# Intelligent Dynamic Simulation of Mechanisms

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## Abstract

Dynamic simulation involves solving sets of equations that represent kinematic (or geometric) and kinetic (or force related) constraints. The kinematic equations are highly nonlinear algebraic equations, and the kinetic equations are coupled differential equations that must be integrated over time. In most simulators, the algebraic and differential equations are solved simultaneously in an iterative manner using sparse matrix techiques and stiff integration schemes. This is not particularly efficient, and can also lead to numerical stability problems. I propose an alternative formulation, based directly on the work of Euler and Lagrange, which allows more efficient solution of dynamics problems at interactive rates. This paper describes the use of a geometric constraint engine based on degrees of freedom analysis (developed for the kinematic simulator TLA), with extensions to incorporate reasoning about dynamics. I present a method for calculating velocity ratios, formulated as a problem in geometry. These calculations, in conjunction with the kinematic solution, allow determination of all kinetic and potential energy terms required for dynamic simulation. The constructed system of dynamics equations are pure differential equations in terms of a minimal set of generalized coordinates. The formulation results in small, dense matrices, rather than large, sparse ones. In principle, large time-steps may be used for coarse dynamic behavior, with smaller time-steps yielding better approximations to the true behavior, at the cost of more computation time.

## Introduction

Understanding complex mechanical devices requires the ability to simulate the behaviors of the devices, and to make reasonable generalizations about those behaviors. Mechanism simulation may be done at a purely qualitative level [Kim, 1990], a mixed numerical/qualitative level [Joskowicz and Sacks, 1991], or at a detailed numerical level followed by qualitative characterization of the results [Gelsey, 1990]. Each of these approaches has advantages and disadvantages with regard to predictive power, generation of incomplete or impossible behaviors, and computational cost. Detailed numerical simulation provides the most accurate level of detail, but at the highest cost. Since multiple simulations are needed to extract data about trends (*e.g.*, velocity *vs.* change in a parameter value), precise numerical simulation is often unattractive, particularly at the earliest stages of design.

In this paper I propose an intelligent computational methodology for performing accurate dynamic analysis of rigid-body mechanisms in an efficient, interactive manner. This methodology will allow fast evaluation of design alternatives, and provide information for sensitivity analysis and force analysis. The approach separates the problem into kinematics and kinetics, as first proposed by Euler [Hartenberg and Denavit, 1964]. The kinematic analysis is efficiently computed using degrees of freedom analysis [Kramer, 1992]. Velocity analysis, required for kinetics, is performed using screw theory [Ball, 1900], formulated as another geometric constraint problem. This analysis allows easy derivation of the Lagrangian form of the dynamics equations in terms of a minimal set of generalized coordinates, thereby resulting in efficient and stable computation methods. In addition, by adjusting the time-step, coarse grained dynamic analysis is possible in an even more efficient manner.

## Related work

### Qualitative kinematics and dynamics

Simple processes and mechanical systems have been described in a number of qualitative reasoning systems [Weld and de Kleer, 1990]. Most systems handle only the simplest cases of nonlinearity, due to the coarse structure of the qualitative representations. More specialized qualitative reasoning, like trigonometric reasoning, can give more precise results, such as whether or not a particular link in a mechanism can rotate completely with respect to another [Kim, 1990]. It cannot be used to describe specific attributes of the space curves traced by arbitrary points on the mechanism,

yet this information is essential to designers.

As the description of the model is more finely discretized, more detail can be obtained from the simulation. Configuration spaces have been used to model a variety of mechanisms [Joskowicz and Sacks, 1991]. The representation becomes computationally intractable for more complicated non-fixed-axis mechanisms. However, within its domain of applicability, the configuration space approach is able to deal with topology changes during the operation of the mechanism. Simple approximations to dynamic behavior, incorporating models of steady-state forces, allow a number of mechanisms to be simulated at a coarse dynamic level [Sacks and Joskowicz, 1992].

Detailed numerical simulation, followed by abstracting the results into qualitatively interesting regions, is the most accurate and general approach to describing dynamics in qualitative terms [Gelsey, 1990]. However, the process can be time-consuming due to long runtimes for general-purpose simulators. Since many simulation runs may be needed to extract data about behavioral trends, this approach can be computationally infeasible.

