Using Diagrammatic Representations in Mathematical Modeling: The Sketches of Expert Modelers

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Abstract

Modelers create descriptions of a problem in part to assess their understanding of the problem situation. We postulate that these descriptions, i.e., models, are often constructed in our minds, i.e., mental models, and that their complexity forces the modeler to use external memory aids such as pencil and paper. The purpose of this paper is to report on an exploratory study of the use of sketches, i.e., diagrammatic representations, by modelers expert in the construction of mathematical models in the context of management science. In addition, we suggest a theoretical framework for the role of diagrammatic representations in the process of constructing mathematical models.

Introduction

"Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality." So began Minkowski's famous lecture "Space and Time" (1923, p. 75). The union he referred to with this dramatic prose was his space-time diagram, introduced during this lecture. Such diagrammatic representations have long been used to assist problem solving in physics (Gardin & Iwaski, 1995; Nersessian, 1992; Nerssesian & Greeno, 1990), biology (Kindfield, 1992), medicine (Egar, Puerta, & Musen, 1992; Rogers, 1995), and mathematics (Barker-Plummer & Bailin, 1992; McDougal, 1995).

Constructing mathematical models is a process of abstracting reality. Throughout this process, problems must be solved and decisions made. Modelers create descriptions of a problem in part to assess their understanding of the problem situation. We postulate that these descriptions, i.e., models, are often constructed in our minds, i.e., mental models. In addition, the complexity of these representations usually forces the modeler to use external memory aids such as pencil and paper (see Reisberg & Logie, 1993 for a discussion of the memory limitations associated with mental imagery).

The purpose of this paper is to report on an exploratory study of the use of sketches, i.e., diagrammatic representations, by modelers expert in the construction of mathematical models in the context of management science. In addition, we will suggest a theoretical framework for the role of diagrammatic representations in the problem solving process involved in constructing mathematical models.

The process of developing a mathematical model involves problem solving. A mathematical model is a systems of relations that represents another system of relations (Allison, Charnes, Cooper, & Sueyoshi, 1994); a problem is a perceived gap between present and desired states (Smith, 1989), and problem solving is the process of closing that gap. In order for that gap to be a problem, it must be difficult to close (Smith, 1988). The process of constructing a mathematical model is a problem in that the modeler is trying to close the gap between the reality that she is trying to represent and the mathematical expressions that she is using to develop the system of relations to describe reality.

Note that a mental model has been described as an internal mental replica that has the same "relation-structure" as the phenomena it represents (Johnson-Laird, 1983). This led us to investigate the mental model construct as an explanation of the expert modelers' use of sketches in the process of model building.

Bell (1996) surveyed the use of visualization in management science models. Jones (1994) described how the type of visualization usually changes as the modeling process proceeds from the early stage of problem conceptualization to the final stage of results presentation. He noted that combinations of natural language and informal diagrams are often used early in the modeling process; it is these representations that we study in this paper. A survey of expert modelers (Willemain, 1994) showed a strong consensus about the value of sketches in constructing mathematical models in the management and decision sciences.

Sketching by Expert Modelers

Background

Willemain (1994; 1995) used think-aloud protocols to try to better understand the process of problem solving that goes into the activity of mathematical modeling. Twelve expert modelers worked four model formulation exercises while their verbal descriptions of their problem solving processes were recorded and later transcribed. Four core modelers worked all four exercises, and the remaining eight modelers each worked one of the four exercises. The exercises concerned solving abstract problems related to graduate admissions, alumni donations, television commercials, and global ecology. The modelers had pencil and paper while working the exercises, and most made notes, graphs, sketches, outlines, and diagrams. We refer to this graphical output as "sketches," even though some of it is sentential, that is, in words. We use the word "sketch" in the sense of the dictionary definition of "a rough drawing representing the chief features of an object or scene, often made as a preliminary study" (Merriam-Webster, 1980, p. 1079). We use the terms "diagrammatic representation" and "sentential representation" in the fashion of Larkin and Simon (1987). We define a diagrammatic representation as a visualization whose meaning is determined by the location of marks on a two-dimensional plane. A sentential visualization is one that could be scanned as ASCII text without significant loss of meaning. We define "visualization" as an image in the external world that enhances insight or serves as a memory aid.

The Modeling Exercises

The "Admissions" exercise concerned the development of a graduate admissions process based on background data such as grade point average, Graduate Record Exam scores, class rank, college major, and the like. The modelers were asked to develop 1) an automated admissions system and 2) a model to evaluate both the traditional and automated admissions processes. The "Alumni" exercise addressed the forecasting of annual donations from alumni based on data that included number of alumni in class, number of givers, average donation, and so on. The modelers were asked to develop a model to estimate how much money the alumni would donate in each of the next five years. The "Commercials" exercise involved allocating budget dollars between the production of draft commercials and the airing of the final commercial. The modelers were asked to develop a model to determine the optimal number of draft commercials to produce. The "Trees" exercise entailed the implementation of environmental and economic development policy, whose stated objective was to "first attain and then maintain an 'ecologically prudent' number of trees while avoiding drastic changes in current production and consumption levels" (Willemain, 1995, p. 919). The modelers were asked to develop a model to address the issue of how many trees should be planted and harvested each year throughout the world.

The design of these problems was intended to run the gamut for complexity of the system to be modeled (the problem's context) and for clarity of goals (the problem's purpose). The Commercials exercise has a simple system and clear goals and in some respects may be considered the "easiest" problem. The Trees exercise has a complex system and vague goals, and perhaps may be considered the "hardest" problem. The Admissions exercise has a simple system and vague goals, while the Alumni exercise has a complex system and clear goals.

