Spatial and Temporal Reasoning: Beyond Allen's Calculus

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Abstract

Temporal knowledge representation and reasoning with qualitative temporal knowledge has now been around for several decades, as formalisms such as Allen's calculus testify. Now a variety of qualitative calculi, both temporal and spatial, has been developed along similar lines to Allen's calculus. The main object of this paper is to point to open questions which arise when, leaving the now well-chartered waters of Allen's, we venture into rougher sea of these formalisms. What remains true among the properties of Allen's calculus? Partial answers are indeed known, but numerous new problems also arise. We try to point to the main issues in this paper.

Introduction

Temporal knowledge representation and reasoning with qualitative temporal knowledge has now been around for several decades, as tense logics and formalisms such as Allen's calculus testify. Applications are numerous, including natural language processing, scheduling, planning, database theory, diagnosis, circuit design, archaeology and genetics. Representing and reasoning about spatial knowledge in a qualitative way has developed more recently, as previous work had mainly considered spatial knowledge from a purely quantitative way. In many applications, such as robotics, geographic information systems, and computer vision, the main motivations for qualitative approaches are robustness (robotics), naturalness and user-friendliness (GISs), and high-level adequacy (computer vision).

Many questions have been raised for each individual formalism, and some have been answered in each particular case. Basic problems pertain to the following issues:

- Develop suitable languages of representation for temporal and/or spatial knowledge.
- Propose methods for managing and reasoning about that knowledge. In particular, maintain consistent knowledge bases, and answer queries.
- Investigate the computational cost of basic operations on the knowledge basis, such as testing for consistency.
- Characterize the models (in a logical sense) of a given calculus. More precisely, if the calculus can be expressed as a first order theory, describe all models of that theory.

As a case in point, consider the formalism proposed by Allen in 1983 (Allen 1983):

- The language uses 13 basic relations between temporal objects interpreted as intervals. The intended interpretation is in terms of intervals understood as ordered pairs of time points on the time line. In logical terms, the corresponding first order theory has 13 dyadic predicates satisfying suitable axioms.
- Allen's main innovation was to propose to reason about intervals using constraint networks, and providing explicit algorithms for doing so: the knowledge about a (finite) number of temporal intervals is expressed in terms of a network whose nodes stand for the intervals, and where arcs are labeled by disjunctions of the basic relations. The basic reasoning mechanism operates by propagating constraints using the operation of *composition* of relations.
- In the same paper where he introduced the formalism, Allen proposed an algorithm which maintains path-consistency of the network, which is a necessary condition for consistency. He also exhibited a network which, although it is path-consistent, is not consistent. An important body of work since that time has been devoted to determining under what conditions path-consistency would imply consistency, a question which has been completely solved by Nebel and Bürckert (Nebel & Bürckert 1995) and Drakengren and Jonsson (Drakengren & Jonsson 1998).
- Concerning the complexity of testing consistency, early results of Vilain and Kautz (Vilain & Kautz 1986) (completed by van Beek (van Beek 1990; van Beek 1992)) showed that the full problem of consistency is NP-complete. As a consequence of Nebel and Bürckert and others' results mentioned above, subclasses of Allen's algebra are either NP-complete, or tractable by using the path-consistency method, that is, in cubic time.
- Allen's composition table for the 13 basic relations embodies a "strong" logical theory. Ladkin and Maddux, in particular (Ladkin & Maddux 1994), emphasized the fact that it can be interpreted in algebraic terms. Using logical methods (quantifier elimination) Ladkin (Ladkin 1987) proved that this theory is in fact syntactically complete. This implies for instance that the only countable

model of the theory, up to isomorphism, is the set of intervals on the rational numbers.

As far as the problems we raised at the outset are concerned, then, navigating in the framework of Allen's calculus is navigating in smooth waters. Now a variety of qualitative calculi, both temporal and spatial, has been developed along similar lines to that calculus. Here is a tentative list of calculi:

1. On the temporal side:
   (a) Generalized intervals (Ladkin, Ligozat, Osmani, Condotta (Ladkin 1986; Ligozat 1991b; Balbiani *et al.* 1998; Balbiani, Condotta, & Ligozat 2000b)).
   (b) Partially ordered time (Anger, Mitra and Rodriguez (Anger, Mitra, & Rodriguez 1998))[1].
   (c) Cyclic intervals (Balbiani, Osmani (Balbiani & Osmani 2000)).
   (d) The INDU calculus (Pujari and Sattar (Pujari & Sattar 1999))[2].
2. On the spatial side:
   (a) RCC calculi (RCC5 and RCC8, Randell, Cui and Cohn (Randell, Cui, & Cohn 1992)).
   (b) n-dimensional point calculi (Balbiani, Condotta (Balbiani & Condotta 2001)).
   (c) Rectangle and n-block calculi (Balbiani, Condotta).
   (d) The oriented interval calculus (Renz (Renz 2001)).
   (e) $2n$-star calculi (Mitra (Mitra 2002)).

