Diagrammatic Reasoning about Allen's Interval Relations

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Abstract

We explore different ways in which a computational model of diagrammatic reasoning can capture imagery experiences which subjects reported in an experiment on spatial reasoning. The algorithmic reconstruction of this particular reasoning process suggests that computational models of diagrammatic reasoning should be formulated at different levels of abstraction. Four levels are distinguished – ordinal positioning, metrical positioning, pictorial merging and object depiction. Finally, consequences for the computational modelling of individual differences in imagery are discussed.

1 Introduction

This article addresses the issue of formulating a computational theory which describes a specific kind of reasoning, namely inferences about spatial interval relations. The data that the theory claims to account for comes from an experiment on spatial three-term series described in Knauff, Rauh & Schlieder (1995). A suitable framework for an algorithmic reconstruction of the reasoning process is provided by the theory of mental models (Johnson-Laird and Byrne, 1991; Evans, Newstead and Byrne, 1993). According to this theory, the premises of an inference task are integrated into a unified representation, the mental model, which is then inspected for drawing the conclusion.

There are four facts that a computational account within the framework of mental model theory should consider. (1) The existence of preferred mental models – section 3 summarizes the relevant empirical results. (2) The presence of order effects. As demonstrated in section 4 the process of model construction is sensitive to a special kind of change in premise order. (3) That a computational theory based on ordering information alone is able to account for the preferences. This theory, which has been described in detail in Schlieder (in prep.), operates on purely spatial representations. It does not assume that any metrical information is encoded or processed (section 5). While this is found to correspond well to the data on model preference it does not explain the last fact. (4) That the subjects reported having imagery experiences while being involved in reasoning. Psychologists have good reasons to be very careful with such introspective data, but is not the intention of this article to state a psychological hypothesis. We will instead explore a computational question. What alternative ways are there to extend a purely spatial model construction process into a pictorial one? Answers are provided in sections 6, 7 and 8.

2 Interval relations and three-term series reasoning tasks

A representational formalism that is now widely used in temporal and spatial reasoning has been described by James Allen (1983). It consists of a system of 13 relations that encode the relative position of two intervals on a line. Originally, the interval relations had a temporal interpretation, i.e. intervals represent events, but they were soon used also in spatial reasoning (Güsgen, 1989; Mukerjee & Joe, 1990, Hernández 1994). Fig. 1 illustrates all the situations that can be distinguished with respect to the linear order of the startpoints (s_X, s_Y) and endpoints (e_X, e_Y) of two intervals (X, Y). The point ordering is used to define the relations. We will write $X\ r\ Y$ to express that the relation r holds between the interval X and the interval Y.

A simple reasoning problem arises for configurations of three intervals, X, Y and Z, where two interval relations, $X\ r_1\ Y$ and $Y\ r_2\ Z$, are known and the third one is unknown. In general, r_1 and r_2 impose a restriction on the relation r_3 that may hold between X and Z. If *X overlaps Y* and *Y overlaps Z* then the relation between X and Z is restricted to one of the following: *X before Z*, *X meets Z* or *X overlaps Z*. To put it differently, there are only three ways to arrange the startpoints and endpoints of X, Y and Z into a linear order consistent with the definitions of r_1 and r_2. We will call the point orderings *models of the problem's premises* $X\ r_1\ Y$ and $Y\ r_2\ Z$. In each model a different interval relation r_3 holds between X and Z. Any of these relations is considered a correct answer, a *"conclusion" of the problem*. Thus, there is a one-to-one correspondence between models of the premises and conclusions.

name	relation	inverse	diagram	point ordering
equals	X = Y	Y = X		$s_X = s_Y < e_Y = e_X$
before	X < Y	Y > X		$s_X < e_X < s_Y < e_Y$
meets	X m Y	Y mi X		$s_X < e_X = s_Y < e_Y$
overlaps	X o Y	Y oi X		$s_X < s_Y < e_X < e_Y$
starts	X s Y	Y si X		$s_X = s_Y < e_X < e_Y$
finishes	X f Y	Y fi X		$s_Y < s_X < e_Y = e_X$
during	X d Y	Y di X		$s_Y < s_X < e_X < e_Y$

Fig. 1: The 13 interval relations, adapted from Allen (1983). Diagrams and point orderings for the inverse relation are obtained by exchanging the roles of X and Y.