Results

The sketches drawn by the modelers were coded by location and type of representation. Location was determined by line numbers within the protocol transcript. The types were diagrammatic and sentential. Sketches of mixed type were coded according to which aspect of the sketch better conveyed the core of its meaning. For convenience, each sketch also was given a brief descriptive name, such as "AHP" or "Cost vs. value" or "Regionalize".

Coding the sketch locations involved several steps: dividing a page up into discrete sketches, finding the general location of each sketch, and determining the starting and ending points in the text. Sketch text (the text associated with a sketch) is not necessarily a continuous chunk of text. Sometimes a modeler would interrupt the drawing and discussion of a particular sketch with another idea. Finding the general location of each sketch in the text turned out to be a relatively easy task, since the modelers tended to talk about their sketches with enough specificity that there usually was little to no question about where in the text of the verbal protocol a given sketch was drawn.

Each sketch was coded as diagrammatic or sentential. A sketch was coded sentential if its meaning was conveyed primarily by words or equations and thus could be translated to ASCII code without significant loss of meaning. A sketch was coded diagrammatic if its meaning was conveyed primarily by a drawing. The following examples of sketches that contain both diagrammatic and sentential elements illustrate the criteria for classification.

"First shot," seen in Figure 1, contains a number of diagrammatic elements, such as the arrows and the small chart in the upper right; nonetheless, it is an example of a sentential sketch. The modeler's explanation was that the admissions process should begin with a "first shot" that

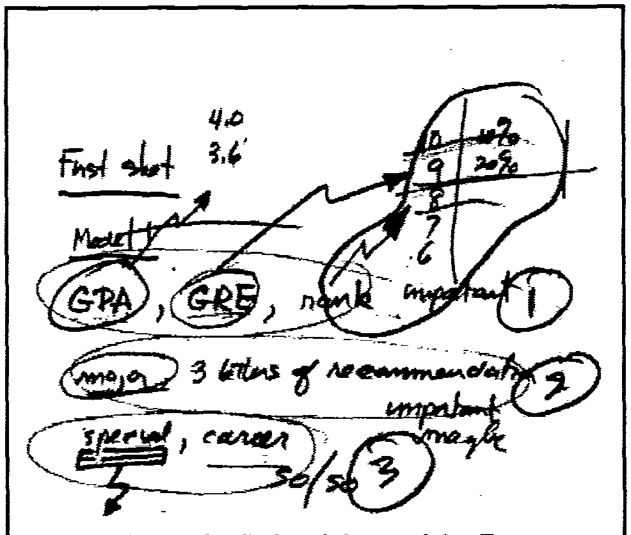

Figure 1. "First shot" sketch by modeler E on Admissions.

takes a quick look at the elements listed in the sketch in the order specified. The essence of the sketch's meaning, then, is given by the words in the sketch. The diagrammatic elements are secondary in importance; therefore, it was classified as sentential.

"Rank B," on the other hand (see Figure 2) is a diagrammatic sketch with a number of sentential elements. It illustrates the idea of ranking the proposals for commercials according to a subjective measure of quality and then allocating portions of the budget to produce each of the commercials. Although the meaning of this sketch could be expressed in ASCII, the diagrammatic part of the sketch conveys the primacy of the ranking idea, which is central to this sketch.

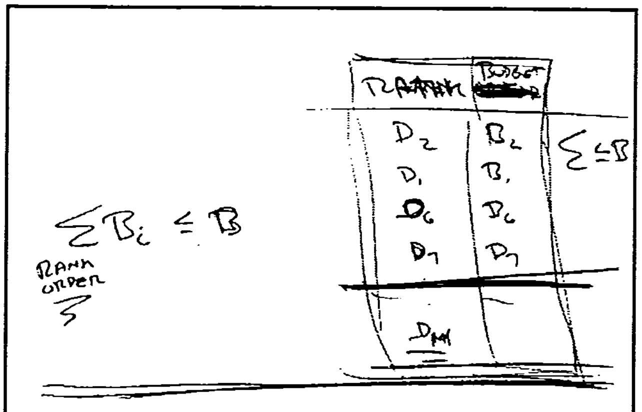

Figure 2. "Rank B" sketch by modeler B on Commercials.

Figure 3 summarizes mean counts of diagrammatic and sentential data for all modelers. When the verbal protocols were transcribed, the lines of text were numbered, starting with "1" for the beginning of each protocol. "Number of Lines of Total Text" is the mean total number of lines in the protocol for a given modeler working a given exercise. "Number of Lines Text for All Sketches" is the mean number of lines of text that were associated with sketches, both diagrammatic and sentential. "Number of Lines Diagrammatic Text" is the mean number of lines of text that were associated with diagrammatic sketches, and "Number of Lines Sentential Text" is the mean number of lines of text that were associated with sentential sketches. "Diagrammatic Proportion" is equal to "Number of Lines Diagrammatic Text" divided by "Number of Lines Text for All Sketches." "Sketch Proportion" is equal to "Number of Lines Text for All Sketches" divided by "Number of Lines of Total Text." Figure 4 shows that Commercials (the easiest problem) had the lowest proportion of sketch text, .34, and Trees (the hardest problem) had the highest, .45. This may be related to the difficulty of the problem. Easier problems may require less visualization than harder problems. In general, as we see from Figure 4, a grand total of about 39% of the protocol lines were associated with sketches. Of these lines, about 52% involved diagrammatic