### Matrix-based dynamics methods

In the matrix-based approach to dynamics, the kinematic equations and the differential equations are solved together. One approach is to use a maximally redundant set of generalized coordinates (six positional and six velocity per rigid body), and solve using sparse matrix techniques [Orlandea *et al.*, 1977]. The resulting equations are stiff, and therefore require small integration time-steps. Other approaches attempt to reduce the number of generalized coordinates in order to improve computational efficiency, and to help reduce the stiffness of the matrix [Haug, 1985].

General-purpose simulators usually deal with fixed topology mechanisms. Cremer describes a simulator with built-in capabilities for detecting collisions and other topological changes. His simulator reformulates the equations when changes occur, and then continues dynamic simulation [Cremer, 1989].

### Symbolic derivation of dynamics equations

An alternative to detailed numerical solution of the mixed algebraic/differential equations is the symbolic generation of a set of pure differential equations, which describe the system in terms of a minimal set of generalized coordinates corresponding to the system's true degrees of freedom. The Dyne system [Brown and Leifer, 1991] uses symbolic reasoning to derive such equations, guided by a set of algebraic transformation rules and meta-level control rules. Such systems are hard to design, maintain, and debug. However, the equations derived using Dyne are useful to designers performing sensitivity analysis at selected points in the mechanism's behavior, and for describing qualitative regions of behavior.

Symbolic geometric solution of kinematics problems using degrees of freedom analysis is described in [Kramer, 1992], with extensions to other kinds of geometry presented in [Kramer, to appear]. The next section extends this technique to generate dynamics equations more efficiently than rule-based systems which directly manipulate algebraic and differential equations.

## A direct method for dynamics

Direct methods for dynamics can be traced to Euler, who advocated treating dynamics by partitioning the problem into two parts: kinematics and kinetics [Hartenberg and Denavit, 1964]. Kinematics deals with the positions of the parts of the mechanism as constrained by geometric relationships. Relative (but not absolute) velocities and accelerations often can be calculated kinematically. Kinetics deals with how physical objects move under the effect of forces, and deals with absolute velocity, acceleration, mass, inertia, *etc.*

Euler also demonstrated how an object in three-dimensional space can be moved from one position to an arbitrary second position by a combination of a single translation and a single rotation, where the rotation axis is parallel to the translation vector, resulting in a screw-like motion. At any instant in time, a body in motion may be thought of as moving about an *instantaneous screw* in space; this screw's position, orientation, and pitch changes over time. The theory of screws was treated in depth in a geometric manner in [Ball, 1900].

With the instantaneous screws known for each body in a rigid-body mechanism, all relative velocities in the system are related by ratios of distances from the appropriate screws. Specifying one absolute velocity then allows finding all absolute velocities. The velocities provide the kinetic coenergy terms in the lagrangian formulation of dynamics, and the kinematic information provides the potential energy terms [Crandall *et al.*, 1982].

All terms required in the lagrangian are attainable from simple geometric constructions. This leads to the following algorithm for constructing the lagrangian and using it in dynamic analysis:

1. Calculate the kinematic information in terms of the generalized position coordinates.
2. Find the instantaneous screw axes by geometric construction.
3. Calculate angular and linear velocities as ratios of distances betwen points on the mechanism and the screw axes.
4. Calculate the time derivatives of the screw axes.
5. Use the above information to construct the lagrangian directly.
6. Solve the lagrangian for accelerations, and integrate over time.

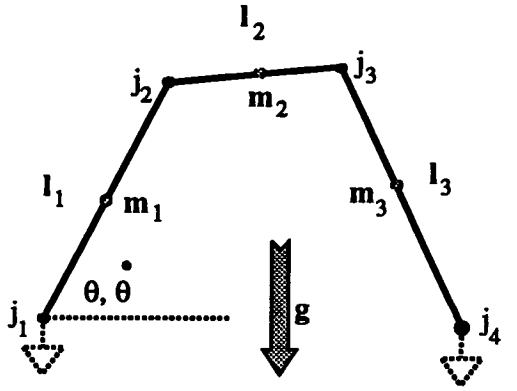


Figure 1: A four-bar linkage.

The remainder of this paper describes the solution of a simple dynamics problem using this algorithm. At present, the mathematics have been solved, but the solution process has not been automated.

## A four-bar linkage

Figure 1 describes a four-bar planar linkage. One of the links is grounded, or fixed to the global reference frame, and is not shown. The moving links have lengths $l_1, l_2$, and $l_3$, and are modeled with masses $m_1, m_2$, and $m_3$ and rotational inertias $I_1, I_2$, and $I_3$. The revolute joints $j_1$ and $j_4$ connect the cranks to the ground; the remaining two joints connect the cranks to the coupler. The linkage has one degree of freedom; the position of all links are determined fully by the crank angle $\theta$, and the velocities by the first derivative of $\theta$ with respect to time ($\dot{\theta}$).