The problem considered is closely related to a type of inference task that has been investigated since long in the psychology of reasoning: the three-term series task (Hunter, 1957; DeSoto, London & Handel, 1965; Huttenlocher, 1968; Johnson-Laird, 1972). We will call therefore call the task of finding a conclusion X r_3 Z for given premises X r_1 Y and Y r_2 Z, where r_1, r_2 and r_3 are interval relations, the *Allen three-term series* (X r_1 Y, Y r_2 Z) $\rhd$ X r_3 Z. A total of 12 x 12 = 144 Allen three-term series can be build from the interval relations, not counting series in which the trivial relation *equal* appears in the premises. It turns out that half of the problems have a single model (conclusion). The 72 multiple-model problems fall into four classes: those with 3, 5, 9 and 13 models (conclusions).

3 Empirical evidence for preferred mental models

A general assumption underlying the mental model account of inference is that of premise integration. Mental model theory assumes that the information conveyed by the premises of a reasoning task is integrated into a unified representation, the *mental model.* For Allen three-term series it is possible to state precisely what kind of spatial information the mental model will have to encode: a specific linear order on the startpoints and endpoints of the three intervals, in other words, what we called a (logical) model of the problem's premises. However, the mental model is not restricted to encode just the point ordering. Additional spatial information may be represented. This would be the case if a mental image of the interval configuration (which specifies the relative size of the intervals) is used as mental model.

According to mental model theory two processes are involved in solving an Allen three-term series task. Firstly, a model construction process which integrates the premises, secondly, a model inspection process which determines the conclusion that holds in the model. As we have seen, certain reasoning tasks allow for the construction of different models (conclusions). Knauff, Rauh and Schlieder (1995) have conducted an experiment on Allen three-term series which provides evidence that in reasoning not all these mental models play the same role. Since the present article is concerned with the problem of specifying the computational theory of the model construction process, the experimental method and procedure is only briefly described.

The experiment was devided into three consecutive phases. In the *definition phase* the subjects read definitions of the interval relations. They were also showed pictures with intervals in an appropriate position. A *learning phase* followed in which one-sentence descriptions of premises (e.g. "The red interval lies to the left of the blue interval") were presented and the subjects had to graphically localize one interval with respect to the other on a computer screen. The learning phase lasted as long as it took the subject to reach a certain success criterion. In the *inference phase* of the experiment subjects had to solve all 144 Allen three-term series tasks that can be formed without using the *equal* relation. Presentation in the inference phase was randomized. The premises of the three-term series were displayed verbally using their one-sentence descriptions

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o	<	<	<	<	m	o	o	o	o,d	=	oi	>
fi	<	m	o	fi	di	di	o	d	=,f	oi	oi	>
di	<	o	o	di	di	di	o	=	oi	oi	oi	>
si	[illegible]	o	o	di	di	si	=	d	oi	oi	mi	>
s	<	<	o	o	fi	si	s	d	d	oi	mi	>
d	<	<	o	o	=	oi	d	d	d	oi	>	>
f	<	m	o	fi	di,oi	oi	d	d	f	oi	>	>
oi	[illegible]	o	=	oi	mi	mi	d	oi	oi	>	>	>
mi	[illegible]	si	oi	mi	>	>	oi	oi	mi	>	>	>
>	=	oi	oi	>	>	>	oi	>	>	>	>	>

Tab. 1: Table of empirical model preferences, from Knauff, Rauh & Schlieder (1995).

and the conclusion had to be characterized graphically by the subjects just as in the learning phase.

Tab. 1 summarizes the results of the experiment. The preferred conclusion (model) for an Allen three-term series with the premises X r_1 Y and Y r_2 Z is found in the row indexed by r_1 and the column indexed by r_2. Three cells of the table contain two relation symbols. In those cases two preferred conclusions with equal answer frequency were found. It is interesting to note that there was no preferred answer which was not a conclusion. In other words, the table of the empirical model preferences is a "simplification" of Allen's original composition table: for the single-model entries (white cells in Tab. 1), both tables are identical and for the multiple-model entries (shaded cells in Tab. 1), the table of empirical model preferences selects one out of the several alternative models. Note that such a simplification has dramatic consequences. The meaning of the interval relations (i.e. their definition in terms of point orderings) is tied to the original table. Changing the table changes the meaning of the interval relations!