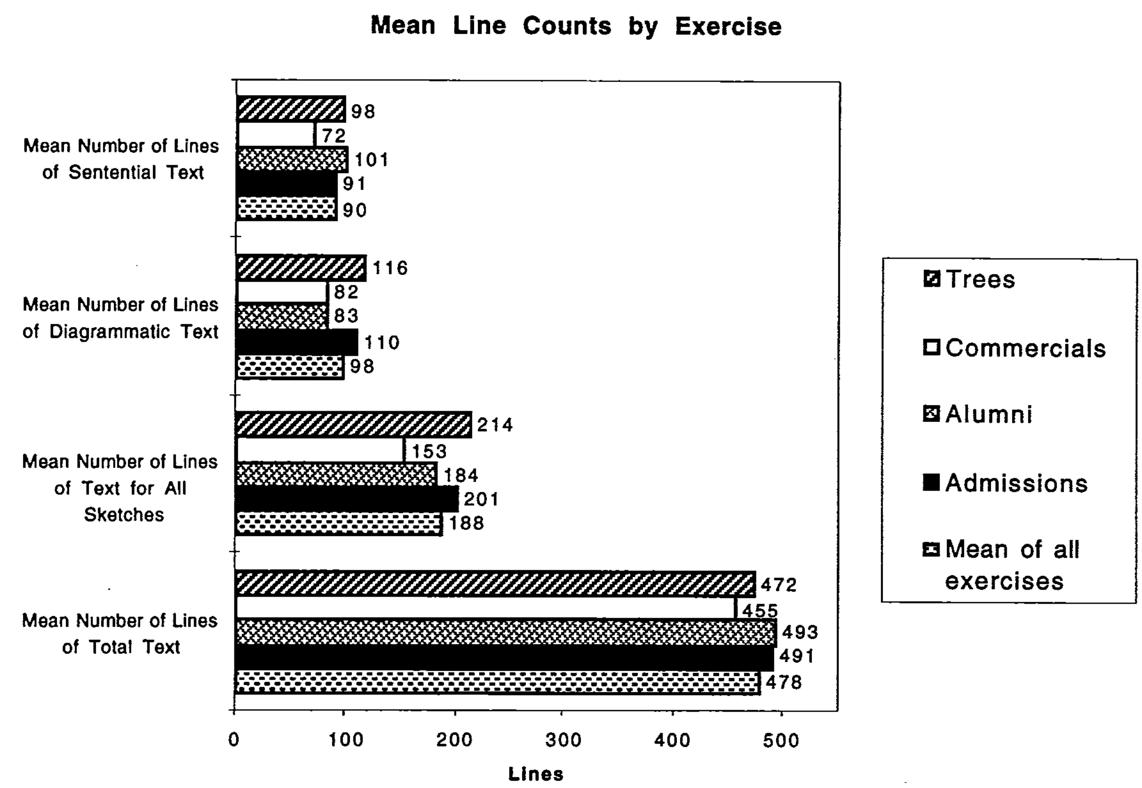

Figure 3. Summary of diagrammatic and sentential data, including means for all exercises.

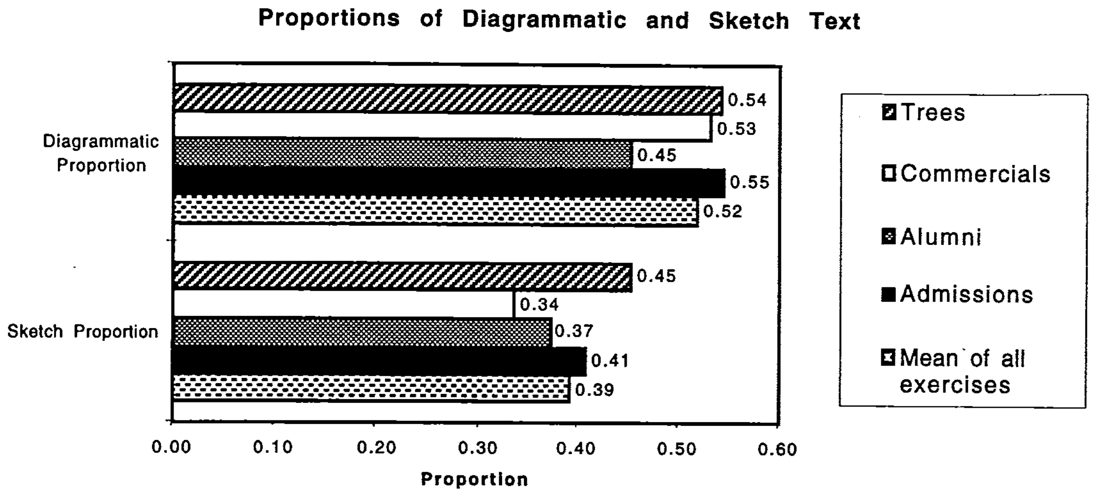

Figure 4. Summary of diagrammatic and sketch text proportions.

sketches.

A Theoretical Framework for Diagrammatic Representations in Mathematical Modeling

Visual Imagery in Modeling

We wish to propose a theoretical framework about the interaction between the modelers and their sketches during the problem solving process. We start with Johnson-Laird's mental models theory (1983; 1988; Johnson-Laird & Byrne, 1991) and, accordingly, let a "mental model" be a mental system of relations that represents another system of relations; a "propositional representation" be a set of natural language formulations, such as a paragraph of text; and "mental images" be pictures in the mind that "correspond to views of mental models" (Johnson-Laird, 1983, p. 157) -- as such, they are mental images of mental models.

We are aware of the controversy surrounding the theoretical use of mental images as internal representations (see, for example, Pylyshyn, 1973) and present Kosslyn's argument in favor of the existence of these representations and their role in problem solving. Kosslyn (1995) makes the following observations:

- Some visual areas of the brain are known to be topographically organized, and these regions of the brain roughly preserve the spatial structure of the retina.
- The parts of the brain that are topographically organized are active during visual mental imagery, even when subjects' eyes are closed.
- The areas of the brain that provide long-term storage for visual memories are not topographically organized.