The main object of this paper is to point to open questions which arise when, leaving the now well-chartered waters of Allen's, we venture into rougher sea. What remains true among the facts listed above for Allen's calculus? Partial answers are indeed known, but numerous new problems also arise. We try to point to the main issues in what follows.

Looking for general methods We understand this paper as a contribution to the more general context of devising general methods in the field of qualitative spatial and temporal reasoning. We claim that such general methods are to be developed inside the framework of general algebraic methods, as exemplified by Ladkin and Maddux (Ladkin & Maddux 1994), Ligozat (Ligozat 1998b), Krokhin and Jeavons (Krokhin & Jeavons 2001) (as opposed to computer-based calculations). Geometrical and topological methods have also proved to be useful, as shown by the geometrical characterizations of tractable classes, see Balbiani, Condotta, Ligozat (Balbiani, Condotta, & Ligozat 2000a). The structure of the paper is as follows: Section 1 introduces the basic algebraic properties of the calculi, as well as the notion of a weak representation, which plays a central role in the sequel. Section 2 discusses the relationships between weak representations (which generalize the familiar notion of scenarios) and configurations in a domain of interpretation. Finally, Section 3 discusses the models of the strong theories in terms of representations of the corresponding algebras.

[1]This calculus has four basic relations, which may be denoted by $\prec$ (precedes), $\succ$ (follows), eq (equality), and $||$ (not related).

[2]The INDU calculus refines Allen's calculus by using basic relations which denote the relative size of the intervals. For instance, the m relation splits into three relations: $x \mathsf{m}_{>} y$, $x \mathsf{m}_{=} y$, $x \mathsf{m}_{<} y$, which mean that $x \mathsf{m} y$ and that the length of x is strictly greater, equal or smaller than that of y, respectively. On the other hand, since $x \mathsf{s} y$ implies that the length of x is strictly less than that of y, there is only one relation $\mathsf{s}_{<}$.

Calculus: The algebra behind the scenes

Algebraic properties of qualitative spatial and temporal reasoning

The calculi we consider share most or all of the following features:

1. The calculus is based on a set $\mathbf{B}$ of basic relation symbols denoting temporal or spatial binary qualitative relations between some entities.
2. The set of basic relations is JEPD (jointly exhaustive and pairwise disjoint).
3. Exchanging roles is expressed by a conversion operator.
4. Composition of knowledge is expressed by a composition operation.
5. Indefiniteness (which inevitably arises in most cases when composing knowledge) can be represented by disjunctive relations (subsets of $\mathbf{B}$).
6. The Boolean algebra $\mathbf{A}$ of subsets of $\mathbf{B}$, augmented by conversion and composition (with suitable elements as neutral elements for composition) is a *relation algebra* in the sense of Tarski (Tarski 1941).

Relation algebras

For all the formalisms we consider here, $\mathbf{A}$ is the set of subsets of $\mathbf{B}$, hence it is a Boolean algebra. The operation of transposition sends each basic relation on a basic relation (notice that in the case of Mitra's star relations, this is only true for an even number of sectors in the plane, as observed by Mitra).

The composition table of each calculus results from the necessary conditions which are valid in a standard interpretation of the calculus: for the temporal calculi 1(a), the entities of this standard interpretation are generalized intervals of the real line; the composition table for partially ordered time expresses conditions valid in any partial order; for cyclic intervals, a standard model is the real circle; for the INDU calculus, again, the interpretation is based on intervals on the real line.

In the case of the composition tables of RCC5 and RCC8, various standard models for determining the composition table have been considered (Düntsch 1999), such as e.g. circles in the plane.