4 Order effects on the construction of preferred mental models

From a logical point of view none of the models of an Allen three-term series is more preferable than the others. An explanation for the empirical model preference must be sought in the cognitive processes that construct the mental model. A seemingly simple way to explain the preferences consists in assuming that the subjects used some kind of metrical prototypes of the relations to build their mental model. As Fig. 2 illustrates, such prototypes would specify the distances by which intervals of unit length standing in a specific interval relation must overlap or separate. We will assume that the distance parameters d_1, d_2, d_3 and d_4 are chosen in a way to achieve the best fit with the data.

A classical imagery account of spatial reasoning which postulates the existence of pictorial prototypes for relations would be very close to this type of parametrized theory. In fact, most of the subjects reported imagery experiences. There are several sources where the distance information needed to build up mental images could be drawn from. Possibly, the subjects used the distances of the illustrations of the relations that were presented to them in the definition phase. Of course, the model preferences would then have to be considered an experimental artifact with a trivial cognitive explanation. We will find, however, that no distance-parametrized theory of the described type can give a satisfactory account of the preferences found.

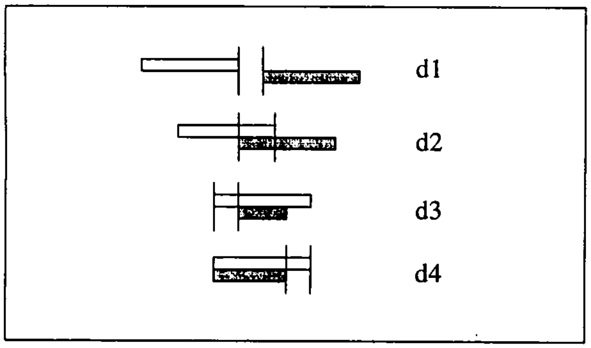


Fig. 2: Metrical relation prototypes

If we choose the parameters in a way that accounts for the preferred model of a specific Allen three-term series task this choice has immediate consequences on the predictions for a number of other three-term series. There is a general dependency between three-term series problems due to symmetry. The relevant group of symmetry transformations is known from the analysis of the computational properties of the interval relations (see Ligozat, 1990). This group is generated by two transformations which we will call *reorientation* and *transposition*. When diagrams of interval configurations similar to those in Fig. 3 are used as models then the symmetry transformations can be given a simple geometrical interpretation. Reorientation is equivalent to reflection of the diagram about the horizontal axis and transposition is equivalent to reflection about the vertical axis. For some problems applying reorientation or transposition amounts to the same. That is to say, a maximum of four models can be generated from a model by applying the symmetry transformations.

As can easily be seen, the distance-parametrized account of model construction predicts symmetry with respect to both symmetry transformations, reorientation and transposition. This twofold symmetry is not reflected in the data. The empirical model preferences may show the first kind of symmetry (4 symmetry violations) but certainly not the second (13 symmetry violations).

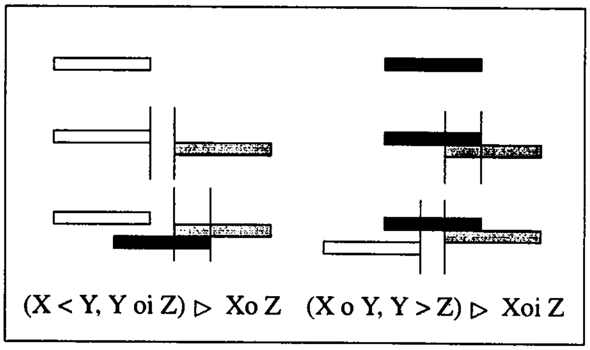


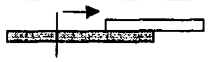
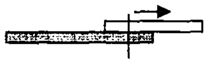
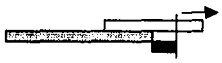
Fig. 3: Example of an order effect

Fig. 3 illustrates the consequences of the violation of transposition symmetry for the distance-parametrized theory of model construction. Both three-term series are related by transpositions symmetry. The left model is constructed by first using the distance d_1 then d_2. While this model corresponds to the empirical model preference the right model does not. It is build by using the distances in the inverse order: first d_2 then d_1. Obviously, the order in which the distances are processed matters for the empirical model construction process. We will therefore call the remarkably high number of violations of transposition symmetry violations the *order effect*.

