Local Extent in Diagrams

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Abstract

Most diagrams, even if they can be interpreted as embeddings in infinite objects, are nevertheless finite in extent if they are to be presented conventionally. In many applications where diagrams are used to explain, instruct, communicate, cogitate or conjecture, this finiteness implies local extent. By this we mean that even if they are used to reason about potentially unbounded domains or constructs, the fact that they can be used at all suggests that only local properties are being examined. We are interested in how the features of such diagrams can be described in logic, and how diagram manipulations that represent actions can be justified. In particular, we establish a correspondence between a substructure construction which formalizes local extent and a class of sentences preserved under extension from, and reduction to, this substructure. The sentence class is called EnE because it has the form of successively nested existential and negated existential subsentences, and appear to cover most of the applications so far encountered. The hope is that this understanding can be used to mark out the local regions of diagrams that can be safely isolated for manipulations.

Keywords: local extent in diagrams, actions, Gaifman metric, preservation theorems, embeddings, genericity.

Introduction

In two recent papers (Foo, et.al. 97; Foo 98) we examined the use of diagrams in reasoning about blocks world and link lists respectively. In both domains we picked a simple action and looked at how state transitions were represented in pictorial form. Some features that we believe are typical of such diagrams are:

1. They are generic or schematic, hence can be parametised;
2. They represent local regions of concrete or abstract domains;
3. All changes due to actions are confined to such regions;
4. Objects (and their attributes) are conflated with icons.

On reflection, none of this is surprising, nor even peculiar to diagrams. In programming languages, for instance, each instruction (or even procedure) or rule execution normally changes only a finite number of things (values, pointers, bindings, etc.). The genericity of procedures is taken for granted. And there is a strong move to visual programming in which iconic tokens play a central role. That such properties are also inherent in many uses of diagrams probably says a lot about the psychology of procedural thinking. While in programming languages these features are there by design (there is nothing inherent in instruction sets that forces finite effects — just consider the proposed biological computers), the fact that diagrams have to be finitely displayed largely constrains them to have the features. If we can understand in rigorous terms what these features actually mean, we may be able to use such diagrams more confidently to explain, instruct, communicate, cogitate or conjecture. This paper is an attempt to do this. Some of it provides a different perspective on work already done; some are extensions and generalizations of prior results; the remainder connects model theory with the idea of local tests and effects.

The Role of Logic

We are aware of strong evidence from cognitive psychology that human processing of diagrams have many aspects, not the least of which is the fact that some form of *immediate comprehension* is involved. Moreover, as Barwise and Etchemendy (Barwise 95) have pointed out, there is in general no interlingua between logic and diagrams that is fully adequate. As this paper addresses the ostensibly perceptual feature of local extent – humans presumably have an intuitive grasp of which objects are locally relevant to given objects in the context of an application – it is legitimate to ask why logic should be invoked.

There are at least two justifications for using logic (more precisely, model theory) to examine local extent. The first is well-known, and it is the analogy of nonmonotonic logics that are used to explicate commonsense reasoning. The roles of such logics are (i) to realize commonsense reasoning in computational models, and (ii) to classify the various nuances of commonsense approaches by reference to different modes of inference. Likewise, there may well be various nuances of local extent that can be classified and given computational realizations using logical models. The second is less traditional, but perhaps more persuasive from the viewpoint of applications. If it is possible to give simply declarative and realizable accounts of reasoning about local extent - and logic may be one way to do so - then interfaces that display diagrams can have local regions automatically constructed and highlighted. This would be a congenial aid in diagrammatic reasoning. Conversely, if a local region were to be user-selected, a logic could be used to confirm its validity. Logic is therefore used to explain and sanction intuition.

Basic Definitions

For completeness we recapitulate some basic definitions and establish notation in this section. A model theory text such as (Chang and Keisler 73) should be consulted for more details.