Besides transposition and composition, a relation algebra requires a unit element for composition $\mathbf{1'}$. This element is the equality relation, which is a basic relation (an atom of the Boolean algebra) when all entities considered are of the same type. Again, notice that this last fact is no longer true when the entities are of various types. For instance, in the case of the algebra of generalized intervals $\mathbf{A_{1,2}}$ corresponding to time points and intervals on the time line (Ligozat 1991a), $\mathbf{1}'$ is the sum of two atomic relations: $\mathbf{1}'_{\mathbf{1},\mathbf{1}}$ (equality between time points) and $\mathbf{1}'_{\mathbf{2},\mathbf{2}}$ (equality between time intervals).

Recall the general definition of a relation algebra:

Definition 1 *A relation algebra* $\mathbf{A} = (A, +, \mathbf{0}, \cdot, \mathbf{1}, \circ, \mathbf{1}', \smile)$ *is a Boolean algebra* $(A, +, \mathbf{0}, \cdot, \mathbf{1})$ *together with a unary operation of converse (denoted by* $\alpha \mapsto \alpha^{\smile}$*), a binary operation of composition (denoted by* $(\alpha, \beta) \mapsto (\alpha \circ \beta)$*), and a distinguished element* $\mathbf{1}'$*, such that the following conditions hold:*

1. *For all* $\alpha, \beta, \gamma, \quad (\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$*;*
2. $\mathbf{1}'$ *is a unit element for composition:* $(\alpha \circ \mathbf{1}') = (\mathbf{1}' \circ \alpha) = \alpha$*, for all* α*;*
3. *For all* α*,* β *and* γ*, the following conditions are equivalent:*
   $(\alpha \circ \beta) \cdot \gamma = \mathbf{0}$*;* $(\alpha^{\smile} \circ \gamma) \cdot \beta = \mathbf{0}$*;* $(\gamma \circ \beta^{\smile}) \cdot \alpha = \mathbf{0}$

Relation algebras were introduced by Tarski in order to axiomatize the structural properties of binary relation algebras (BRA), whose elements are actual binary relations, with transposition, binary relation composition and the identity relation as a neutral element.

The algebras of qualitative spatial and temporal reasoning

Is $\mathbf{A}$ actually a relation algebra? In all cases where standard interpretations of the calculus are based on entities on the real line, it is a relatively easy matter to check that the composition table is indeed the composition table of an algebra of relations: in other terms, the algebra can be realized as an actual subset of the set of binary relations on some set (in algebraic terms, it is a representable algebra). For instance, thinking of Allen's algebra as a typical instance, it is easy to check that it is isomorphic to the algebra of binary relations between rational intervals, where a rational interval is a pair (q_1, q_2) in $\mathbf{Q} \times \mathbf{Q}$ with $q_1 < q_2$. The hard (although easy) part consists in checking that the composition table expresses necessary *and* sufficient conditions. To take an easy example, the table asserts that $\mathsf{p} \circ \mathsf{p}$ is p. The necessary part asserts that if an interval x precedes another interval z, and if z precedes y, then x precedes y. The sufficient part asserts that, if x precedes y, then there exists an interval z such that x precedes z and z precedes y: this is true only because the real line is a dense ordering.

Similarly, the algebras 1(a), 2(b-f), which are directly based on entities on the time line, can be shown to be relation algebras. For 1(c), the calculus of cyclic intervals, similar arguments obtain, using the density of the real circle. The same is true for 2(a) (RCC5 and RCC8), because suitable interpretations exist (Gotts 1996; Düntsch 1999).

Partial orderings

Proposition 1 *The algebra of the calculus of partial orderings is a relation algebra.*

The composition table of the calculus of partial ordering is given as Table 1.

$\circ$	eq	$\prec$	$\succ$	$\|\|$
eq	eq	$\prec$	$\succ$	$\|\|$
$\prec$	$\prec$	$\prec$	$\mathbf{1}$	$\{\prec, \|\|\}$
$\succ$	$\succ$	$\mathbf{1}$	$\succ$	$\{\succ, \|\|\}$
$\|\|$	$\|\|$	$\{\prec, \|\|\}$	$\{\succ, \|\|\}$	$\mathbf{1}$

Table 1: Composition in $\mathbf{M_4}$

Proof We claim that $\mathbf{Q}^2 = \mathbf{Q} \times \mathbf{Q}$, the set of pairs of rational numbers equipped with the product ordering, which is a partial ordering, provides an algebra of relations whose composition table is Table 1.

