An Axiomatic Framework for Interval Probability*

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Abstract

This paper suggests a new axiomatic framework for interval probability, which is novel in two aspects: 1) this framework can degenerate into the classical probability model; and 2) the sound mathematical properties can be derived from this framework. The above virtues do not exist in the old framework for interval probability, nor in the D-S evidence theory which can also deal with an interval representation of uncertainty.

Introduction

In the majority of the existent knowledge-based systems real numbers are used to estimate the experts' degrees of uncertainty in different statements. However, the real number based representation fails to satisfactorily distinguish between the uncertainty due to the measure error (experimental or expert's), and the uncertainty due to the lack (even absence) of knowledge (Simoff 1995), or between certainty and confidence. Therefore, some interval-based formalisms (Shafer 1976; Ruspini 1987; Baldwin 1986; Kyburg 1988; Luo *et al.* 1994; Simoff 1995; Zhang & Luo 1997) have been proposed to overcome this problem. One of them is the interval probability representation (Suppes & Zanotti 1977; Ruspini 1987; Tessem 1992).

For the interval probability representation, many investigators put their attention on it, and its applications are rapidly increasing. But, the axiomatic system proposed for interval probability functions appears to be too general to hold suitable properties. Maybe because of this, it cannot be applied to Bayesian networks well . To overcome this problem, we will propose a new axiomatic framework for interval probability functions in this paper. From our framework, some sound mathematical properties can be derived, and our framework can degenerate into the system of traditional probability. These are advantages over the old axiomatic system for interval probability as well as D-S evidence theory (Shafer 1976).

This paper is organized as follows. To begin with we briefly describe several basic operations on interval numbers. Next we present an axiomatic definition for the interval probability representation. Then from our axiomatic framework, we derive some basic properties of the interval probability representation. Finally, we provide the summary of this paper.

The Operations on Interval Numbers

For reference purpose, we start with a brief description of a few basic operations on interval numbers.

A trapezoid fuzzy number $\tilde{N} = (a, b, \alpha, \beta)$ is defined by:

$$\mu_N(x) = \begin{cases} 0 & \text{if } x < a - \alpha \\ \frac{1}{\alpha}(x - a + \alpha) & \text{if } x \in [a - \alpha, a] \\ 1 & \text{if } x \in [a, b] \\ \frac{1}{\beta}(b + \beta - x) & \text{if } x \in [b, b + \beta] \\ 0 & \text{if } x > b + \beta \end{cases} \tag{1}$$

where $\alpha \geq 0$, $\beta \geq 0$ and $b \geq a$.

Some of operations on trapezoid fuzzy numbers are given (Bonissone & Decker 1986). For a trapezoid fuzzy number $\tilde{N} = (a, b, \alpha, \beta)$, if $\alpha = \beta = 0$, it degenerates into an interval number $[a, b]$. So we can easily derive the corresponding operations on interval numbers from those operations on trapezoid fuzzy numbers as follows. Let $\tilde{m} = [a, b]$ and $\tilde{n} = [c, d]$, then

$$-\tilde{n} = [-d, -c] \tag{2}$$

$$\min\{\tilde{m}, \tilde{n}\} = [\min\{a, c\}, \min\{b, d\}] \tag{3}$$

$$\max\{\tilde{m}, \tilde{n}\} = [\max\{a, c\}, \max\{b, d\}] \tag{4}$$

$$\tilde{m} + \tilde{n} = [a + c, b + d] \tag{5}$$

$$\tilde{m} - \tilde{n} = [a - d, b - c] \tag{6}$$

* This research is supported by the large grant from the Australian Research Council (A49530850).