Definition 1 *Suppose $\mathcal{L}$ is a first-order language with equality but no function symbols. An $\mathcal{L}$-structure $\mathcal{U}$ is a triple $\langle \mathcal{A}, \mathcal{R}, C \rangle$ where $\mathcal{A}$ is a non-empty set called the domain of $\mathcal{U}$, $\mathcal{R}$ is a map that assigns relations of the appropriate arity on $\mathcal{A}$ to predicate symbols in $\mathcal{L}$, and C is a map that assigns elements of $\mathcal{A}$ to constant symbols in $\mathcal{L}$.*

The relationship between diagrams and structures has been much discussed. If diagrams are regarded as alternatives to textual forms conventional to first-order logic (terms, formulas, etc.) then they should both be interpretable into structures. If so, it should be the case that there are *calculi* of diagrams as well, and indeed a number have been actively researched (Sowa 84; Shin 94). We accept the broad thrust of this view (but see below for our perhaps idiosyncratic additions). It is our intention that diagrams have structures as meanings, and we will even admit infinite structures as standard models for some diagrams.

Definition 2 (Sub-structure) *If $\mathcal{U}_1$ and $\mathcal{U}_2$ are $\mathcal{L}$ structures with domains $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively such that $\mathcal{A}_1 \subseteq \mathcal{A}_2$, and the relation and constant assignment maps of $\mathcal{U}_1$ are that of $\mathcal{U}_2$ restricted to $\mathcal{A}_1$, then $\mathcal{U}_1$ is a sub-structure of $\mathcal{U}_2$, and conversely $\mathcal{U}_2$ is an extension of $\mathcal{U}_1$.*

Sub-structures will be the meanings of sub-diagrams. Sub-diagrams can arise in a variety of ways. In systems like Xfig, a widely used picture-drawing program, it is possible to "border" a region containing drawn objects. It is also possible to "select" non-contiguous objects. These are typical sub-diagrams whose objects give rise to substructures (when there are no function symbols). It may also be necessary to add to such initially chosen objects other objects to form domains of sub-structures, as will be the case below.

Localness and Genericity

This part summarizes work already completed using new insights and vocabulary. The paper (Foo, et.al. 97) used STRIPS and a blocks world to illustrate how localness and genericity can be formalized, and suggested how action invariants can be conjectured from these features. The paper (Foo 98) continued this work in the domain of linked lists but used logic programs as the underlying semantics for the diagrams. Both the papers relied on confining the pre-condition of actions to formulas that had a particular syntax. We generalize this syntax and provide the intuitive reasons for it.

Genericity

Suppose we want to execute an action A on some objects (they need not be physical — linked lists are an example). Typically this action actually belongs to a *class* of similar actions of which this one is an instance. Say, for this instance the objects are $a_1, \ldots, a_n$. If so, it is natural to regard A here as really the instantiated form of a parametised or generic action $A[x_1, \ldots, x_n]$, with the substitutions a_i for x_i. This is the essence of *genericity*, and it sanctions the following methodology. If we were to reason about the *specific* action $A[a_1, \ldots, a_n]$ — let us call it A for short — and come to certain conclusions, so long as we did not use any specific properties of the names $a_1, \ldots, a_n$, we can *generalize* the conclusions by lifting these names to the variables $x_1, \ldots, x_n$. This is a version of the theorem on Generalization from Constants (Shoenfield 67). If we believe that our diagrams are generic, then this is precisely what we are doing, and exactly the caveat that must be observed. This imposes the following design constraint on any system of diagrammatic reasoning that permits lifting of names to variables in the construction of "templates": the variables must be "tagged" with the used properties of the names from which they were lifted. This is not so onerous as it might sound, for it is satisfied by systems in which icons are strongly typed. From these observations the informal notion of a template boils down to the isomorphism of sub-structures constructed from the names of the action. This construction is considered in the next sub-section.