Let $R = \{(x, y) \in \mathbf{Q}^2 \times \mathbf{Q}^2 \mid x < y\}$, Let $\Delta = \{(x, y) \in \mathbf{Q}^2 \times \mathbf{Q}^2 \mid x = y\}$, and $N = \mathbf{Q}^2 \times \mathbf{Q}^2 \setminus (R \cup R^t \cup \Delta)$.

Interpreting $\prec$, $\succ$, eq, and $||$ as R, R^t, and N, respectively, we have to check that Table 1 describes the properties of the four relations R, R^t, Δ and N.

We have to check that all the necessary properties hold. Part of them result from the fact that Q itself is dense and unbounded. Obviously also, as the product of two total orderings, $\mathbf{Q}^2$ is a (distributive) lattice. Hence each pair in $\mathbf{Q}^2$ has a least upper bound and a greatest lower bound. The following facts have to be checked:

1. Necessary conditions:
   (a) $R \circ R \subseteq R$: This is true because of transitivity.
   (b) $R \circ N \subseteq (R \cup N)$: Again because of transitivity. If $R(x, y)$ and $N(y, z)$ hold, $R^t(x, z)$ would imply that $R(z, y)$, a contradiction.
   (c) $R \circ R \subseteq (R \cup R^t \cup \Delta)$ adds no constraint.
   (d) The cases of $R^t \circ R$, $R^t \circ R^t$, $R^t \circ N$ are treated analogously.
2. Sufficient conditions:
   (a) $R \circ R \supseteq R$: This is true because of density.
   (b) $R \circ R^t \supseteq R$: Because the partial ordering is unbounded on the right.
   (c) $R \circ R^t \supseteq R^t$: Again, because the partial ordering is unbounded on the right.
   (d) $R \circ R^t \supseteq N$: Because every pair of elements has a least upper bound.
   (e) $R \circ R^t \supseteq \Delta$: Again, because the partial ordering is unbounded on the right.
   (f) $R \circ N \supseteq R$: Let $x = (x_1, x_2), y = (y_1, y_2)$ in $\mathbf{Q}^2$ such that $R(x, y)$. Then at least one of $x_1 < y_1$ or $x_2 < y_2$ holds. Assume for instance that $x_1 < y_1$. Choose $z = (z_1, z_2)$ such that $x_1 < z_1 < y_1$ and $z_2 > y_2$ (Fig. 1 (a)). Then $R(x, z)$ and $N(z, y)$.
   (g) $N \circ R \supseteq N$: Suppose for instance that $x_1 > y_1$, while $x_2 < y_2$. Choose $z_1 < y_1$ and $x_2 < z_2 < y_2$ (Fig. 1 (b)). Then $N(x, z)$ and $R(z, y)$.
   (h) The remaining facts have similar proofs.

Notice that analogous considerations could be applied to the partial orderings $\mathbf{Q}^n$, for $n > 1$. This raises the question whether the resulting interpretations are isomorphic as partial orderings.

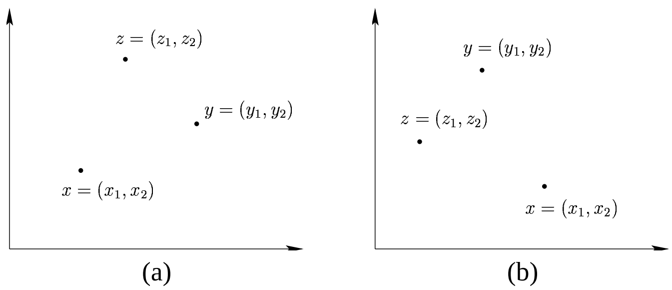

Figure 1: (a) $R(x,z)$ and $N(z,y)$, (b) $N(x,z)$ and $R(z,y)$

The algebra of the INDU calculus The algebra of the INDU calculus, however, is *not* a relation algebra: the axiom of associativity is not true. The following is a counter-example:

Consider the three atomic relations $\mathsf{p}_{>}^{\smile}$, $\mathsf{m}_{>}^{\smile}$, $\mathsf{m}_{>}$. It can easily be checked that:

$$(\mathsf{p}_{>}^{\smile} \circ \mathsf{m}_{>}^{\smile}) \circ \mathsf{m}_{>} = \{\mathsf{o}_{>}^{\smile}, \mathsf{m}_{>}^{\smile}, \mathsf{p}_{>}^{\smile}\}$$

whereas

$$\mathsf{p}_{>}^{\smile} \circ (\mathsf{m}_{>}^{\smile} \circ \mathsf{m}_{>}) = \mathsf{p}_{>}^{\smile}.$$

The algebra of the $2n$-star calculus The 4-star calculus is the Cardinal Direction Calculus studied in (Ligozat 1998a; 1998b) (or the point calculus in two dimensions). The corresponding algebra is a relation algebra.