$$\tilde{m} \times \tilde{n} = \begin{cases} [ac, bd] & \text{if } \tilde{m} > 0, \tilde{n} > 0 \\ [ad, bc] & \text{if } \tilde{m} < 0, \tilde{n} > 0 \\ [bc, ad] & \text{if } \tilde{m} > 0, \tilde{n} < 0 \\ [bd, ac] & \text{if } \tilde{m} < 0, \tilde{n} < 0 \end{cases} \quad (7)$$

$$\tilde{m} \div \tilde{n} = \begin{cases} [\frac{a}{d}, \frac{b}{c}] & \text{if } \tilde{m} > 0, \tilde{n} > 0 \\ [\frac{a}{c}, \frac{b}{d}] & \text{if } \tilde{m} < 0, \tilde{n} > 0 \\ [\frac{b}{d}, \frac{a}{c}] & \text{if } \tilde{m} > 0, \tilde{n} < 0 \\ [\frac{b}{c}, \frac{a}{d}] & \text{if } \tilde{m} < 0, \tilde{n} < 0 \end{cases} \quad (8)$$

Note that in the above last two formulas, the concepts that an interval number is greater or less than 0 are used. In fact, for any interval number $[a, b]$, $[a, b] > 0$ iff $\forall x \in [a, b]$, $x > 0$; $[a, b] < 0$ iff $\forall x \in [a, b]$, $x < 0$.

When two ends of a interval number are equal, it degenerate into a real number. Hence, from formulas (7) and (8), we easily get the following formula:

$$a \times [c, d] = [ac, ad] \quad (9)$$

$$[a, b] \div c = [\frac{a}{c}, \frac{b}{c}] \quad (10)$$

In the next section, we shall propose an axiomatic framework for interval representation of probability. Then when deriving some basic properties for the framework for interval probability, we will mainly exploit formulas (9) and (10). Maybe other operations on interval numbers will be employed in further study on properties of our system for interval probability, and therefore we list all of them here.

The Axiomatic Framework for Interval Probability

In this section, we will give an axiomatic system for interval probability. In the past investigators also gave an axiomatic system for interval probability (Chrisman 1996). But it is more general at additivity than ours. Unfortunately, this seems to lead that some sound mathematical properties are difficult to be derived from this old framework. Another advantage of our system is that it can degenerate to the axiomatic system for classic probability.

Sometimes, it is difficult to supply an exact estimate for the value of a probability P, yet it is relatively easy to provide a pessimistic estimate $\underline{P}$ and an optimistic estimate $\overline{P}$ for this value. In other words, at this moment we offer an interval for the probability, namely, $P = [\underline{P}, \overline{P}]$.

The question is which properties interval-based probability should satisfy. Since we estimate probabilities sometimes by real numbers and sometimes by interval numbers, maybe probabilities in the two forms are operated at the same times. Accordingly, the axiomatic system for interval probability should degenerate into that for real-valued probability. The key to problem is how to express additivity of interval probability, so that when upper boundary and lower boundary of interval probability are equal, the additivity can degenerate into the additivity of point probability. Based on such a consideration, we propose the following axiomatic system for interval probability.

Definition 1 *Let $(\Omega, \mathcal{F})$ be a probability space, and let $\underline{P}$, $\overline{P}$: $\mathcal{F} \longrightarrow [0, 1]$ be set-functions on this space satisfying the following properties for any A, $B \in \mathcal{F}$ with $A \cap B = \emptyset$:*

1. $\underline{P}(\emptyset) = \overline{P}(\emptyset) = 0$
2. $\underline{P}(\Omega) = \overline{P}(\Omega) = 1$
3. $\underline{P}(A \cup B) = \min\{\underline{P}(A) + \overline{P}(B), \underline{P}(B) + \overline{P}(A)\}$
4. $\overline{P}(A \cup B) = \max\{\underline{P}(A) + \overline{P}(B), \underline{P}(B) + \overline{P}(A)\}$

Then $\underline{P}$ and $\overline{P}$ are called lower *and* upper probability functions *respectively. It is always the case that $\underline{P}(A) \leq \overline{P}(A)$.*

In the above definition, properties 3 and 4 specify a sort of additivity for interval probability. In point probability system, the additivity is specified as follows:

$$P(A \cup B) = P(A) + P(B) \quad (11)$$

where A, $B \subset \Omega$ with $A \cap B = \emptyset$. Clearly, when $\overline{P} = \underline{P}$, the additivity in our system for interval probability can degenerate into the one for point probability. And noticing properties 1 and 2 in Definition 1, we can conclude our system for interval probability can degenerate into the system for point probability.