Kosslyn points out that these observations are consistent with the idea that visual memories are stored in a non-visual format (i.e., in the areas of the brain that are not topographically organized) and must be somehow unfolded in the topographically-organized parts of the brain in order to make geometric information about visual memories known to the subject. This concept closely parallels Johnson-Laird's theory, which posits the existence of an inaccessible mental model that requires a translation through some sort of mental imagery in order to be visually accessible to the subject and communicable by the subject to the external world.

In addition, we note Qin and Simon's (1995) finding that mental images used in problem solving are stored in short-term memory (STM), and that subjects form mental images by drawing on mental models stored in long-term memory (LTM). If long-term visual memories are not stored topographically (from Kosslyn), and non-topographic memories are not immediately accessible (also from Kosslyn), and mental models are stored in LTM (from Qin and Simon), then it is reasonable to infer that mental models are inaccessible mental constructs that reside in LTM and require translation to STM to become accessible and communicable.

Based on this argument in favor of the mental models paradigm, we propose a model that expands the theory to include external representations, such as the diagrammatic and sentential representations created by Willemain's modelers.

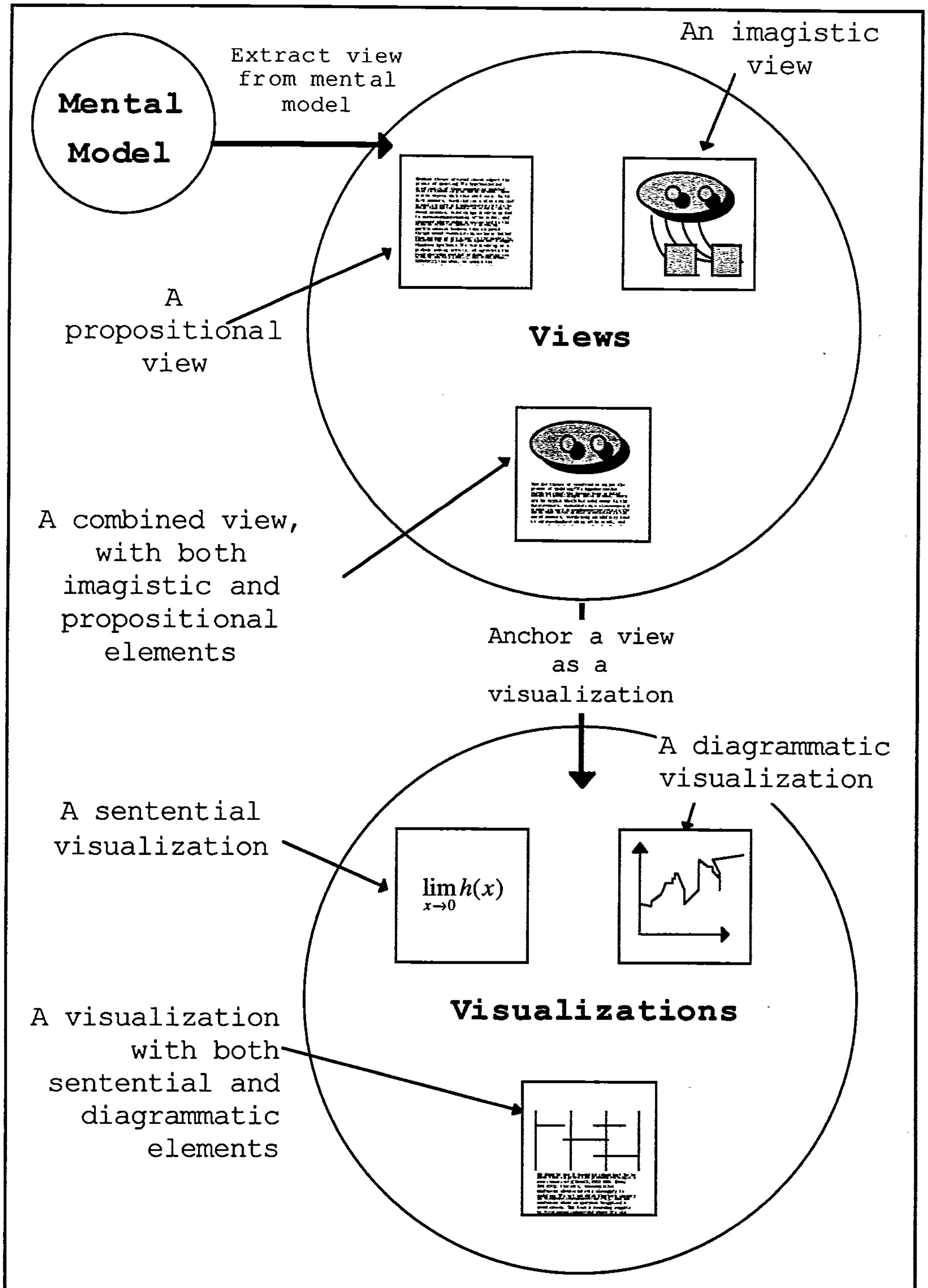

Figure 5. Elements of a framework of modeling with visualization.

Because Willemain's subjects were performing exercises in formulating mathematical models, we view mental models theory in the context of problem solving. We do this by drawing on Johnson-Laird's theory of deduction, which consists of three stages: comprehension, description, and evaluation. During comprehension, one uses available knowledge to construct an "internal model of the state of affairs that the premises describe" (Johnson-Laird & Byrne, 1991, p. 35), that is, a preliminary mental model. Description involves drawing a putative conclusion, that is, developing a revised mental model that not only takes into account the original information, but also makes assertions that had not been explicitly stated previously. During evaluation, one searches for alternative models of the original knowledge that falsify the putative conclusions reached during the description phase.