Local Extent

A generic action $A[x_1, \ldots, x_n]$ can act on a situation or state only if a pre-condition is satisfied. For instance, a node can be inserted into a particular location of an ordered linked list only if its value is in between those of the two nodes sandwiching the location; likewise a block can be moved on top of another only they both do not have other blocks on them. We can test for such pre-conditions procedurally or diagrammatically, but can also express them declaratively in logic. It will be assumed that these are mutually translatable, so we will use logic to explain our intuitions about local extent. Let the pre-condition of the action above be expressed by the formula $\phi[x_1, \ldots, x_n]$. Then for any specific action $A[a_1, \ldots, a_n]$ the corresponding pre-condition is the sentence $\phi[a_1, \ldots, a_n]$. The names $a_1, \ldots, a_n$ refer to the objects of the action, but in general *there may be more objects which the action may influence.* For instance, in the blocks world the pre-condition of a $move(a, b)$ action, meaning move block a to the top of block b may have the pre-condition $\exists x\ on(a, x) \wedge \exists y\ on(b, y) \wedge \neg\exists u\ on(u, a) \wedge \neg\exists v\ on(v, b)$. If this is satisfied, there will be instantiations for the variables x and y — yes, we are interpreting the existential quantifiers substitutionally, but diagrams are presumably constructive. These will normally figure in the post-condition of the action, and hence any diagram that displays the effects of the action will show the objects that are names for the x and y. So, if we wish to reason about the action it is necessary to use not only the names in it but also other names implied by its pre-condition. In this example the extra names were pulled in via the existential quantifiers. We may describe this informally as a kind of *domain expansion* or *closure* with respect to the instantiated action parameters and the predicates that occur in its pre-condition. The definition below is a generalization of the one that appeared in (Foo 98) (that was closure of extent 1; see below).

Definition 3 (Closure) *Given a set S of constants in $\mathcal{U}$, the 1-expansion of S by n-ary predicate P in $\mathcal{U}$, denoted $closure(\mathcal{U}, S, P, 1)$, is*
$\bigcup\{\{a_1, \ldots, a_n\} \mid$ *some* a_i *is in* S, *and* $\mathcal{U} \models P(a_1, \ldots, a_n)\}$.
Inductively,
the j+1-expansion of S by P, $closure(\mathcal{U}, S, P, j+1)$, is
$\bigcup\{\{a_1, \ldots, a_n\} \mid$ *some a_i is in $closure(\mathcal{U}, S, P, j)$, and* $\mathcal{U} \models P(a_1, \ldots, a_n)\}$
$\cup\ closure(\mathcal{U}, S, P, j)$.
The notation $closure(\mathcal{U}, S, P_1, \ldots, P_k, n)$ is the obvious generalization to several predicates in which any of them can be used at any stage.

The last parameter in the closure notation is the *extent* of the closure. It will be omitted if the context is clear.

By a slight abuse of notation we will identify the substructure of $\mathcal{U}$ (restricted to the predicates $P_1, \ldots, P_k$)) with $closure(\mathcal{U}, S, P_1, \ldots, P_k, j)$.

This is really a model-theoretic construction based on a metric introduced by Gaifman (Gaifman 82). He was concerned with defining local regions and did this with respect to predicate P by decreeing distance 0 for the equality predicate, then distance 1 between any two points c and d such that $\mathcal{U} \models P(\ldots, c, \ldots, d, \ldots)$. Higher distances n are defined inductively to reflect the n-closure above. More precisely, if we denote the distance between c and d by $G(c, d)$, then:

Lemma 1 $d \in closure(\mathcal{U}, \{c\}, P, n)$ *iff* $G(c, d) \leq n$.

Notation 1 *When we wish to denote either the positive or the negated form of a formula ϕ without specifying which is the case, we will write $\pm\phi$. In the definition of EnE sentences below, by $\bar{x}$ we mean a sequence of variables, say $x_1, \ldots, x_k$. By $\exists\bar{x}$ we mean a sequence of existential quantifiers, say $\exists x_1, \ldots, \exists x_k$. When we write $P(\bar{x}, \bar{y})$ we mean that predicate P has free variables among the sequences $\bar{x}, \bar{y}$ in no particular order.*

Definition 4 (Primitive EnE sentences) *A primitive EnE sentence of width n based on $\bar{c}$ is $\pm\exists\bar{x}_1\phi_1(\bar{x}_1)$ where:*
$\phi_1(\bar{x}_1)$ *is* $P_1(\bar{c}, \bar{x}_1) \wedge \pm\exists\bar{x}_2\phi_2(\bar{x}_2)$;
$\phi_2(\bar{x}_2)$ *is* $P_2(\bar{x}_1, \bar{x}_2) \wedge \pm\exists\bar{x}_3\phi_3(\bar{x}_3)$;

$\vdots$

$\phi_n(\bar{x}_n)$ *is* $\pm\exists\bar{x}_n P_n(\bar{x}_{n-1}, \bar{x}_n)$

The collection of such sentences is denoted by $EnE(S, n)$ where S is the set of constants from $\bar{c}$. Informally, a primitive $EnE(S, 1)$ sentence is one that has a prefix of the form $\exists x$ or $\neg\exists x$ followed by an atom in which there is at least one constant from S. A primitive $EnE(S, n)$ formula is so defined that its ground instances are conjunctions of ground literals with n-chains as arguments. In section there is an example of EnE formulas of width 3.