Now let us examine the old axiomatic system for interval probability as follows (Chrisman 1996):

1. $\underline{P}(\emptyset) = \overline{P}(\emptyset) = 0$
2. $\underline{P}(\Omega) = \overline{P}(\Omega) = 1$
3. $\underline{P}(A) + \overline{P}(\overline{A}) = 1$
4. $\underline{P}(A) + \underline{P}(B) \leq \underline{P}(A \cup B)$ (Super-additivity)
5. $\overline{P}(A) + \overline{P}(B) \leq \overline{P}(A \cup B)$ (Sub-additivity)

where A, $B \subset \Omega$ with $A \cap B = \emptyset$.

Clearly, there are the difference in additivity between the two systems. The additivity in our system is expressed by equations, whereas the one in old system is represented by inequalities. For the relationship between them, we have the following obvious theorem:

Theorem 1

1. *If* $\underline{P}(A \cup B) = \min\{\underline{P}(A) + \overline{P}(B), \underline{P}(B) + \overline{P}(A)\}$, *then* $\underline{P}(A) + \underline{P}(B) \leq \underline{P}(A \cup B)$

2. *If* $\overline{P}(A \cup B) = \max\{\underline{P}(A) + \overline{P}(B), \underline{P}(B) + \overline{P}(A)\}$, *then* $\overline{P}(A) + \overline{P}(B) \leq \overline{P}(A \cup B)$

This theorem states that the old system for interval probability is more general at additivity than ours. But it seems not to be a good thing. The definition of additivity is vital for us to derive sound mathematical properties. In fact, if it is too general it cannot reveal some sound mathematical properties. Intuitively, from the additivity expressed by inequalities it is difficult to derived some mathematical properties. Instead, it is easy to derive from our system because ours is not so general. We will see this fact in the following section.

In addition, when $\underline{P} = \overline{P}$, our system degenerates into the classic probability axiom, but this property does not exist in the old system for interval probability. This is because when $\underline{P} = \overline{P}$, both super-additivity and sub-additivity turn to the following inequality:

$$P(A) + P(B) \leq P(A \cup B)$$

yet we cannot derive (11) from the above inequality.

Theorem 2

$$\underline{P}(A) + \overline{P}(\overline{A}) = 1 \quad (12)$$
$$\underline{P}(\overline{A}) + \overline{P}(A) = 1 \quad (13)$$

Proof.

$$\begin{aligned} & 1 \\ = \ & \underline{P}(\Omega) \\ = \ & \underline{P}(A \cup \overline{A}) \\ = \ & \min\{\underline{P}(A) + \overline{P}(\overline{A}), \overline{P}(A) + \underline{P}(\overline{A})\} \end{aligned}$$

Similarly, we have

$$1 = \max\{\underline{P}(A) + \overline{P}(\overline{A}), \overline{P}(A) + \underline{P}(\overline{A})\}$$

Thus, we have

$$1 \leq \underline{P}(A) + \overline{P}(\overline{A}) \leq 1$$
$$1 \leq \overline{P}(A) + \underline{P}(\overline{A})\} \leq 1$$

So, the theorem holds. □

Theorem 2 states that the additivity in our system for interval probability can imply property 3 in the old system. Consequently, Theorems 1 and 2 state that our system for interval probability is not contradictory to old one.