The example problem is an initial values problem. Given initial values for $\theta$ and $\dot{\theta}$, find the behavior of the linkage over time. In this problem, gravity exerts a downward force, as illustrated, and the joints are assumed to be frictionless.

## Computing angular and linear velocities

The first portions of the algorithm require calculating the kinematic information as a function of the generalized position coordinates, finding the instantaneous screw axes by geometric construction, and calculating angular and linear velocities as ratios of distances of mass centers from the screw axes. The first item is covered by kinematic analysis [Kramer, 1992]; the next two are covered here.

In planar mechanisms, the axes of all instantaneous screws are normal to the plane, and the pitch is always zero. In this specialized case, the instantaneous screw is known as the *instantaneous center* of motion [Hall, 1961]. Figure 2 illustrates how the instantaneous centers and velocities are calculated. The instantaneous centers of the cranks are the fixed joints ($j_1$ for link 1, and $j_4$ for link 3); these centers do not change over time.

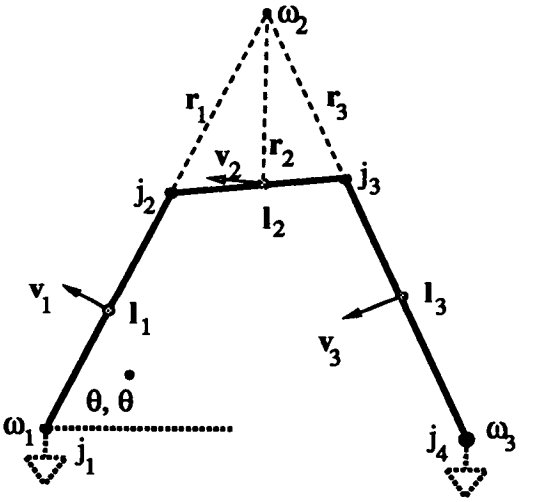


Figure 2: Kinetic analysis of the linkage.

The instantaneous center for link 2 is computed geometrically as follows. Since joint $j_2$ is on the link, it must rotate about the instantaneous center for link 2. However, since $j_2$ is also on link 1, it must rotate about $j_1$, and hence, its linear velocity must be perpendicular to the line describing link 1. Therefore, the instantaneous center must lie on a line through $j_1$ and $j_2$. Using a similar argument for the motion of joint $j_3$, the instantaneous center is found by intersecting the two lines. Link 2 rotates about the intersection point with angular velocity $\omega_2$, which is yet to be determined.

The angular velocity of link 1, $\omega_1$, is specified to be $\dot{\theta}$. The angular velocity $\omega_2$ is found by equating the linear velocities at joint $j_2$: $\omega_1 l_1 = -\omega_2 r_1$. In similar fashion, the remaining angular velocities are calculated:

$$\omega_1 = \dot{\theta} \tag{1}$$

$$\omega_2 = -\omega_1(l_1/r_1) = -\dot{\theta}_1(l_1/r_1) \tag{2}$$

$$\omega_3 = -\omega_2(r_3/l_3) = \dot{\theta}_1(l_1 r_3/l_3 r_1) \tag{3}$$

The directions of the linear velocities are derived from the angular velocities. The magnitudes of the linear velocities (assuming each center of mass is at the center of the link) follow directly from the angular velocities:

$$|\mathbf{v}_1| = \omega_1(l_1/2) = \dot{\theta}_1(l_1/2) \tag{4}$$

$$|\mathbf{v}_2| = \omega_2 r_2 = \dot{\theta}_1(l_1 r_2/r_1) \tag{5}$$

$$|\mathbf{v}_3| = \omega_3(l_3/2) = \dot{\theta}_1(l_1 r_3/2r_1) \tag{6}$$

Thus, all angular and linear velocities are calculated as simple ratios of distances between instantaneous centers and points on the mechanism. For a more compact notation, the ratios are written as $\omega_i = \dot{\theta} k_i$, and $|\mathbf{v}_i| = \dot{\theta} j_i$.