This portrayal of the process of deduction as a three-stage use of mental models can be mapped into many theories of problem solving. We are interested in the description phase, which maps to the stage that has been referred to as problem formulation (Schwenk & Thomas, 1983), design (Simon, 1960), model building (Urban, 1974), model construction (Churchman, Ackoff, & Arnoff, 1957), analysis (Dewey, 1933; Evans, 1991), incubation (Hadamard, 1945; Wallas, 1926), problem structuring (Smith, 1989), problem-finding and idea-finding (VanGundy, 1988), diagnosis (Bartee, 1973), and categorization (Cowan, 1986). This stage marks the point after the initial data have been assimilated and at the start of the creative process of manipulating data and ideas in the effort to find a solution.

We postulate that the mental model is the long-term inaccessible storage site for the modeler's conception of the problem (see Figure 5). At the moment we step in, at the beginning of Johnson-Laird's description phase, the mental model consists of the unprocessed aggregation of the data assimilated during the comprehension phase. In Willemain's exercises, this initial mental model consists of the modeler's comprehension of the exercise instructions and, for one exercise, hardcopy data. Note that the initial mental model contains all known facts about the problem.

We know that in order for the inaccessible mental model to be communicable, it must go through a visual imagery stage. We call that stage a view. The term "view" is an allusion to database technology. We have found it helpful to conceive of the mental model as analogous to a Structured Query Language (SQL) database. The database exists, frequently in many dimensions, but its contents are inaccessible unless the database is queried and a view is extracted. One cannot go "inside" the database, but can pull out data only by extracting one view at a time. Each view of the database is limited to the two-dimensional tables that can be displayed on a computer monitor, much as each view of the mental model is restricted by the limitations of STM and mental imagery.

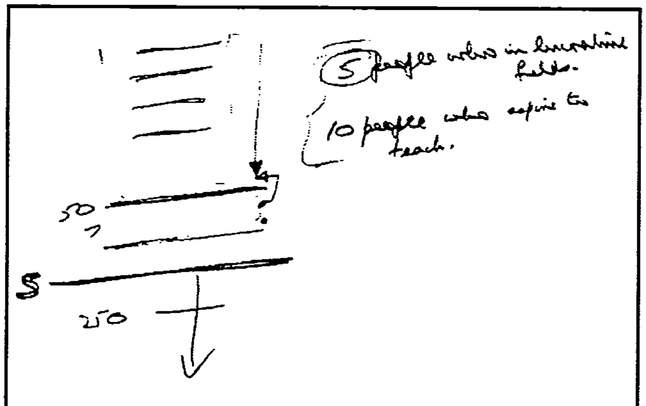

Figure 6. "Rank D" by modeler D on Admissions exercise.

The outputs of this process are the diagrammatic and sentential sketches drawn by Willemain's subjects. Because the output comes in two forms, we postulate that the intermediate translation stage (i.e., the views) also come in two forms: propositional views (i.e., Johnson-Laird's propositional representations) and imagistic views (Johnson-Laird's mental images). The existence of imagistic views follows directly from the argument in favor of mental models presented above. Providing evidence for the existence of propositional views is more problematic. The most we can say at this time is that since the output comes in two forms, we infer that the translation also comes in two forms.

The process described by this argument is iterative. Modelers repeatedly extract views and translate them into visualizations until they are satisfied that the visualizations accurately represent the mental model. We construe these comparisons as within-model testing; that is, the mental model is compared with the visualizations generated by it for internal consistency. Some of the questions modelers may ask themselves include

- Does the visualization accurately represent the picture (view) in my mind?
- Do the visualization and the view accurately represent the mental model?
- Are the view and the visualization consistent with known facts?

If the answers to any of these questions is "no," then the modeler continues the problem solving process by modifying the mental model, extracting a different view, or re-transcribing a visualization. A "no" answer also may prompt the retrieval of additional information from long-term memory, such as a problem solving technique that has been used before. An example of this will be described shortly, with modeler B working the Commercials problem.

Assessment of the Theoretical Framework: Sketches of Expert Modelers

How does this framework fit with our data?

While working the Admissions exercise, modeler D said, "we've got everyone rank ordered . . . and we're going to go through and count up the number of people in the top," and drew the sketch in Figure 6 ("Rank D") to illustrate this idea. After drawing "Rank D," he asked, "How far down the list do you want to go?" D then drew Figure 7 and talked about how he would define a cutoff criterion. We would say that D is working from his mental model of the admissions process. The "Rank D" sketch tells us that he has extracted a view that illustrates the concept of rank-ordering and has translated that view to a diagrammatic visualization. D's question about how far down the list to go implies that after looking at the visualization of the list (the "Rank D" sketch) and comparing it with his mental model, he found that the visualization was missing the concept of a cutoff criterion. Accordingly, he extracted another view, which translated into the diagrammatic visualization "Sigma" in Figure 7.

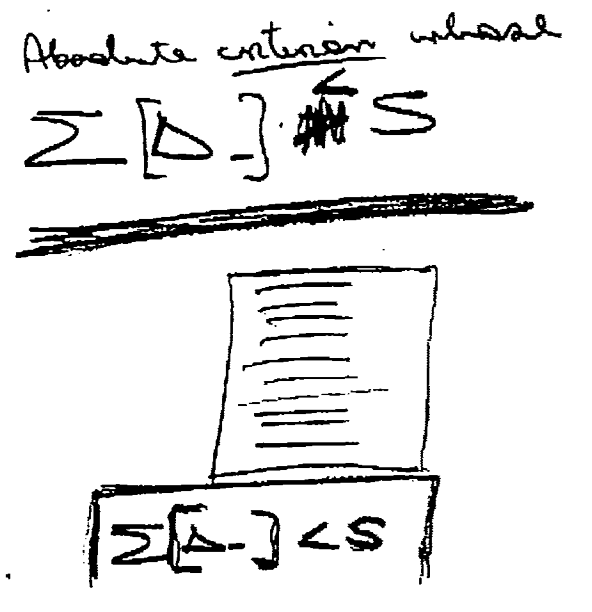

Figure 7. "Sigma" by modeler D on Admissions exercise.