Moreover, Theorem 2 exposes that the relationship between our system and D-S evidence theory. In D-S evidence theory, the uncertainty of a proposition A is assessed by an interval $[Bel(A), Pl(A)]$. The lower boundary, called the *belief*, represents the degree to which the evidence supports this proposition, while the upper boundary, called the *Plausibility*, denotes the degree to which the evidence fails to refute the proposition, namely the degree to which it remains plausible. The relationship between the belief function and the plausibility function is as follows:

$$Pl(A) = 1 - Bel(\overline{A}) \quad (14)$$

that is,

$$Pl(A) + Bel(\overline{A}) = 1 \quad (15)$$

Comparing (15) and (13), we can see that $\underline{P}(\overline{A})$ appears to be equivalent to $Bel(\overline{A})$, and $\overline{P}(A)$ appears to be equivalent to $Pl(A)$. The most important characteristic of D-S evidence theory is that it is able to represent the ignorance because the following inequality:

$$Bel(A) + Bel(\overline{A}) \leq 1 \quad (16)$$

Our system for interval probability can also represent the ignorance because a similar property also holds in our system for interval probability. In fact, we have the following theorem:

Theorem 3

$$\underline{P}(A) + \underline{P}(\overline{A}) \leq 1 \quad (17)$$

Proof.

$$\begin{aligned} & \underline{P}(A) + \underline{P}(\overline{A}) \\ \leq \ & \underline{P}(A \cup \overline{A}) && (by\ Theorem\ 1) \\ = \ & \underline{P}(\Omega) \\ = \ & 1 && (by\ property\ 2\ in\ Definition\ 1) \end{aligned}$$

□

In short, our system for interval probability can do what D-S evidence theory can do. Rather, unlike our system for interval probability but like the old system for interval probability, clearly D-S evidence theory fails to degenerate into the system for point probability, and some sound mathematical properties are difficult to be derived from the axiomatic system for D-S evidence theory, which is as follows.

1. $Bel(\emptyset) = 0$

2. $Bel(\Omega) = 1$

3. $Bel(A_1 \cup \cdots \cup A_n) \geq \sum_{i=1}^{n} Bel(A_i) - \sum_{i,j:1 \leq i < j \leq n} Bel(A_i \cap A_j) + \cdots + (-1)^{n+1} Bel(A_1 \cap \cdots \cap A_n)$

for every positive integer n and every collection $A_1, \cdots, A_n$ of subsets of Ω.

Basic Properties

In this section we will examine some of the basic properties of our interval probability function. We can easily see that these properties are the extension of the corresponding ones in classic probability theory.

Definition 2 *The interval conditional probability $P(A|B)$ is defined as follows:*

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{18}$$

that is,

$$[\underline{P}(A|B), \overline{P}(A|B)] = [\frac{\underline{P}(A \cap B)}{P(B)}, \frac{\overline{P}(A \cap B)}{P(B)}] \tag{19}$$

where $P(B) \in (0, 1]$.

Clearly, the definition is consistent with that in classic probability theory. The following theorem shows that such a definition is reasonable.

Theorem 4 *The interval conditional probability satisfies the axiomatic definition of interval probability.*

Proof. 1) In the case $A = \emptyset$,

$$\begin{aligned}
& P(A|B) \\
= \ & [\underline{P}(A|B), \overline{P}(A|B)] \\
= \ & [\frac{\underline{P}(A \cap B)}{P(B)}, \frac{\overline{P}(A \cap B)}{P(B)}] \\
= \ & [\frac{\underline{P}(\emptyset \cap B)}{P(B)}, \frac{\overline{P}(\emptyset \cap B)}{P(B)}] \\
= \ & [\frac{\underline{P}(\emptyset)}{P(B)}, \frac{\overline{P}(\emptyset)}{P(B)}] \\
= \ & [\frac{0}{P(B)}, \frac{0}{P(B)}] \\
& \qquad (by\ property\ 1\ in\ Definition\ 1) \\
= \ & [0, 0]
\end{aligned}$$

2) In the case $A = \Omega$,

$$\begin{aligned}
& P(A|B) \\
= \ & [\underline{P}(A|B), \overline{P}(A|B)] \\
= \ & [\frac{\underline{P}(A \cap B)}{P(B)}, \frac{\overline{P}(A \cap B)}{P(B)}] \\
= \ & [\frac{\underline{P}(\Omega \cap B)}{P(B)}, \frac{\overline{P}(\Omega \cap B)}{P(B)}] \\
= \ & [\frac{\underline{P}(B)}{P(B)}, \frac{\overline{P}(B)}{P(B)}] \\
= \ & [\frac{P(B)}{P(B)}, \frac{P(B)}{P(B)}] \\
& \qquad (because\ P(B) = \underline{P}(B) = \overline{P}(B)) \\
= \ & [1, 1]
\end{aligned}$$