## Generating the lagrangian

The lagrangian of the system is then described as follows (assuming a positive $y$ axis moving upward in Figure 2, and with $v_i = |\mathbf{v}_i|$):

$$\mathcal{L} = \frac{1}{2}\sum_i (m_i v_i^2 + I_i \omega_i^2) - g\sum_i m_i y_i \qquad (7)$$

Here, $I_i$ is the moment of inertia for mass link $i$, and $g$ is the acceleration of gravity. The differential equation describing the motion of the linkage is then [Crandall *et al.*, 1982]:

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right] - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \qquad (8)$$

The second term in Equation 8 is the potential energy term, and is calculated as follows:

$$\frac{\partial \mathcal{L}}{\partial \theta} = -g\frac{\partial}{\partial \theta}\sum_i m_i y_i = -g\sum_i m_i \frac{\partial y_i}{\partial \theta} \qquad (9)$$

The change in height ($y$) for each point mass is:

$$\frac{\partial y_i}{\partial \theta} = \frac{\partial \mathbf{p}_i}{\partial \theta} \cdot \hat{\mathbf{y}} \qquad (10)$$

where $\partial \mathbf{p}_i / \partial \theta$ is the change in position of point mass $m_i$ with respect to a $\theta$. This vector is always in the direction of $\mathbf{v}_i$. The value of $\partial \mathbf{p}_1 / \partial \theta$ is calculated from $\theta$ as follows:

$$\frac{\partial \mathbf{p}_1}{\partial \theta} = -\hat{\mathbf{x}}\left(\frac{l_1}{2}\right)\sin\theta + \hat{\mathbf{y}}\left(\frac{l_1}{2}\right)\cos\theta \qquad (11)$$

The remaining position change magnitudes are related to each other directly as the magnitude of the linear velocities, which have already been computed geometrically:

$$\frac{|\partial \mathbf{p}_i|}{|\partial \mathbf{p}_j|} = \frac{|\mathbf{v}_i|}{|\mathbf{v}_j|} \qquad (12)$$

The first term in Equation 8 is the kinetic coenergy term, and is calculated as follows. The derivative with respect to $\dot{\theta}$ is:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= \sum_i (m_i v_i \frac{\partial v_i}{\partial \dot{\theta}} + I_i \omega_i \frac{\partial \omega_i}{\partial \dot{\theta}}) \\ &= \dot{\theta}\sum_i (m_i j_i^2 + I_i k_i^2) \end{aligned} \qquad (13)$$

The time derivative of this quantity is:

$$\begin{aligned}\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right] &= \ddot{\theta}\sum_i (m_i j_i^2 + I_i k_i^2) \\ &\quad + 2\dot{\theta}\sum_i (m_i j_i \frac{dj_i}{dt} + I_i k_i \frac{dk_i}{dt}) \end{aligned} \qquad (14)$$

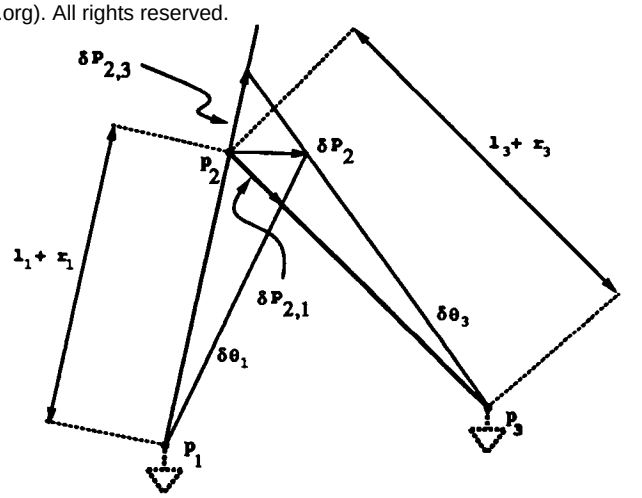


Figure 3: Calculating the derivative of an instantaneous center.

The first term involves the lengths of geometric entities already constructed. Evaluating the second term involves understanding how the instantaneous centers move over time. In this example, the only instantaneous center that moves over time is the center for link 2.

**Time derivative of the instantaneous center** Using the chain rule, finding the time derivatives of the $j_i$'s (as well as the $k_i$'s) is reduced to finding the derivatives of these quantities with respect to $\theta$:

$$\frac{dj_i}{dt} = \frac{dj_i}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{dj_i}{d\theta} \qquad (15)$$

The quantity $dj_i/d\theta$ may be found through geometric sensitivity analysis, involving simple geometric constructions.[1] Figure 3 illustrates the calculation.