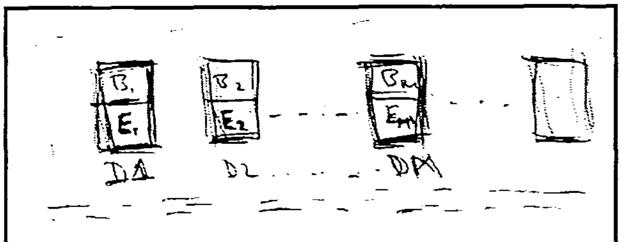

Figure 8. "Budget / Effectiveness" by modeler B working Commercials exercise.

A series of sketches by modeler B working the Commercials exercise illustrates the use of repetition of a visualization to develop a mental model. In "Budget / Effectiveness" (Figure 8), B first visualized the idea of comparing budget with effectiveness for each draft commercial. Studying his visualization stimulated B to modify his mental model to include the analytic hierarchy process (AHP) technique, which he retrieved from LTM and described verbally; he then drew "AHP 1" (Figure 9). After completing this sketch, B stated, "Let me just redraw this thing," implying that this visualization was not adequate to illustrate his mental model and needed to be redone. He then proceeded to draw "AHP 2" (Figure 10).

An example from modeler F also working the Admissions exercise delineates the process of using a series of diagrammatic representations to build different components of a complex mental model. F began by drawing "Membership Function" (see Figure 11) to illustrate the idea of creating a function to determine whether or not applicants are admitted. The column of numbers with "dv" at the top indicates that

> it could be [anything from] a 10 to a 0, with . . . 6 is an admit, and anything below a 6 you don't admit. And a 6 is, well, admit but no aid. And a 9 or a 10 is admit . . . and fly the kid to campus because you're really interested in attracting him. (Modeler F)

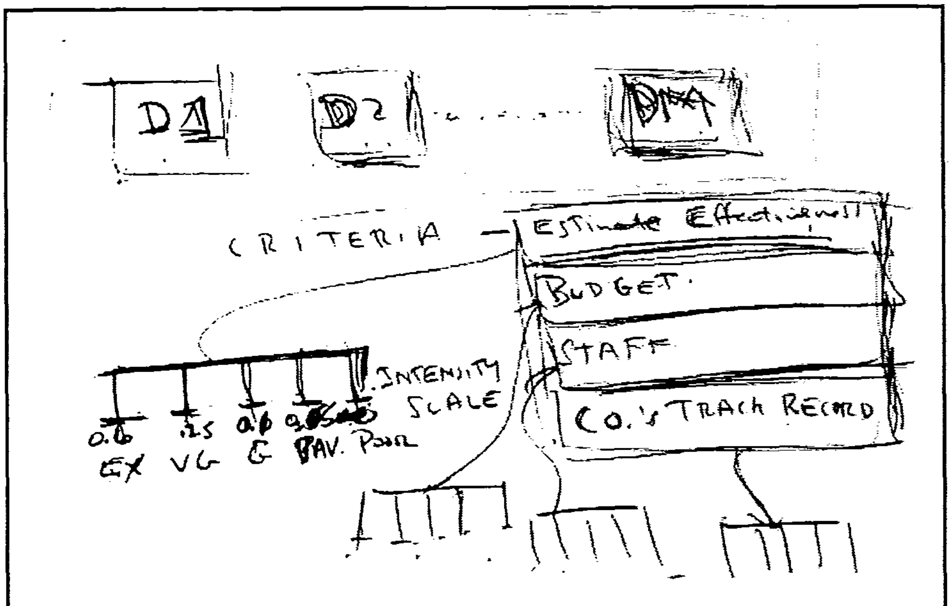

Figure 9. "AHP 1" by modeler B working Commercials exercise.

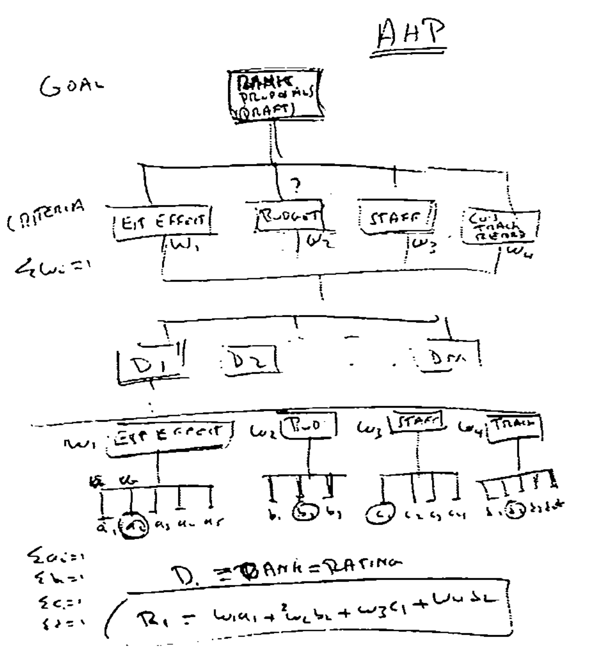

Figure 10. "AHP 2" by modeler B working Commercials exercise.

F's next sketch was "Feedback Loop" (see Figure 12), which shows how a feedback loop between the automated admissions model and student success could be used to evaluate the admissions model and modify it as needed. What appears to have happened is that F began by referring to the initial mental model created by reading the instructions, thinking about how he would decide whom to admit, extracted the view for the membership function idea and sketched it. He then remembered that part of the exercise instructions were to develop a model that would evaluate both the traditional and automated systems, thereby comparing the visualization of the membership function idea with his mental model for the entire task.