3) Additivity

$$\begin{aligned}
& P(A_1 \cup A_2|B) \\
= \ & \frac{P((A_1 \cup A_2) \cap B)}{P(B)} \\
= \ & \frac{P((A_1 \cap B) \cup (A_2 \cap B))}{P(B)} \\
= \ & [\frac{\underline{P}((A_1 \cap B) \cup (A_2 \cap B))}{P(B)}, \\
& \frac{\overline{P}((A_1 \cap B) \cup (A_2 \cap B))}{P(B)}] \\
= \ & [\frac{1}{P(B)} \times \min\{\underline{P}((A_1 \cap B) + \overline{P}(A_2 \cap B), \\
& \qquad \overline{P}((A_1 \cap B) + \underline{P}(A_2 \cap B)\}, \\
& \frac{1}{P(B)} \times \max\{\underline{P}((A_1 \cap B) + \overline{P}(A_2 \cap B), \\
& \qquad \overline{P}((A_1 \cap B) + \underline{P}(A_2 \cap B)\}] \\
& \qquad (by\ properties\ 3\ and\ 4\ in\ Definition\ 1) \\
= \ & [\min\{\frac{\underline{P}((A_1 \cap B)}{P(B)} + \frac{\overline{P}(A_2 \cap B)}{P(B)}, \\
& \qquad \frac{\overline{P}((A_1 \cap B)}{P(B)} + \frac{\underline{P}(A_2 \cap B)}{P(B)}\}, \\
& \max\{\frac{\underline{P}((A_1 \cap B)}{P(B)} + \frac{\overline{P}(A_2 \cap B)}{P(B)}, \\
& \qquad \frac{\overline{P}((A_1 \cap B)}{P(B)} + \frac{\underline{P}(A_2 \cap B)}{P(B)}\}] \\
= \ & [\min\{\underline{P}((A_1|B) + \overline{P}(A_2|B), \\
& \qquad \overline{P}((A_1|B) + \underline{P}(A_2|B)\}, \\
& \max\{\underline{P}((A_1|B) + \overline{P}(A_2|B), \\
& \qquad \overline{P}((A_1|B) + \underline{P}(A_2|B)\}] \\
& \qquad (by\ Definition\ 2)
\end{aligned}$$

□

The following theorem is Bayes' rule in the form of our interval probability. Clearly, it is the extension of the corresponding one in classic probability theory. Moreover, from the old axiomatic system for interval probability we cannot derive this theorem probably because the old one is too general.

Theorem 5

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \tag{20}$$

where $P(A), P(B) \in (0,1]$.

Proof.

$$\begin{aligned}
& P(B)P(A|B) \\
= \; & P(B) \times [\frac{\underline{P}(A\cap B)}{P(B)}, \frac{\overline{P}(A\cap B)}{P(B)}] \\
= \; & [P(B) \times \frac{\underline{P}(A\cap B)}{P(B)}, P(B) \times \frac{\overline{P}(A\cap B)}{P(B)}] \\
& \qquad (by \;\; (9)) \\
= \; & [\underline{P}(A\cap B), \overline{P}(A\cap B)] \\
= \; & P(A\cap B)
\end{aligned}$$

Similarly, we have

$$P(A)P(B|A) = P(B\cap A)$$

Noting that $P(B\cap A) = P(A\cap B)$, we have

$$P(A)P(B|A) = P(B)P(A|B)$$

That is,

$$\begin{aligned}
& [P(A)\underline{P}(B|A), P(A)\overline{P}(B|A)] \\
= \; & [P(B)\underline{P}(A|B), P(B)\overline{P}(A|B)]
\end{aligned}$$