Definition 5 (EnE sentences) *An EnE(S,n) sentence is a Boolean combination of primitive EnE(S,j) sentences where $j \leq n$.*

The connection between such sentences and closure is the key to local extent.

Proposition 1 (EnE Preservation) *Suppose ϕ is an EnE(S,n)-sentence with predicates $P_1, \ldots, P_k$. Then $\mathcal{U} \models \phi$ iff $closure(\mathcal{U}, S, P_1, \ldots, P_k, n) \models \phi$.*

This proposition, whose proof is given in the Appendix, is the basis for local extent if we confine pre-conditions (or any property of interest to EnE-sentences. Observe that unless the set of ground atoms involving S that satisfies $\mathcal{U}$ is sparse, there is little to be gained from considering the closure rather than the entire structure. In many applications, not only is this set sparse but many interesting properties are expressible as EnE sentences. The informal paraphrase of this proposition is that for sparse relations and EnE sentences it suffices to consider the local extent marked out by the closure of the constants given. The proof of this proposition is by induction on n, and will be included in later version of this paper.

An example

In this section we illustrate the above ideas with an example. We consider floor plans, much like those in (Myers and Konolige 95), and assume that the predicates *person(p)*, *in_room({o,p},r)*, *next_to(r,s)*, *connected_to(r,s)* to describe the topology in logic. We have indicated the types as follows: o are objects, p are people, r, s are rooms. The action is to locate a room not occupied by a person but is next to one with a fax machine. The pre-condition sentence is expressible as

$\exists r(in\_room(fax, r) \wedge \exists s(next\_to(r, s) \wedge \neg\exists p(person(p) \wedge in\_room(p, s))))$ which is an EnE($\{fax\}$,3) sentence with respect to the mentioned predicates. The closure of *fax* will pull in the names (constants) of following items: all rooms with a fax machine, all rooms next to these, then all people who are in these latter rooms. The order in which they are pulled in reflects the increasing width j of the closure. Indeed, once this highly intuitive idea is grasped, the proof of the EnE Preservation proposition is almost immediate. In the next section we describe a common way to represent the effect of actions. Using the vocabulary of that section, if the action here is to put person c in the room, the add-list would be $in\_room(c, r1)$ where $r1$ is a room that satisfies the pre-condition, and the delete-list is empty. The cut-and-paste operation described therein is also clear . These remarks are illustrated in figure 1. The room $r1$ is the one picked up as the one next to the fax room. Other objects, including the person in the other room next to the fax room, are also picked up and used in testing the pre-condition.

Now, consider the predicate $connected\_to(r, s)$ which did not figure in the pre-condition. Is that the kind of property that is easily represented in diagrams, and how do they figure in closure and EnE sentences? A hint can be obtained from a Prolog-like definition of this predicate in terms of the $next\_to(r, s)$ predicate:

$$connected\_to(r, s) \leftarrow next\_to(r, s)$$
$$connected\_to(r, s) \leftarrow next\_to(r, v) \wedge connected\_to(v, s)$$

In the fixed-point semantics of logic programs this says that $connected\_to(r, s)$ is the transitive closure of $next\_to(r, s)$, and this is known to be inexpressible in first-order languages as it is a global property. For the usual diagrams though, because of local extent, connectedness is usually quickly apparent. In figure 1 for instance, if we add a hallway to connect the rooms, the predicate is visually trivial. We do not fully understand the implications of these observations, and are exploring them.

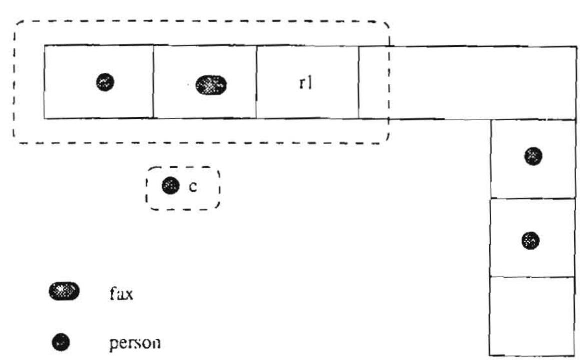

Figure 1: A floor plan showing the closure in dashed boundaries.

