

# A Semantic Analysis of Inference Involving Venn Diagrams

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My thesis, *Valid Reasoning and Visual Representation* (1991), challenges a general prejudice against visualization in the history of logic and mathematics by providing a semantic analysis of two graphical representation systems — a traditional Venn diagram representation system and an extension of it. I present these two diagrammatic systems as standard formal representation systems equipped with their own syntax and semantics. I also show that these two systems are sound and complete. In the following, I lay out an outline of the thesis.

## 1 Introduction

Suppose that the students in a logic class are asked to *prove* that the following argument is valid:

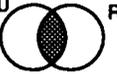
There is no unicorn.  
Therefore, no unicorn is red.

Suppose the students gave the following two kinds of answers:

### Answer 1

- |   |                            |
|---|----------------------------|
| 1. $\forall x \neg Ux$                  | premise                    |
| 2. $\neg Ua$                            | 1 Universal Instantiation  |
| 3. $\neg Ua \vee \neg Ra$               | 2 Disjunction Introduction |
| 4. $Ua \rightarrow \neg Ra$             | 3 Implication              |
| 5. $\forall x (Ux \rightarrow \neg Rx)$ | 4 Universal Generalization |

### Answer 2

- |    |   |                         |
|----|---|-------------------------|
| 1. |  | Premise                 |
| 2. |  | 1 Adding a closed curve |
| 3. |  | 2 Erasing a shading     |

It is true that many of us use Venn diagrams (like answer 2) in logic classes. However, answer 2 has not been considered a valid proof but a heuristic tool to find a real formal proof such as answer 1. The question I want to raise is as follows: Is this evaluation of Venn diagrams justified? Or, have we ever shown that answer 2 is not a valid proof? Nobody has shown that using Venn diagrams is not a valid proof. The reason why answer 2 has not been considered a valid proof is just that nobody has shown that using Venn diagram is a valid proof.

Venn started the chapter of his *Symbolic Logic* where he introduced his new diagrammatic method with the following passage:

The majority of modern logical treatises make at any rate occasional appeal to diagrammatic aid, in order to give sensible illustration of the relations of terms and propositions to each other.<sup>1</sup>

After presenting his own method, he expressed his faith in the diagrammatic method in the following passage:

Of course, we must positively insist that our diagrammatic scheme and our purely symbolic scheme shall be in complete correspondence and harmony with each other. . . . But symbolic and diagrammatic systems are to some extent artificial, and they ought to be constructed as to work in perfect harmony together. This merit, so far as it goes, seems at any rate secured on the plan above.<sup>2</sup>

Considering that Venn did not have an appropriate tool, that is, semantics, we understand why Venn had to have only this unproven belief that a diagrammatic system, if correct, is at least as good as a correct symbolic system.

The most interesting point is that even after semantics was introduced in logic, nobody (including Euler, Venn, Peirce and Prior) has attempted to prove that answer 2 is a valid proof just as answer 1 is, despite the fact that they have been using Venn diagrams for a long time. Peirce modifies Venn's system and introduces the rules of transformation, but he never justifies his rules. At the end of his discussion of this modified new system, he says that "Euler's diagrams are the best aids in such cases [non-relative deductive reasoning], being natural, . . ."<sup>3</sup> This remark clearly shows that Peirce has never approved of using Venn diagrams in a valid proof, since he attributes the role of 'aids' to this diagrammatic system. That is, there has been no justification for the steps taken in answer 2. Why? The answer to this question is directly related to the following two related prejudices which reside in the history of logic and mathematics: (1) a general prejudice for linguistic representation: Only linguistic representation systems can be used in valid proofs. (2) a general prejudice against visualization: Fallacies often have arisen from the misuse of diagrams in mathematical proofs. Accordingly, diagrams can be only heuristic tools.