Thus,

$$\underline{P}(B|A) = \frac{P(B)\underline{P}(A|B)}{P(A)}$$
$$\overline{P}(B|A) = \frac{P(B)\overline{P}(A|B)}{P(A)}$$

Therefore,

$$\begin{aligned}
& P(B|A) \\
= \; & [\underline{P}(B|A), \overline{P}(B|A)] \\
= \; & [\frac{P(B)\underline{P}(A|B)}{P(A)}, \frac{P(B)\overline{P}(A|B)}{P(A)}] \\
= \; & P(B) \times [\frac{\underline{P}(A|B)}{P(A)}, \frac{\overline{P}(A|B)}{P(A)}] \quad (by \;\; (9)) \\
= \; & \frac{P(B) \times [\underline{P}(A|B), \overline{P}(A|B)]}{P(A)} \quad (by \;\; (10)) \\
= \; & \frac{P(B)P(A|B)}{P(A)}
\end{aligned}$$

□

The following theorem is the version of the total probability formula in interval based probability. Clearly, it can degenerate to the corresponding one in classical probability theory.

Theorem 6

$$\begin{aligned}
\underline{P}(H) = \min\{& \underline{P}(H|E)P(E) + \overline{P}(H|\overline{E})P(\overline{E}), \\
& \overline{P}(H|E)P(E) + \underline{P}(H|\overline{E})P(\overline{E})\} \quad (21) \\
\overline{P}(H) = \max\{& \underline{P}(H|E)P(E) + \overline{P}(H|\overline{E})P(\overline{E}), \\
& \overline{P}(H|E)P(E) + \underline{P}(H|\overline{E})P(\overline{E})\} \quad (22)
\end{aligned}$$

where $P(E) \in [0,1]$.

Proof. We check only (21). (22) can be similarly proved.

$$\begin{aligned}
& \underline{P}(H) \\
= \; & \underline{P}(H\cap\Omega) \\
= \; & \underline{P}(H\cap(E\cup\overline{E})) \\
= \; & \underline{P}((H\cap E)\cup(H\cap\overline{E})) \\
= \; & \min\{\underline{P}(H\cap E) + \overline{P}(H\cap\overline{E}), \\
& \qquad \overline{P}(H\cap E) + \underline{P}(H\cap\overline{E})\} \\
& \qquad (by \; property \; 3 \; in \; Definition \; 1) \\
= \; & \min\{\frac{\underline{P}(H\cap E)}{P(E)} \times P(E) + \frac{\overline{P}(H\cap\overline{E})}{P(\overline{E})} \times P(\overline{E}), \\
& \qquad \frac{\overline{P}(H\cap E)}{P(E)} \times P(E) + \frac{\underline{P}(H\cap\overline{E})}{P(\overline{E})} \times P(\overline{E})\} \\
= \; & \min\{\underline{P}(H|E)P(E) + \overline{P}(H|\overline{E})P(\overline{E}), \\
& \qquad \overline{P}(H|E)P(E) + \underline{P}(H|\overline{E})P(\overline{E})\} \\
& \qquad (by \; Definition \; 2)
\end{aligned}$$

□

Conclusions

The probability in the form of interval is very important. In fact, numerous researchers put their attention on it. However, the axiomatic system for interval probability proposed in the past is not good enough. It is so general that some sound mathematical properties cannot be derived from the axiomatic system. (D-S evidence theory seems to suffer from the similar shortcoming.) Perhaps because of the above drawback it is difficult that the interval probability representation is used to capture epistemological independence for Bayesian network.

In this paper, we propose an axiomatic system for the interval probability representation. It is not contradictory to the old axiomatic system. Furthermore, it is not too general like the old one. Some sound mathematical properties can be derived from the axiomatic system. In addition, it can degenerate into the axiomatic system for classical probability.

We believe that the work in this paper is an important step to apply the interval probability representation to Bayesian networks.

Acknowledgement

The authors would like to thank the anonymous referees for their comments.

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