No matter how the distance parameters are chosen in this computational theory, the order effect will never be reproduced. We may therefore say that no such theory can give a satisfactory account of the empircal data. A process with more dynamic behavior is needed to reproduce the effect: the result of integrating a premise into the mental model should somehow depend on the structure of the model that has been build up so far.

5 A computational theory based on ordering information

It is common practice to distinguish different types of spatial information according to the degree of determination. A relation conveys *metrical information* if it can be defined in terms of metrical invariants, e.g. distances between points. If the relation is already definable in terms of the linear order relation on a finite set of points we will say that it encodes *ordering information* – this is just a working definition limited to the one-

model construction	image generation
$\blacktriangleleft\ s_Y < s_X < e_Y < e_X \blacktriangleright$	
$\blacktriangleleft\ s_Y < s_X < e_Y < e_X \blacktriangleright$	
$\blacktriangleleft\ s_Y < s_X < e_Y = s_Z < e_X \blacktriangleright$	
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Tab. 2: Model construction and image generation

dimensional case. In this sense, the interval relations encode ordering information about the relative position of intervals. It should be mentioned that the relations are sometimes classified among the topological relations. (We will not bother with conceptual issues here – but for a general definition of n-dimensional ordering information and a discussion of the differences to topological information see Schlieder, 1995).

We have seen that a computational theory of the model construction process based on metrical information failed to explain the empirical preferences. This raises the question of whether a more abstract account of model construction which uses a less specific type of spatial information could be more successful. Schlieder (in prep.) formulates such a computational theory of the model construction process in terms of ordering information. We cannot go into the details of this theory, but we will state some of its central representational assumptions.

The mental model is encoded by means of a *point ordering representation* in which points are the representational primitives. They occur in two types, as *startpoints* and *endpoints* of intervals. Only the direct succession and the identification of points is represented explicitly. These relations are encoded by two kinds of relational elements, *identification links* and *successor links*. A peculiarity of the representation is the *spatial focus*. Modifications of the representation always occur at the focus position. In order to insert a point, the focus has to be moved to the appropriate place, an operation which is called *scanning*. Essentially, the computational theory describes the sort of scanning which is required to insert the startpoint and the endpoint of the interval for each Allen relation. The left column of Tab. 2 illustrates the kind of scanning and insertion steps that may occur during model construction. Only the integration of the second premise during the three-term series task (X oi Y, Y m Z) $\triangleright$ X o Z is shown.

Most empirical model preferences are matched by the theoretical model construction process but there are others the theory cannot reproduce. Assuming that strategy use accounts for the three-term series which arise from the composition of inverses, and assuming further that the model construction process does essentially work the same way in left-to-right direction as in right-to-left direction, the computational theory is left with only 2 out of the remaining 127 preferences which cannot be reproduced. It may therefore be considered descriptively adequate. Since this account of the model construction process also predicts the preferred models for Allen four-term series (generally: n-term series) it could easily be falsified.

Descriptive adequacy and predictive power of a theory will probably not compensate for a lack of explanatory appeal. One does not only want to know "how" but also "why" preferences in mental model construction arise. To put the answer into a brief though necessarily sketchy form: the mental model is constructed in such a way that further information can "easily" be integrated into it. Integration of a premise is considered easy if the required focus movements are realized by search operations which are computationally elementary in a sense that can be precisely defined. It seems for instance that model construction follows what could be called the *principle of linearization*. We will consider an interval configuration to be linearized if for all startpoints and endpoints holds: $s_X < s_Y$ if and only if $e_X < e_Y$. If a configuration is linearized then not only the points but also the intervals themselves can be brought into a linear order. The principle of linearization states that linearized models are preferred over non-linearized models. Fig. 4 illustrates the principle for the models of the example from Tab. 2. A diagrammatic notation for the models is used but the principle's definition is purely ordinal.