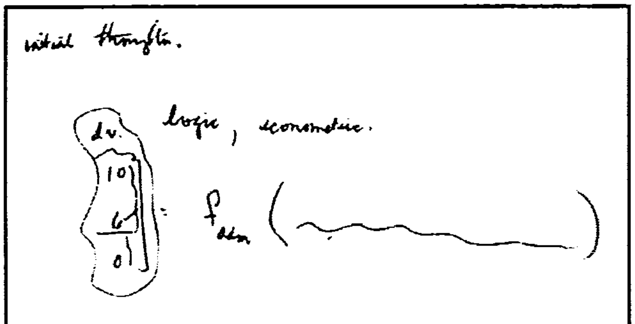

Figure 11. "Membership Function" by modeler F working Admissions exercise.

Realizing that his initial visualization did not incorporate the evaluation part of the instructions, F extracted the view and visualized the idea for the feedback loop. The next sketch in the series is "Vector Y" (Figure 13), which describes an alternative way to formulate the admissions

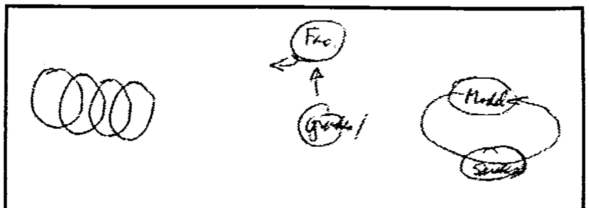

Figure 12. "Feedback Loop" by modeler F working Admissions exercise.

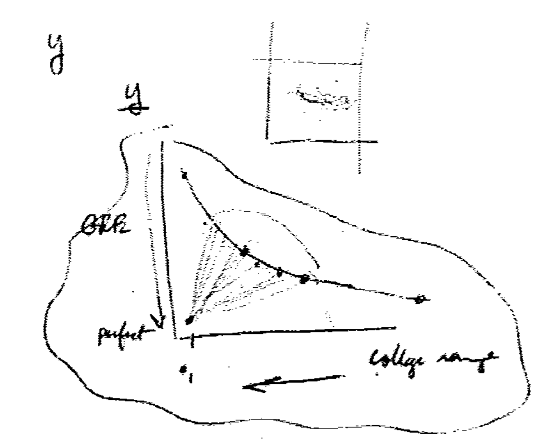

Figure 13. "Vector Y" by modeler F working Admissions exercise.

decision. We infer that this sketch resulted from a comparison of the "Membership Function" sketch with the mental model, a comparison which showed that "Membership Function" was perhaps not the best way to achieve the goals of the exercise. The final sketch in the

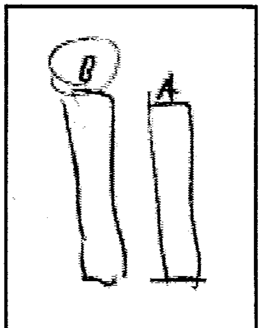

Figure 14 . "Satisficing" by modeler F working Admissions exercise.

series is "Satisficing" (Figure 14), in which F further explicates his mental model by visualizing his conception of this problem as satisficing, meaning that (in F's words) "if they meet all the criteria sufficiently, then there isn't a good reason to reject them." The sketch of the two bars, labeled "A" and "B" depicts a situation where B is better than A in one respect and no worse in any; F's point is that both A and B should be admitted, as opposed to admitting B and rejecting A in comparison with B.

This analysis is consistent with our theoretical framework and suggests that problem solvers' visualizations reflect their use of mental models. We will now offer additional evidence for our approach by using a "cognitive-historical" analysis (Nersessian, 1992) as applied to Charles Darwin's use of visualization to help develop a complex mental model, the Theory of Evolution.

Further Assessment: Darwin's Use of Visualization

Darwin drew and redrew tree diagrams over the years as he developed the Theory of Evolution. Gruber (1978; 1974) calls this kind of persistent significant image of creative work an "image of wide scope." Three diagrams (see Figure 15, Figure 16, and Figure 17) found in Darwin's manuscripts illustrate the process of modeling with diagrammatic representations and can be interpreted in light of our theoretical construct.

In July 1837, Darwin conceived of a model of the development of species that included the concepts of monadism, adaptive equilibrium, and an "irregularly branching tree of nature" (Gruber & Barrett, 1974). Monadism refers to the spontaneous generation of life from inanimate matter, such as the belief that rotting meat produces maggots. Adaptive equilibrium means that organisms adapt as needed to changing environments. Darwin extracted a view of a tree structure to visually express his model. The first tree diagram (see Figure 15) has three branches; it depicts adaptive equilibrium with separate branches and depicts monadism by having a point of origin for the tree. Before drawing the first tree diagram in his notes (see Figure 15), Darwin wrote (Gruber & Barrett, 1974, p. 442),

> Would there not be a triple branching in the tree of life owing to three elements air, land & water, & the endeavour of each typical class to extend his domain

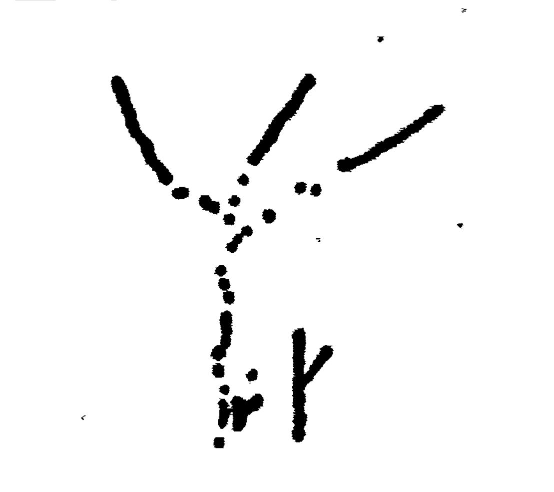

Figure 15. Darwin's first tree diagram.

