A^+ \cup A^- \succeq A^+$. On the other hand, by negative monotony, $A^+ \succeq A$. Hence $A^+ \sim A$.
  One can prove symmetrically that $A^- \sim A$.
  So, $A^+ \sim A$ and $A^+ \succ B^+$. The assumptions of completeness and quasi transitivity then imply $A \succeq B^+$. By negative monotony, this implies $A \succeq B$
  Similarly, from $A^- \sim A$ and $A^- \prec B^-$. we get $A \preceq B^-$. By positive monotony, this implies $A \preceq B$
  Thus $A \succeq B$ and $A \preceq B$, i.e. $A \sim B$.
- The case $Max_{a \in A}\pi(a) < Max_{b \in B}\pi(b)$ is symmetric to the previous one. We get $A \sim B$ in a similar manner.
- $Max_{a \in A}\pi(a) = Max_{b \in B}\pi(b)$. Then $OM(A^+ \cup B^-) = OM(A^- \cup B^+)$ implies that either (i) there exist $a^+ \in A^+$, $b^+ \in B^+, \forall x \in A \cup B, \pi(a^+) = \pi(b^+) \geq \pi(x)$ or (ii) there exist $a^- \in A^-$, $b^- \in B^-$, $\forall a \in A, \pi(a^-) = \pi(b^-) \geq \pi(a)$ and $\forall a \in A, \pi(a^-) = \pi(b^-) \geq \pi(b)$.
  Let us consider case (i). By GNEG, we know that $\{b^+\} \cup B^- \succeq \emptyset$ and by positive monotony, $B^+ \succeq \{b^+\}$. Then GCLO implies that $B \succeq \{b^+\}$. On the other hand, GCLO imply that $\{b^+\} \sim B^+$, so by negative monotony $\{b^+\} \succeq B$. Hence we get $\{b^+\} \sim B$. The same kind of reasoning allows to derive $\{a+\} \sim A$. By $\mathbb{X}$-monotonicity we get $\{b^+\} \sim A$ and $\{a+\} \sim B$. GCLO then derives $A \sim B$.
  Case (ii) is symmetric: By GNEG, we know that $\{b^-\} \cup B^+ \preceq \emptyset$ and by negative monotony, $B^- \preceq \{b^-\}$. Then GCLO implies that $B \preceq \{b^-\}$. On the other hand, GCLO imply that $\{b^-\} \sim B^-$, so by positive monotony $\{b^-\} \preceq B$. Hence we get $\{b^-\} \sim B$. The same kind of reasoning allows to derive $\{a-\} \sim A$. By $\mathbb{X}$-monotonicity we get $\{b^-\} \sim A$ and $\{a-\} \sim B$. GCLO then derives $A \sim B$.

**Proof of Proposition 9.** As said in the text, The proposition is obvious using the classical encoding of the leximax (unipolar) procedure by a capacity, e.g. $\sigma^+(V) = \sigma^-(V) = \sum_{i \in L} |V_i| \cdot |X|^i$.

**Proof of Proposition 10.** Suppose that $A \succ^{Biposs} B$: then $\exists x^* \in A^+ \cup B^-$ such that $\forall x \in A^- \cup B^+, \pi(x^*) > \pi(x)$ – if many arguments satisfy this property, let $x^*$ be one of those maximizing $\pi$.

This $x^*$ is thus not in $A^- \cup B^+$. So, it is either in $A^+ \setminus B^+$ or in $B^- \setminus A^-$. Since $A^+ \setminus B^+ = (A \setminus B)^+$ and $B^- \setminus A^- = (B \setminus A)^-$, we can write $x^* \in (A \setminus B)^+ \cup (B \setminus A)^-$. On the other hand $(B \setminus A)^+ \cup (A \setminus B)^- \subseteq A^- \cup B^+$ and no element in $A^- \cup B^+$ has a higher degree than $x^*$. So, $OM((B \setminus A)^+ \cup (A \setminus B)^-) < \pi(x^*)$. So $A \succ^{Discri} B$. Hence $\succeq^{Discri}$ refines $\succeq^{Biposs}$

Suppose that $A \succ^{Discri} B$: then $\exists x^* \in A^+ \setminus B^+ \cup B^- \setminus A^-$ such that $\forall x \in B^+ \setminus A^+ \cup A^- \setminus B^- \pi(x^*) > \pi(x)$. If many elements satisfy this property, let $x^*$ be one of those maximizing $\pi$.

So, at any level $i > \pi(x^*)$, $(A^+)_i = (B^+)_i$ and $(A^-)_i = (B^-)_i$. – otherwise the discri rule would have conclude to an indifference between $A$ and $B$. So, at these levels i, we also have the equality of the cardinalities.

On the other hand, at level $lb = \pi(x^*)$, there is no element of $B^+ \setminus A^+ \cup A^- \setminus B^-$. So, $Card((B^+ \setminus A^+)_{lb}) = 0$ and $Card((A^- \setminus B^-)_{lb}) = 0$. And there is at least one element in $A^+ \setminus B^+ \cup B^- \setminus A^-$ so $Card((A^+ \setminus B^+)_{lb}) \geq 1$ or $Card((B^- \setminus A^-)_{lb}) \geq 1$ (or even both).

Suppose $Card((A^+ \setminus B^+)_{lb}) \geq 1$: from $Card((B^+ \setminus A^+)_{lb}) = 0$ and adding the common elements, we get $Card((B^+)_{lb}) = Card((A^+ \cap B^+)_{lb})$. Since $Card((A^+ \setminus B^+)_{lb}) \geq 1$ we get $Card((A^+)_{lb}) > Card((B^+)_{lb})$. From $Card((A^- \setminus B^-)_{lb}) = 0$ and adding the common element we get $Card(A^-) = Card(B^- \cap A^-)$ thus $Card(A^-) \leq Card(B^-)$. So $A \succ^{Bilexi} B$.

We get $A \succ^{Bilexi} B$ in the same way from $Card((B^- \setminus A^-)_{lb}) \geq 1$. Hence $\succeq^{Bilexi}$ refines $\succeq^{Discri}$

Finally, suppose that $A \succ^{Bilexi} B$, i.e. at any level higher $i$ than $lb$, $Card((A^+)_i) = Card((B^+)_i$ and $Card((A^-)_i) = Card((B^-)_i$ and that at level $lb$, there is a difference in favor of $A$. Then necessarily, at any level higher $i$ than $lb$ $Card((A^+)_i) - Card((A^-)_i) = Card((B^+)_i - Card((B^-)_i)$. If the difference is made on the positive scale, i.e. $Card((A^+)_{lb}) > Card((B^+)_{lb}$ and $Card((A^-)_{lb}) \leq Card((B^+)_{lb}$. Summing the inequalities we get $Card((A^+)_{lb}) - Card((A^-)_{lb}) > Card((B^+)_{lb} - Card((B^-)_{lb})$. The proof is similar if the difference is made on the negative side. So, $A \succ^{Lexi} B$. Hence $\succeq^{Lexi}$ refines $\succeq^{Bilexi}$