For $n \geq 3$, it is not clear whether the $2n$-star algebra is a relation algebra or not.

A challenge: When does the algebra determine the calculus?

Suppose conversely that we are given such a relation algebra. The algebra may be abstracted from some qualitative calculus we are not aware of. The following question arises: How much we can recover, from the knowledge of the algebra, of the calculus or calculi it originates from?

Before trying to answer this question, we begin by reformulating some of the basic notions in algebraic terms.

Constraint networks and weak representations

Much of the effort in the study of qualitative temporal and spatial calculi has been devoted to the context of binary constraint networks (BCN): A BCN on $\mathbf{A}$ is a network (an oriented graph) whose arcs are labeled by elements $\mathbf{A}$. Each node represents some temporal or spatial entity, while the labels express binary constraints between the nodes.

More specifically, a constraint network $(N, (r_{i,j}))$ consists of a set N of nodes, and for each pair (i,j) in N, of an element $r_{i,j}$ in $\mathbf{A}$. In this paper, we assume that for all pairs (i,j) in N $r_{i,i}$ is the identity, and that $r_{j,i} = r_{i,j}^{\smile}$.

If all labels $r_{i,j}$ are atoms of the algebra $\mathbf{A}$, we say that the network is atomic.

Finally, recall the notion of path-consistency:

Definition 2 *A network is* path-consistent *if none of its labels is the zero element of* $\mathbf{A}$*, and for each 3-tuple* (i,j,k) *of elements of* N*, we have:*

$$r_{i,k} \circ r_{k,j} \supseteq r_{i,j}$$

Scenarios

Suppose we use one of the formalisms to describe a configuration of spatial or temporal entities (in finite number). The configuration can then be represented by a constraint network: for each pair of objects, the corresponding label is the actual relation between the two entities. Then this particular network is atomic. It is also path-consistent, because we know that the composition table describes (at least necessary) constraints.

Such a network is usually called a scenario in the literature:

Definition 3 *A scenario is a constraint network which is atomic and path-consistent.*

In the case of Allen's algebra, it is a fact (usually admitted without proof) that a scenario entirely defines a qualitative configuration of intervals (say, on the time line).

In what follows, we slightly generalize the notion of scenario in order to include infinite sets of entities. The resulting notion is called a weak representation.

Weak representations

Consider a scenario. For each basic relation $b \in \mathbf{B}$, there is a set of pairs (i,j), say $\Phi(b)$, of nodes of the network such that:

1. Every pair of nodes (i,j) belongs to exactly one $\Phi(b)$, for some $b_{i,j} \in \mathbf{B}$.
2. If $(i,j) \in \Phi(b)$, then $(j,i) \in \Phi(b^{\smile})$.
3. If $(i,j) \in \Phi(b_{i,j})$ and $(j,k) \in \Phi(b_{j,k})$, then $(i,k) \in \Phi(b)$ for some $b \in (b_{i,j} \circ b_{j,k})$.
4. For each i, $(i,i) \in \Phi(b)$ for some $b \in \mathbf{1'}$.

Because of (1), we can extend Φ to the whole algebra $\mathbf{A}$. Then this map is a homomorphism of Boolean algebras. Because of (2), Φ preserves transposition, that is, $\Phi(\alpha^{\smile}) = \Phi(\alpha)^t$. Because of (3), it also preserves composition in a "weak" sense, that is, for any pair of elements α and β we have:

$$\Phi(\alpha) \circ \Phi(\beta) \subseteq \Phi(\alpha \circ \beta)$$

.

More generally, we define weak representations (Ligozat 1990):

Definition 4 *A* weak representation *of* $\mathbf{A}$ *is a pair* (U, Φ) *where* U *is a non empty set, and* Φ *is a map of* $\mathbf{A}$ *into a product of algebras of subsets of* $U \times U$*, such that:*

1. *Φ is an homomorphism of Boolean algebras.*
2. *$\Phi(\alpha \circ \beta) \supseteq \Phi(\alpha) \circ \Phi(\beta)$.*
3. *$\Phi(\mathbf{1'}) = \Delta$.*
4. *$\Phi(\alpha^{\smile})$ is the transpose of $\Phi(\alpha)$.*

Intuitively, a weak representation is just a set U of elements, which stand for objects, together with the assignment to each atomic relation of a set of pairs (u, v) of elements in U (i.e. a binary relation in U). To be interpreted as a model, these binary relations should satisfy the axioms corresponding to the algebraic properties.