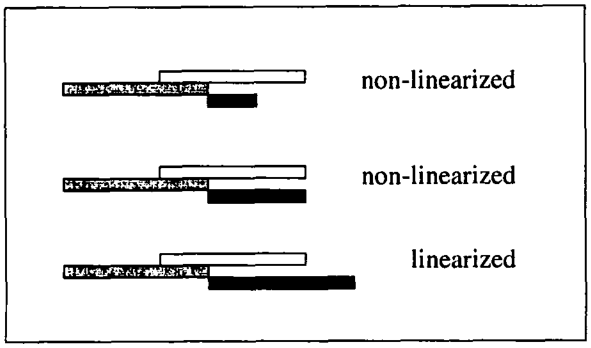

Fig. 4: The linearization principle

6 Model construction and image generation

Reasoning based on mental models bears some resemblance to diagrammatic reasoning. In both cases a spatial representation is build up from problem information and inspected to find a solution. In principle, mental images could be used as mental models in spatial reasoning tasks such as the three-term series. Mental model reasoning would then simply be a kind of diagrammatic reasoning. From a computational perspective it seems a minor task to extend the algorithmic description of the model construction process into a description of an image generation process, that is, a process which operates on a *pictorial representation*.

We will not need a precise definition of pictorial representations here, because, for our purposes it suffices to refer to just one of the defining features: a pictorial representation encodes metrical information. To simplify the discussion we will only consider cases in which the metrical information is completely determined. Fragmented or partial images are thus excluded. Even without this restriction, the point ordering representation introduced in the preceeding section cannot be considered a pictorial representation since it does not represent any metrical information at all. To transform this spatial representation into a pictorial representation, one has to specify the distances between all represented points. Since metrical information was found not to be relevant for model construction, any way of providing these distances will do.

As illustrated in Tab. 2 such an *image generation process* parallels model construction exactly. In a sense this is just the opposite of what happens in Anderson's (1978) famous mimicry argument where operations on an analogical representation are simulated by operations on a propositional representation. The right column of the table shows one of the many realizations of the image generation process compatible with the model construction process.

How convincing is such a diagrammatic reasoning account of answer preferences for Allen inferences? Nothing is gained with respect to explanatory power. Image generation produces exactly the same results as model construction. The only difference lies in the fact that image generation operates on a representation which is metrically determined and thus compatible with the imagery experience which was reported by almost all (31 of 33) subjects in the experiment. Generally, cognitive psychologists adopt a sceptical attitude towards such introspective or phenomenal data:

> "The tokes of mental models may occur in a visual image, or they may not be directly accessible to consciousness. What matters is, not the phenomal experience, but the structure of the models. This often transcends the perceptible."
> Johnson-Laird & Byrne (1991, p. 39)

Following the principle of explanatory parsimony, representational assumptions should be linked to what proves to be causally effective in experiments. In our case, the ordering information, not the metrical information, is causally effective. One could therefore argue that the imagery experience which accompanies model construction constitutes an epiphenomenon. However, the question that we will address is computational rather than psychological: Is the "mimicry" of model construction by image generation really that trivial?

7 Levels of representation in diagrammatic reasoning

Looking at the trace in Tab. 2 gives some impression of the alternative ways in which metrical information can be integrated into the model construction process. One finds that different sets of parameters have to be specified to obtain a metrically determined representation: the horizontal position of startpoints and endpoints, the vertical layout of the whole configuration and finally, the shape of the intervals. There are many ways to set the parameters, but the decisions one takes are not completely independent. It turns out, for instance, that not every layout is compatible with every shape. Obviously, the task of assigning metrical information is not as trivial as it first appeared. To characterize dependencies such as the one mentioned, it is useful to distinguish four *levels of specificity* at which the image generation process can be described: ordinal positioning, metrical positioning, pictorial merging and object depiction. Every level specifies some spatial information which is not determined at the previous one, the least specific being the level of ordinal positioning and the most specific, the level of object depiction. They will be discussed individually in the following passages.