A misconception of valid reasoning is mainly responsible for the first prejudice mentioned above. Valid reasoning is a process of extracting information, not a process of manipulating symbols. Accordingly, the essence of valid reasoning lies in the relation among pieces of information, not in the relation among syntactic objects by which information is conveyed. If so, there is no legitimate reason to conclude that only symbolic representation can provide valid proofs. Then, why have we related validity almost exclusively to linguistic representation? The second prejudice listed above has been one of the main assumptions which underlie this misconception of validity. Here, we observe an interesting vicious circle: The worry about the misuse of diagrams has convinced logicians and mathematicians that only symbolic representation, not diagrams, can be used in valid proofs. In turn, this conviction has prevented us from making an attempt to give a proper place to visualization in the logical tradition. Again, that there has been no claim that diagrams can be used in valid proofs has enforced the prejudice for linguistic representation systems.

I aim to break this circle by claiming that the worry about the misuse of diagrams has been mistakenly related to the nature of diagrams. If my claim is correct, the general prejudice for linguistic systems loses one of the main sources it has heavily relied on. At the same time, my claim opens the possibility that diagrams can be used more than just as heuristic tools.

My claim is proven through two diagrammatic representation systems — a traditional Venn diagram representation system and an extension of it. By giving a semantic analysis of these two visual systems, I show that we can have valid transformation rules in this diagrammatic system just as we have valid inference rules in symbolic systems. That is, if we apply only the transformation rules of this system, fallacies would not arise in using Venn diagrams.

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<sup>1</sup>Venn, p. 110.

<sup>2</sup>Venn, p. 139.

<sup>3</sup>Charles Hartshorne and Paul Weiss, p. 317. Peirce calls his modified Venn's system Euler's diagrams.

I present the Venn diagrams which have been used in logic as a formal system of representation equipped with its own syntax and semantics. (In the following I name this formal system of Venn diagrams Venn-I.) I extend Venn-I to what I call Venn-II and also present it as a standard representation system with its own syntax and semantics. I show that Venn-I and Venn-II, with the rules of transformation that I specify, are sound and complete. The soundness of these two diagrammatic systems assures us the following: fallacious inferences we often get in the use of diagrams do not originate in the nature of diagrams but in a lack of clarity about legitimate manipulation of diagrams.

In the next section, I introduce the way I formalized the syntax and the semantics of Venn-I. In §3, I mention the new syntactic aspects of Venn-II, that is, a new clause of well-formedness and four new transformation rules.

## 2 Venn-I

Peirce adopts Venn's diagrams, for the representation of sets, replaces Venn's shading with the symbol 'o' and adds 'x' and a line to connect o's and x's. However, the connected o's in Peirce's system are so confusing that the system cannot maintain much visual power.<sup>4</sup> Peirce's modification of the Venn system increases its expressive power, while sacrificing its visual power. I suggest that the Venn system which I call Venn-I and show to be sound and complete should strive for half of Peirce's modification of Venn's system: we want to increase its expressiveness, while maintaining its visuality. Therefore, Venn-I, of which I am about to establish the syntax and the semantics, consists of the following: (i) Venn's representation of sets and the empty set: closed curve and shading (ii) Peirce's representation of a non-empty set: 'x' and the connected x's (iii) the representation of a background set: rectangle.

### 2.1 Primitive objects

We assume we are given the following sequence of distinct diagrammatic objects to which we give names as follows:

Diagrammatic Objects	Name
	closed curve
	rectangle
	shading
	x
	line

### 2.2 Well-formed diagrams

The set of well-formed diagrams, say  $\mathcal{D}$ , is the smallest set satisfying the following rules:

1. Any rectangle drawn in the plane is in set  $\mathcal{D}$ .
2. If  $D$  is in the set  $\mathcal{D}$ , and if  $D'$  results by adding a closed curve interior to the rectangle of  $D$  by the partial-overlapping rule (described below), then  $D'$  is in set  $\mathcal{D}$ .

<sup>4</sup>Refer to the diagrams in p. 311 or p. 313 of Charles Hartshorne and Paul Weiss.

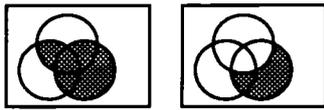
Partial-overlapping rule: A new closed curve should overlap *every* existant minimal region, but *only* once and *only* part of each minimal region.