Remark 1 Equivalently, weak representations are models of the associated first-order weak theories: U is the domain of interpretation, and Φ is the interpretation function for the predicates associated to the symbols in $\mathbf{B}$.

Remark 2 When U is a finite set, a weak representation can be represented by a network, as already shown in the examples. This network is atomic, and none of the atoms is equality. Moreover, it is *path-consistent*: For any 3-tuple of vertices (i, j, k), the corresponding labels $a_{i,j}$, $a_{j,k}$ and $a_{i,k}$ are such that $(a_{i,j} \circ a_{j,k})$ contains $a_{i,k}$.

Conversely, if a network is such that it is atomic, that none of its labels is equality, and if it is path-consistent, then it defines a weak representation.

Domains and configurations

As already mentioned, most calculi come with (sometimes implicit) domains of interpretation, together with a standard way of interpreting the basic relations in terms of this domain. In (Ligozat 2001), the corresponding package is called a configuration. For instance, a configuration for Allen's calculus is a linear ordering W together with a subset U of the intervals on W, that is, of pairs (w_1, w_2), where $w_1 < w_2$.

A configuration defines in a canonical way a weak representation. Intuitively, this weak representation is the description of the configuration using the language provided by the calculus. For instance, a configuration (W, U) for Allen's calculus defines the weak representation whose universe (think of it as a set of nodes of a network) is U, and where the arc from u to v is labeled by the actual relation holding between u and v.

The important fact for all the calculi based on linear orderings is that, conversely, each weak representation is uniquely associated to a configuration. This is true for calculi 1(a) and 2(b-e):

Proposition 2 *For all calculi mentioned above, there is a well-defined construction F which, to any weak representation $\mathcal{U} = (U, \Phi)$ of $\mathbf{A}$, associates a configuration for $\mathbf{A}$.*

Example The construction of F was first introduced in (Ligozat 1990) for generalized intervals (which includes Allen's special case). It is described in detail in (Ligozat 1999) for the spatial calculi 2(b-d).

We illustrate it on a special case. Consider Fig. 2, which represent a configuration of three intervals (left part of the figure). Using Allen's language, this configuration is described by the weak representation (or scenario, in this case) on the right.

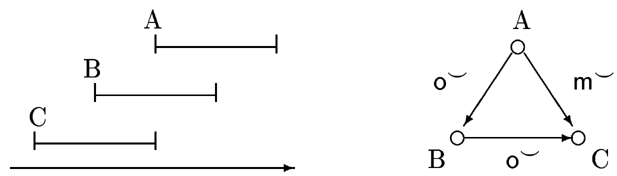

Figure 2: A configuration for Allen's calculus and its associated description

The "inverse" construction F associates to the weak representation a set of five elements (the end-points of the the intervals) $W = \{a_1, a_2, b_1, b_2, c_2\}$, which is linearly ordered by $c_1 < b_1 < a_1 < b_2 < a_2$, and two canonical maps from $\{A, B, C\}$ to W, which can be interpreted as "starting point" and "ending point" respectively.

Starting from a weak representation, using F, then considering the weak representation of all intervals on the resulting set of endpoints yields a weak representation which contains the original one. In the case of the example, we get all ten intervals on the five end-points.

The resulting operation on weak representation is a closure operation, which can be described in terms of the theory of categories. In particular, closed weak representation are fully equivalent to point-based weak representations.

Weak representations of the calculi

What can be said about the weak representations of other calculi?

RCC5 and RCC8 The case of RCC5 and RCC8 is examined in detail in (Ligozat 1999). The results may be summarized as follows:

1. A configuration for RCC5 is a family C of subsets of a set W. It defines a weak representation of RCC5.
2. Conversely, given a weak representation of RCC5, there is an explicit way of constructing a configuration to which it is associated. In this case, however, there is no canonical way of choosing one particular configuration.
3. A configuration for RCC8 is a family C of non empty closed regular subsets of a topological space W. It defines a weak representation of RCC5.
4. Conversely, given a weak representation of RCC8, there is an explicit way of constructing a configuration to which it is associated. Here again, there is no canonical way of choosing one particular configuration.