Ordinal positioning schemes

The *level of ordinal positioning* is the level at which the model construction process is described in Schlieder (in prep.). Processes at this level are characterized as operating on representations which encode ordering information. If this is sufficient to account for the empirical data, then the process is often considered to be spatial rather than pictorial. However, one should not mistake the descriptive levels which are levels of abstraction for levels of implementation. It is conceivable that a genuine pictorial process is best described at the level of ordinal positioning. Such would be the case if further experiments on model inspection in Allen three-term series tasks provided evidence for a metrically determined representation. Model construction would then still have its best (most parsimonious) description at the level of ordinal processing.

Metrical positioning schemes

Basic metrical information about the location of pictorial objects is provided at the *level of metrical positioning*. In our case, the pictorial objects are intervals. The location of each interval is made explicit by fixing the distance between neighboring points in the linear point ordering. As was already observed, any values may be assigned to these distances – the data from the experiment does not constrain them. Fig. 5 shows three possible schemes for the metrical positioning of a point at the end of the interval configuration. The position could for example be determined without regarding any other point, as in the first scheme where the point is located at unit distance from the last point. Other schemes determine the point's position by referring to another point, its predecessor (scheme 2) or the corresponding startpoint (scheme 3). It is not claimed that these are psychologically plausible schemes for metrical positioning. They only serve to illustrate that different schemes generally lead to different predictions of the point's location. It should therefore not prove too difficult to devise experiments on model inspection which discriminate between alternative positioning schemes in model construction

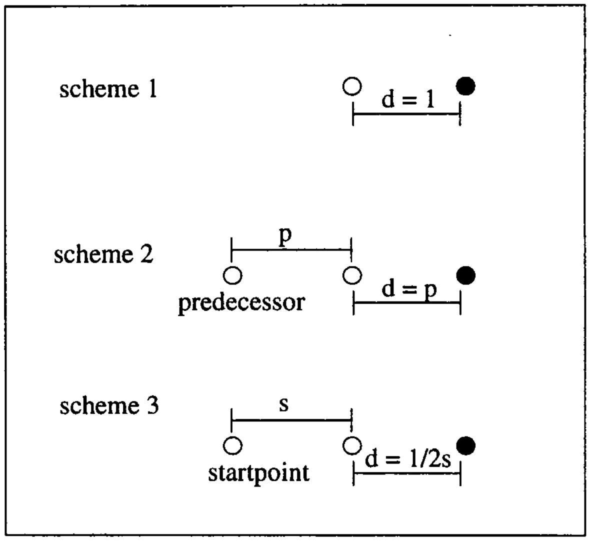

Fig. 5: Metrical positioning schemes

Pictorial merging schemes

Although the location of the intervals is now metrically determined, the complete image cannot be generated yet. In general, overlapping locations will be assigned to the intervals and it is not clear how to combine the pictures of the single intervals into a picture of the whole configuration. A scheme for *pictorial merging* defines how to combine object depictions sharing a location. The simplest way to merge pictures consists in superposing them (graphical XOR operation). Scheme 1 in Fig. 6 realizes superposition. Obviously, the correspondence of startpoints and endpoints may get lost under this merging scheme or, to put it differently, the intervals are not uniquely identifiable in the merged picture. But this is exactly what the description of the model construction process at the ordinal positioning level requires – therefore, we may not use scheme 1 for merging pictures. The other two merging schemes guarantee that the intervals remain identifiable either by shifting the interval shape in vertical direction or by scaling it. Pictorial merging is certainly not involved in every kind of imagery. However, it seems to play an important role in spatial reasoning whenever configurations of entangled objects have to be visualized. In many of these cases the superposition scheme will work, but, as we have seen, not in all cases.