Cut and Paste

What we would like to do after determining a local extent is to use it for *cut and paste* operations. That is, the action specification should permit us to construct a "replacement diagram" (local state update) that can be used to supplant the original one. Under what circumstances can this be done?

In (Foo, et.al. 97) we used STRIPS as the action specification. This has the merit of great simplicity and ease of implementation. The *post-condition* of an action in STRIPS are two lists, the add- and the delete-lists, which are ground atoms that are added and deleted respectively from the prior state to yield the new state. The assumption is that all states are Herbrand models. With STRIPS the following property is usually satisfied — every constant in the post-condition is already implied in the pre-condition. The add- and delete-lists are certainly expressible as EnE sentences, so it follows that the closure structure constructed from the pre-condition suffices for the post-condition, and consequently a Preservation proposition for the post-condition EnE sentence also holds with the same closure domain. This is the essence of the "paste" part of state update.

This does not answer the question more generally. As a number of action theories admit causal propagation rules, even fixed point constructions, we suspect that more elaborate constructions will be necessary for closure. But as this is on-going work, we hope to report progress later.

Conclusion

It has become evident in our work that finte model theory plays an important role in the model theory of diagrams and can be used to validate proposed calculi. Our work in the use of diagrams to convey state change has raised some interesting issues. What is the status of closures that have pictorially disjoint sub-diagrams? These are common in illustrating, say, two relevant local portions of a large diagram in which the "in-between" parts are irrelevant. Logically they correspond to having two far-apart constants as starting points for closure chains. What is the best way to illustrate recursion in algorithms? Can ellipses in, say, infinite series, be given model theoretic meanings and reasoned about? Does the idea of "telescoping" diagrams have a semantics in local extent? We believe our report here is a pointer to some of the techniques that may be fruitful.

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Appendix

This appendix contains the proof of the main proposition in the paper. It uses subsidiary propositions which may be independently interesting.

Notation 2 *We abbreviate closure*$(\mathcal{U}, S, P, n)$ *by* $C(\mathcal{U}, S, P, n)$. *When the language is fixed, so the set of predicates is unambiguous, we write* $C(\mathcal{U}, S, n)$ *for* $C(\mathcal{U}, S, \mathcal{P}, n)$ *where* $\mathcal{P}$ *is the set of predicates. Also, if* S *is a singleton* $\{a\}$, *we write* $C(\mathcal{U}, a, n)$ *for* $C(\mathcal{U}, S, n)$.

Lemma 2 $a_n \in C(\mathcal{U}, a_0, n)$ *iff there is a sequence of predicates (not necessarily distinct)* $P_1, \ldots, P_n$ *such that the ground atoms below satisfy the conditions:*

$$\mathcal{U} \models P_1(\ldots, a_0, \ldots, a_1, \ldots)$$
$$\mathcal{U} \models P_2(\ldots, a_1, \ldots, a_2, \ldots)$$
$$\vdots$$
$$\mathcal{U} \models P_n(\ldots, a_{n-1}, \ldots, a_n, \ldots)$$

Definition 6 *The sequence* $\langle a_0, \ldots, a_n \rangle$ *in the above lemma is called an n-chain.*

Corollary 1 $\langle a_0, \ldots, a_k \rangle$ *and* $\langle a_k, \ldots, a_n \rangle$ *are chains iff* $\langle a_0, \ldots, a_n \rangle$ *is a chain.*

Definition 7 *An n-chain* $\langle a_0, \ldots, a_n \rangle$ *is P-extendable if in addition to the conditions for an n-chain there is a predicate* P *and constant* a_{n+1} *such that* $\mathcal{U} \models P(\ldots, a_n, \ldots, a_{n+1}, \ldots)$.
An n-chain that is not P-extendable for some predicate P *is an n-P-cecum. In this case, for arbitrary position of the variable* x, $\mathcal{U} \models \neg\exists x P(\ldots, a_n, \ldots, x, \ldots)$.