The cyclic interval calculus In the case of the calculus on cyclic intervals, the situation is still worse. Some weak representations may have more than one configuration: consider the case of a network where all the constraints are *ppi* (expressing that the intervals are non intersecting). Then any permutation of the configuration has the same corresponding weak representation.

Moreover, some weak representations have no configuration associated to them. Indeed, there is a stronger result:

Proposition 3 *For any integer $n \geq 2$, there is a path-consistent atomic network with $n+1$ nodes which is not consistent.*

Indeed, consider the configuration where the circle is split into n intervals. Hence two distinct intervals are in the relation mmi (if $n = 2$), or m, mi, or *ppi* (for $n > 2$). To the corresponding weak representation, add a fresh node with the constraint *ppi* relative to each old node. Then the new weak representation is n consistent, because leaving out one of the n intervals permits to insert the new one in the "hole". However there is no room for it if all $n+1$ intervals are considered, hence the new network is not consistent. See Figure 3 for an example with $n = 5$.

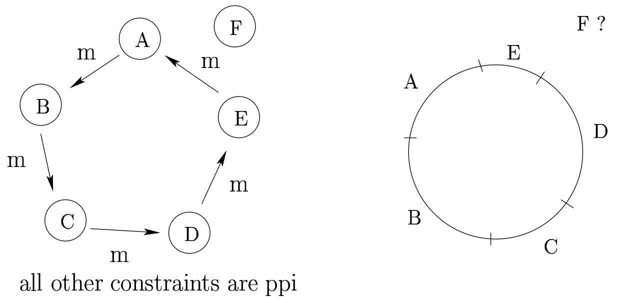

Figure 3: A path-consistent and non-consistent arc cyclic network with six variables

Partially ordered time In the case of the partial ordering algebra $\mathbf{M_4}$, Anger, Mitra and Rodriguez (Anger, Mitra, & Rodriguez 1998) have shown, with the help of some examples, that path-consistency does not imply consistency in the general case. Nevertheless, concerning atomic path-consistent networks on $\mathbf{M_4}$, for all considered models,the following question is still open (at the present state of our knowledge) : can we assert that a such network is always consistent ? We can just remark that the examples given by Anger *et al.* cannot be adapted to the atomic case. For instance, they show that each network with n variables ($n > 3$) defined by one "cycle" of length n labeled with $\{\prec, \succ\}$ while all other relations are || will be path-consistent and inconsistent if n is odd (Fig. 4 (a)). In the case of the atomic network, we can remark that each network with n variables ($n > 3$) defined by one "cycle" of length n labeled with the atomic relation $\prec$ or the atomic relation $\succ$ will not be path-consistent if n is odd (Fig. 4 (b)).

As a consequence, we do not know whether each weak representation of $\mathbf{M_4}$ can be associated to a configuration (in any model).

The INDU calculus In (Vijaya Kumari 2002) (see also (Pujari, Kumari, & Sattar 1999)), properties such as 4-consistency are examined for the INDU calculus. The simple question arises:

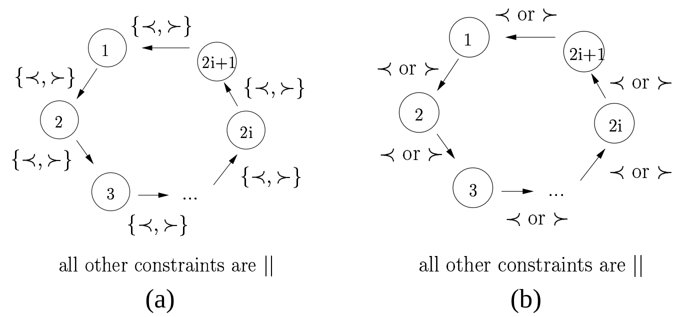

Figure 4: (a) A path-consistent and inconsistent $\mathbf{M_4}$ network, (b) a non path-consistent and inconsistent $\mathbf{M_4}$ atomic network

Question: Is any path-consistent atomic network for the INDU calculus consistent?

The $2n$-star calculus The 4-star calculus is the Cardinal Direction Calculus studied in (Ligozat 1998b). It behaves in the same way as Allen's calculus. In particular, each weak representation is canonically associated to a configuration set of points in the plane.