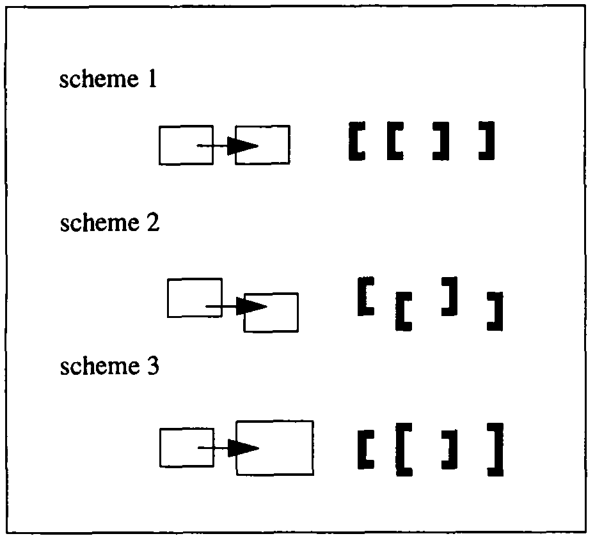

Fig. 6: Pictorial merging schemes

Object depictions

Finally, the exact shapes of the pictorial objects are needed to generate the image. If a pictorial process is described at the *level of object depictions* then these shapes are specified in the form of a scalable pattern. However, generally the object depictions may not be chosen in a completely arbitrary fashion. This even holds if it is not necessary, as in our case, to recur to the level of object depictions for giving an adequate account of the empirical data. One constraint derived from the description at the level of ordinal positioning is that a point must be (locally) recognizable as a start-point or endpoint from the object depiction. Depictions 2 and 3 meet this constraint, depiction 1 does not. There also are constraints linking the level of object depiction to the level of pictorial merging. Merging scheme 1, which could not generally guarantee that intervals remain identifiable, may very well work with certain kinds of object depictions. If instead of the rectangular shape of depiction 3 an ellipsoid shape is used, then point correspondence can be established even with the superimposition merging scheme.

8 Conclusions

Distinguishing the four levels of specificity has an important methodological consequence: computational models of diagrammatic reasoning do not need to be specific up to the level of object depictions. Such *abstract imagery* is useful in cognitive modelling. Less specific models have the advantage that they allow the abstraction of individual differences in imagery. Even if we (naively) assume that at the level of biological implementation images are completely determined in metrical respects we cannot conclude that the suitable level of abstraction for describing image generation is the level of object depictions. Different subjects may generate different images without these differences becoming causally effective in the experimental setting under consideration. Before deciding on the intriguing question of whether or not a certain mental representation can be considered pictorial, the representation has to be studied in connection with various processes – generation, inspection and transformation. For pictorial representations, metrical information should be causally effective in at least some of these processes.

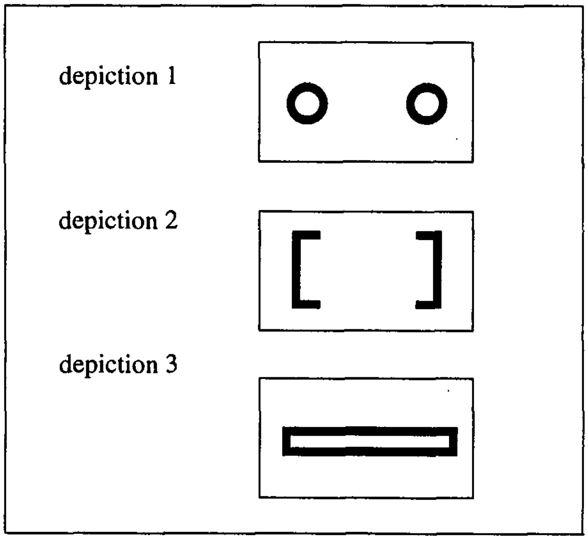

Fig. 7: Object depictions

A separation of representational levels is also necessary to state more precisely in what way diagrammatic reasoning is computationally efficient. For instance, it is well known from computational geometry that many of the asymptotically optimal algorithms rely on a preprocessing phase which makes the ordinal positioning of point data explicit. A good example is the sweep line algorithm which preprocesses the data by sorting points according to their orthogonal projection on a coordinate axis. What happens during model construction or image generation in Allen inferences is just a kind of preprocessing which determines (or enforces) a linear order on point data. It would be a strong argument in favor of diagrammatic reasoning if computational advantages would not just appear on the level of ordinal positioning but also on higher levels.

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