3. If  $D$  is in the set  $\mathcal{D}$ , and if  $D'$  results by shading some entire region of  $D$ , then  $D'$  is in set  $\mathcal{D}$ .
4. If  $D$  is in the set  $\mathcal{D}$ , and if  $D'$  results by adding an X to a minimal region of  $D$ , then  $D'$  is in set  $\mathcal{D}$ .
5. If  $D$  is in the set  $\mathcal{D}$ , and if  $D'$  results by connecting existing X's by lines (where each X is in different regions), then  $D'$  is in set  $\mathcal{D}$ .

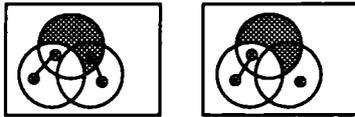
### 2.3 Rules of transformation

In this subsection, I aim to define what it is to obtain a diagram from some other diagrams. A process of transformation in this system is analogous to a process of derivation in deductive systems. We introduce the following six transformation rules into this system:<sup>5</sup>

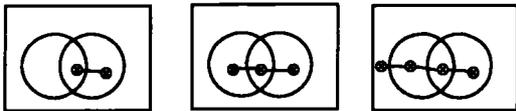
**R1:** *The rule of erasure of a diagrammatic object:* We may copy a *wfd* omitting one of the following diagrammatic objects: a closed curve or a shading or a whole X-sequence.



**R2:** *The rule of erasure of part of an X-sequence:* We may copy a *wfd* omitting any subpart of an X-sequence only if that part is in a shaded region.



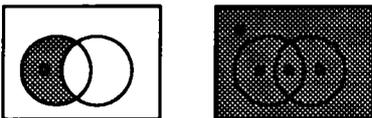
**R3:** *The rule of spreading X's:* If *wfd*  $D$  has an X-sequence, then we may copy  $D$  with  $\otimes$  drawn in some *other* region and connected to the existing X-sequence. Since we should get a *wfd*, the new  $\otimes$  should also be in a minimal region.



**R4:** *The rule of introduction of a basic region:* We may copy a *wfd* introducing a basic region.



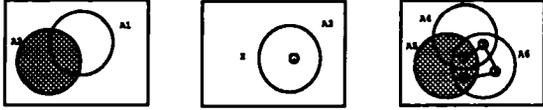
**R5:** *The rule of conflicting information:* If a diagram has a region with both a shading and an X-sequence, then we may transform this diagram to any diagram.



**R6:** *The rule of unification of diagrams:* We may unify two diagrams into one diagram. (The third

<sup>5</sup>One example is given for each rule. For a more accurate explanation of each rule, refer to Shin, *Valid Reasoning and Visual Representation*, p. 50-58.

diagram is the unfication of the first and the second diagrams.)



**Definition:** Let  $\Delta$  be a set of *wfds*. Let  $\Delta \rightsquigarrow D$  iff there is a rule of transformation such that it allows us to transform the diagrams of  $\Delta$  to  $D$ . Then, *Wfd*  $D$  is *obtainable* from a set  $\Delta$  of *wfds* ( $\Delta \vdash D$ ) iff there is a sequence of *wfds*  $\langle D_1, \dots, D_n \rangle$  such that  $D_n \equiv_{cp} D$  and for each  $k$  (where  $1 \leq k \leq n$ ) either (a) there is some  $D'$  such that  $D' \in \Delta$  and  $D' \equiv_{cp} D_k$ , or (b) there is some  $D'$  such that  $\exists_{\Delta'} (\forall_{D_i \in \Delta'} (i < k) \wedge \Delta' \rightsquigarrow D')$  and  $D' \equiv_{cp} D_k$ .

## 2.4 Semantics

This system aims to represent sets and certain relations among those sets. We intended to represent sets by means of regions, the emptiness of a set by means of a shading and the non-emptiness of a set by means of an X-sequence.