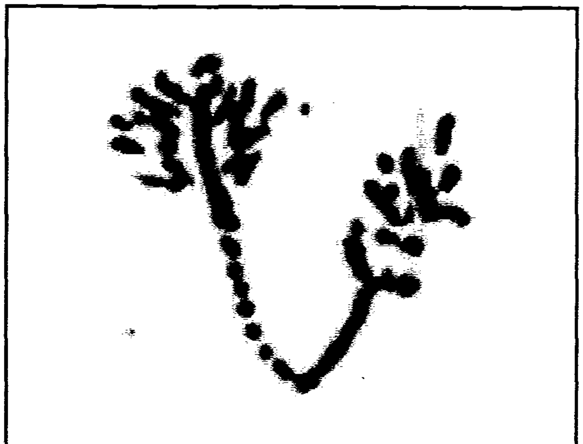

Figure 16. Darwin's second tree diagram.

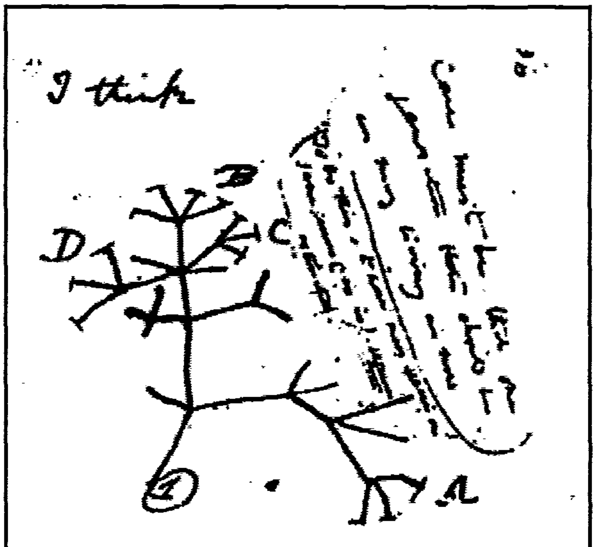

Figure 17. Darwin's third tree diagram.

> into the other domains & subdivision . . . The tree of life should perhaps be called the coral of life, base of branches dead; so that passages cannot be seen.

With these words, Darwin appears to be translating his mental model into a view, which he then transcribed as the visualization in his first tree diagram.

Immediately following the first diagram and immediately preceding the second (Figure 16), Darwin stated (Gruber & Barrett, 1974, p. 442), "Is it thus fish can be traced right down to simple organization. -- birds -- not." He appears to have compared his view with his mental model and found that the visualization did not account for observed discontinuities in nature (some species appear to have a traceable history; others do not). Upon realizing that the discontinuities were unaccounted for in the diagram, Darwin re-extracted the view and/or regenerated the visualization to accommodate this concept, resulting in the second tree diagram, which shows by dotted and solid lines that some species can be traced directly back to simpler organisms, while others cannot (Figure 16). Just before and surrounding the third tree diagram (Figure 17) are Darwin's observations (Gruber & Barrett, 1974, p. 443) that

> If we grant similarity of animals in one country owing to springing from one branch, & the monucle has definite life, then all die at one period, which is not case . . . Case must be that one generation then should have as many living as now. To do this & to have many species in same genus (as is) REQUIRES extinction.

In other words, Darwin has extracted a propositional view from his mental model; this view contains a chain of reasoning that says that if monadism and adaptive equilibrium coexist, then extinction of species must occur regularly. Comparing his second tree diagram with this latest view, Darwin saw that the visualization could be modified to include the view containing the extinction reasoning and thus drew the third tree diagram, Figure 17, which depicts the feature of extinction as crossbars at the ends of the species branches.

It is clear that the process of drawing and modifying the tree diagrams in response to the changing mental models were an integral part of Darwin's creative thought processes. Having demonstrated that the work of both current modelers, such as Willemain's subjects, and historical figures of science, such as Darwin, can be characterized within our theoretical framework, we believe that additional analysis within this framework can lead to greater insight about the process of mathematical modeling, which in turn may lead to more effective ways to teach modeling to future scientists.

Conclusions

We have presented an analysis of sketches done by expert modelers and considered the sketches as examples of both diagrammatic and sentential representations. We have provided a theoretical framework for the role of diagrammatic and sentential representations in mathematical modeling. This framework is based on mental models theory, which has been found useful in the teaching of modeling (Powell, 1995), and we review one possible defense against the criticism that internal representations are unverifiable. A cognitive-historical analysis shows that this framework is consistent with Charles Darwin's use of sketches during his development of the Theory of Evolution.

We are pursuing this line of inquiry by planning experiments to assess the application of our theoretical framework to modelers using computer-generated diagrammatic visualizations instead of sketches. We have selected as the target of our visualization a component that is central to the modeling process and to our understanding of it: the heuristics used in constructing models. In order to assess our theory, we intend to build visualization systems for different types of modeling tasks. Our objectives for these experiments are to 1) operationalize and test our theoretical framework, 2) explore the application of our theoretical framework to the analysis of computer-generated diagrammatic visualizations, and 3) explore the impact of providing diagrammatic visualizations of modeling heuristics as a modeling aid.

The role of visuals such as pictures, graphs, and diagrams in enhancing comprehension in mathematics is well-recognized. The perhaps 30 million users of spreadsheet software (Savage, 1996) must be educated about using diagrammatic representations to better construct and use models. Increasing our understanding of the role of diagrammatic representations in the model building process will enhance both our ability to use visual displays in teaching and communicating and our ability to design and develop human-machine technology.

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