Corollary 2 $\langle a_0, \ldots, a_k \rangle$ *is a chain and* $\langle a_k, \ldots, a_n \rangle$ *is a cecum iff* $\langle a_0, \ldots, a_n \rangle$ *is a cecum.*

Observation 1 *An n-P-cecum may well be extendable by some other predicate* Q, *so that* $\langle a_0, \ldots, a_n \rangle$ *is only a "dead-end" as far as predicate* P *is concerned.*

From the preceding definitions and lemmas, the next corollary is immediate.

Corollary 3 *For any n-chain or n-cecum* $\langle a_0, \ldots, a_n \rangle$, $\mathcal{U} \models \langle a_0, \ldots, a_n \rangle$ *iff* $C(\mathcal{U}, a_0, n) \models \langle a_0, \ldots, a_n \rangle$.

Definition 8 *A positive EnE sentence is one in which there is no negated quantifier. A tail-negative EnE sentence is one in which the only negated quantifier is the deepest nested.*

Observation 2 *The next two remarks are entailed by the standard semantics of existential quantifiers. If* α *is a positive EnE sentence, then* α *is equivalent to a prenex sentence which has the same form as* α *except that all the existential quantifiers have been moved outwards to the prefix. Likewise, if* α *is a tail-negative EnE sentence, it is equivalent to a "near-prenex" sentence in which all but the deepest (the only negated) existential quantifier has moved outwards to the prefix.*

The following propositions are easy consequences of this observation.

Proposition 2 $\langle a_0, \ldots, a_n \rangle$ *is an n-chain iff* $\mathcal{U} \models \alpha$ *where* α *is equivalent to a positive* $EnE(a, n)$ *sentence.*

Proposition 3 $\langle a_0, \ldots, a_n \rangle$ *is an n-P-cecum iff* $\mathcal{U} \models \alpha$ *where* α *is equivalent to a tail-negative* $EnE(a, n+1)$ *sentence with* P *as the deepest nested predicate.*

The principal result of this paper, viz., proposition 1, is a direct consequence of the next proposition and corollary 3.

Proposition 4 *Every EnE(a,n) sentence is equivalent to a disjunction of sentences from the class of positive EnE(a,n) and tail-negative EnE(a,k) sentences where* $k < n$.

Proof If the EnE(a,n) sentence is positive the assertion is trivially true. So, suppose it is not positive. For simplicity of exposition, let us assume that the predicates occurring in the sentence are binary — the proof structure for the general case is similar. We use strong induction on n. If $n = 1$, then the sentence is $\neg\exists x P(a,x)$ for some predicate P, say. Then $\mathcal{U} \models \neg\exists x P(a,x)$ iff $\langle a \rangle$ is a *1-P-cecum*. Assume the assertion for for at most n, consider an EnE(a,n+1) sentence in which the first occurrence of a negated existential is $\neg\exists x_k(P_k(x_{k-1},x_k) \wedge \pm\exists x_{k+1}\phi(x_k,x_{k+1}))$, which is equivalent to $\forall x_k(P_k(x_{k-1},x_k) \rightarrow \pm\exists x_{k+1}\phi(x_k,x_{k+1}))$. Then there is a $k-1$ chain $\langle a_0,\ldots,a_{k-1}\rangle$ and either (i) $\mathcal{U} \models \neg\exists x_k P_k(a_{k-1},x_k)$ or (ii) for some a_k $\mathcal{U} \models P_k(a_{k-1},a_k)$. Case (i) holds iff $\langle a_0,\ldots,a_{k-1}\rangle$ is a $k-P_k-$*cecum*, hence is equivalent to a tail-negative EnE(a,k) sentence. Case (ii) holds if there is some a_k such that $\mathcal{U} \models P_k(a_{k-1},a_k)$, which implies that $\mathcal{U} \models \pm\exists x_{k+1}\phi(x_k,x_{k+1})$, where this is by assumption an EnE(a,n-k) sentence. By the induction hypothesis $\pm\exists x_{k+1}\phi(x_k,x_{k+1})$ is equivalent to a disjunction of sentences from the two classes above. But by propositions 2 and 3 each such disjunct holds iff their corresponding chains or ceca hold. Moreover $P_k(a_{k-1},a_k)$ corresponds to a chain $\langle a_{k-1},a_k\rangle$, so each of the chains or ceca appended to it is, by corollary 1 also chains or ceca. Hence, using propositions 2 and 3 again, the EnE(a, n+1) sentence is equivalent to a disjunction of sentences from the two classes, so completing the induction.