Let us define the set BRG to be the set of all basic regions of well-formed diagrams and  $U$  be a given domain. Let situation  $s$  be a function from the set BRG of all basic regions into the power set of domain  $U$ . Then  $s$  is defined as follows:

$$s: \text{BRG} \rightarrow \mathcal{P}(U), \quad \text{where}$$

- (i) if  $R$  is a basic region enclosed by a rectangle,  $s(R) = U$ ,
- (ii) if  $\langle A, B \rangle \in cp$ , then  $s(A) = s(B)$ ,

Using this definition of situation  $s$ , let us define function  $s'$  such that  $s'$  assigns sets to minimal regions in the following way:

$$s': \text{MRG} \rightarrow \mathcal{P}(U), \quad \text{where}$$

MRG is the set of minimal regions of *wfds*,

$$s'(A) = (s(A_1^+) \cap \dots \cap s(A_i^+)) - (s(A_j^-) \cup \dots \cup s(A_k^-)), \quad \text{where}$$

1.  $A_1^+, \dots, A_i^+$  are the basic regions  $A$  is part of, and
2.  $A_j^-, \dots, A_k^-$  are the basic regions  $A$  is not part of.

We extend this function to assign sets to regions and name it  $\bar{s}$ . That is,

$$\bar{s}: \text{RG} \rightarrow \mathcal{P}(U), \quad \text{where}$$

$$\bar{s}(A) = \bigcup \{s'(A_0) \mid A_0 \text{ is a minimal region which is part of } A\}$$

We can fix the interpretation of a shading or of an X-sequence across situations. Therefore, we define what it is for situation  $s$  to support infon  $\alpha$  [i.e.  $s \models \alpha$ ] in the following way:

$$s \models \langle \langle \text{Shading}, A; 1 \rangle \rangle \quad \text{iff} \quad \bar{s}(A) = \emptyset.$$

$$s \models \langle \langle \otimes^n, A; 1 \rangle \rangle \quad \text{iff} \quad \bar{s}(A) \neq \emptyset$$

Now, the consequence relation among diagrams can be defined as follows:

*Wfd*  $D$  follows from set of *wfds*  $\Delta$  [i.e.  $\Delta \models D$ ] if and only if every situation that supports every member of  $\Delta$  also supports  $D$ . [i.e.  $\forall_s (\forall_{D' \in \Delta} s \models D' \rightarrow s \models D)$ ]

## 3 Venn-II

Venn-I is extended so that this new system has a general way to convey disjunctive information. The following new formation rule is introduced:

6. If  $D_1$  and  $D_2$  are in the set  $\mathcal{D}$ , and if  $D'$  results by connecting these two diagrams by a straight line, then  $D'$  is in set  $\mathcal{D}$ . (We say  $D' = D_1 - D_2$ .)

The following new transformation rules are added.<sup>6</sup>

**R7:** *The rule of splitting X's*

**R8:** *The rule of the excluded middle*

**R9:** *The rule of connecting a diagram*

**R10:** *The rule of construction*

The semantics of Venn-II is formalized as an extension of the semantics of Venn-I. The set of well-formed diagrams is extended. We defined this set of *wfds* (i.e. atomic diagrams and compound diagrams) inductively. Accordingly, we want to define the support relation between situation  $s$  and diagram  $D$  inductively as well.

Situation  $s$  supports *wfd*  $D$ ,  $s \models D$ , iff

1. If  $D$  is atomic, then  $\forall \alpha \in RF(D)(s \models \alpha)$ .

2. If  $D$  is compound, say,  $D = D_1 - D_2$ , then  $s \models D_1$  or  $s \models D_2$ .

The definition for the consequence relation among diagrams is the same as in Venn-I:

*Wfd*  $D$  follows from set of *wfds*  $\Delta$  [i.e.  $\Delta \models D$ ] if and only if every situation that supports every member of  $\Delta$  also supports  $D$ . [i.e.  $\forall_s(\forall D' \in \Delta s \models D' \rightarrow s \models D)$  ]

## References

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<sup>6</sup>For the explanation and the examples of each rule, refer to Shin, *Valid Reasoning and Visual Representation*, Ch. 4.