Consider three points $\mathbf{p}_1, \mathbf{p}_2$, and $\mathbf{p}_3$, where points $\mathbf{p}_1$ and $\mathbf{p}_3$ are fixed in space. If the line through $\overline{\mathbf{p}_1\mathbf{p}_2}$ is rotated through a small displacement $\delta\theta_1$, and $\mathbf{p}_2$ is constrained to lie on $\overline{\mathbf{p}_3\mathbf{p}_2}$, it will move in direction $\delta\mathbf{p}_{2,1}$. If, on the other hand, line $\overline{\mathbf{p}_3\mathbf{p}_2}$ rotated by a small displacement $\delta\theta_3$, $\mathbf{p}_2$ will move in direction $\delta\mathbf{p}_{2,3}$. For a composite change of both $\delta\theta_1$ and $\delta\theta_3$, the movement of point $\mathbf{p}_2$ will be the vector sum of the two independent movements (for infinitesimal displacements, the quadrangle on which the three vectors lie becomes a parallelogram).

In the case where $\mathbf{p}_1$ and $\mathbf{p}_3$ are the grounded joints of the four-bar linkage, and point $\mathbf{p}_2$ is the instantaneous center of link 2, the length $\overline{\mathbf{p}_1\mathbf{p}_2}$ is $l_1 + r_1$, and $\overline{\mathbf{p}_3\mathbf{p}_2}$ is $l_3 + r_3$. Since $l_1$ and $l_3$ are parameters of the linkage, and constant, the value of $\delta\mathbf{p}_2$ yields $\delta r_1$ and $\delta r_3$ trivially. The only remaining detail is to determine $\delta\theta_3$ in terms of $\delta\theta_1$, so only one independent quantity

[1] Work on geometric sensitivities is being pursued by Jahir Pabon at Schlumberger. This section employs his techniques.

is used when taking the derivative. The ratio of the angular velocities provides this information:

$$\frac{\delta\theta_1}{\delta\theta_3} = \frac{\omega_1}{\omega_3} \tag{16}$$

### Integration of the differential equations

Integration schemes have yet to be explored, but since the system of equations is purely differential, with no algebraic equations, stiffness should not be an overriding concern. Traditional integration schemes, such as adaptive step size Runge-Kutta and predictor-corrector methods, will be explored.

If integration uses large time-steps, the effect should be to have "approximate" dynamics. There is no danger of the mechanism "flying apart," since the kinematic constraints are not considered in the time integration. Thus, there is likely to be a convenient trade-off between computation time, accuracy, and interactiveness.

## Theoretical analysis

Generating the plan of geometric constructions to find velocities need only be done once for a given mechanism. After that, the plan may be reused during each step of the dynamic simulation. Since there are as many instantaneous screws as there are bodies in the mechanism, evaluation of the kinetic coenergy terms of the lagrangian takes time linearly proportional to the size of the mechanism. Kinematic analysis, necessary for the potential energy terms, is $O(n \log n)$, but typically linear in $n$, where $n$ is the number of bodies in the mechanism [Kramer, 1992].

Solving the lagrangian for the accelerations requires inverting a matrix of size $d$, the number of true degrees of freedom in the mechanism. This contrasts with the standard matrix-based approaches, where the matrix to be inverted is of size proportional to the number of bodies in the mechanism. In the worst case of a mechanism comprised exclusively of open chains, the number of true degrees of freedom will be proportional to the number of bodies; however, the absolute number of generalized coordinates being considered will still be less using the geometric algorithm, since the kinematic constraints are already eliminated.

## Discussion

This work is in its early stages, and much of the detail remains to be worked out. However, the geometric constructions required for the velocity analysis can all be performed in a highly stylized fashion. If the gains in computational efficiency are comparable with the gains in kinematic simulation in [Kramer, 1992], the speedup in dynamic simulation could be substantial, affording interactive simulation speeds for many complex mechanisms.

At present, only mechanisms with fixed topology are being explored. Techniques found in [Cremer, 1989] and [Sacks and Joskowicz, 1992] will be explored as possible means of dealing with changeable topology.

Besides their use in formulating the dynamics equations, the geometric sensitivities can also be used for force analysis at selected points in the mechanism's trajectory, and for kinematic analysis of velocity ratios. These are quantities that could be optimized in a design at interactive rates.

If the dynamic behaviors can be simulated efficiently enough, it may be possible to make multi-dimensional "maps" of simulated behavior as a function of design parameter values. Such behavioral maps could be of substantial benefit in the design and debugging of complex mechanical devices.

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