For the more general $2n$-star calculus ($n \geq 3$) little seems to be known about the basic properties:

Question: Is any path-consistent atomic network for the $2n$-star calculus consistent?

According to Mitra (Mitra 2002) 4-consistency implies consistency for a class of pre-convex relations.

Tractability

Calculi based on linear orderings Again, for most calculi based on linear orderings, including 1(a), 2(b-e), there is a notion of pre-convexity for relations, and so-called strongly pre-convex relations are tractable. In simpler cases, such as Allen's calculus and the cardinal direction calculus, the resulting class satisfies a maximality condition. In the case of 1(a) and 2(b), strongly pre-convex relations are known to coincide with ORD-Horn relations.

Question: Are the strongly pre-convex classes maximal in cases 1(a), 2(b-c)?

Question: What are the maximal tractable subclasses for the oriented interval calculus of Renz (Renz 2001)?

The RCC5 and RCC8 calculi All maximal tractable subclasses of RCC5 and RCC8 have been determined by Renz (Renz 2001). Is there a description of the maximal classes in geometrical terms?

The calculus of cyclic intervals There are some partial results about tractable subsets of the calculus of cyclic intervals in (Osmani 1999).

Models of the calculi: Representations

We introduced weak representations as a particular type of generalization of a class of networks. On the other hand, from the algebraic point of view, weak representations generalize the classical notion of a *representation* of an algebra. Hence the notion constitutes a bridge between the two domains.

Representations

Weak representations, as considered above, are models of the theory associated to the composition table of the calculus in a weak sense. The stronger notion of a model specifies that the axioms embodied in the composition table should be interpreted as necessary and sufficient conditions: namely, if a pair (u, v) belongs to the relation interpreting γ, and if γ can be obtained by composing α and β, then there should exist w in U such that (u, w) is in the interpretation of α, and (w, v) in that of β.

This stronger notion corresponds to the standard notion of a representation in algebra.

Definition 5 *A* representation *is a weak representation (U, Φ) where Φ is one-to-one and condition (2) is replaced by the stronger condition (4):*

5. $\Phi(\alpha \circ \beta) = \Phi(\alpha) \circ \Phi(\beta)$.

Looking back on our discussion of relation algebra, we see that our proof of $\mathbf{A}$ being a relation algebra (except for INDU) was based on exhibiting a representation of that algebra in each case. For the calculi based on linear orderings, the existence of a canonical construction of configurations from weak representations has strong implications. For other calculi, again, this simple situation no longer holds.

Classifying representations

Calculi based on linear orderings In cases 1(a) and 2(b-d), we know that there is a canonical construction F which associates a suitable configuration to any weak representation. It is an easy fact to check that, in case the weak representation is indeed a representation, it is closed in the sense that it coincides with the weak representation based on this configuration. Moreover, it is proved in (Ligozat 2001) that the configuration associated to a representation coincides with the set of relevant objects (generalized intervals, n-tuples of points, n-tuples of intervals) based on a dense linear ordering without end-points.

In particular, there is no finite representation of the algebra in those cases. Since any countable dense and linear orderings without end-points is isomorphic to $\mathbf{Q}$, this implies that the configuration based on $\mathbf{Q}$ is the only one, up to isomorphism:

Theorem 1 *In cases 1(a) and 2(b-d), any countable representation of $\mathbf{A}$ is isomorphic to the corresponding representation based on $\mathbf{Q}$.*

In other terms, all corresponding first-order theories are aleph-zero categorical. As a consequence, they are decidable.

The RCC calculi Representations of RCC5?

For RCC8, we observed that weak representations can be realized (in many ways) as closed regular subsets in a topological space. Gotts (Gotts 1996) has given a characterization of topological spaces which yield a representation of RCC8. Further discussion of the problem can be found in (Düntsch 1999).

The $2n$-star calculus The standard interpretation of the calculus is a representation of the algebra.

Question: Give a classification of all representations of the $2n$-star algebra.

The INDU calculus As observed above, the algebra of the INDU calculus is not a relation algebra. The standard interpretation of this algebra in terms of intervals in $\mathbf{Q}$ satisfies the properties of a weak representation.

Question: Are there objects similar to the standard interpretation of the INDU calculus in terms of intervals on the time line?

Conclusions: Towards general methods in qualitative spatial and temporal reasoning

We have examined some of the most typical qualitative calculi developed in the field of spatial and temporal reasoning, by relating them to their underlying relation algebra. In many respects, Allen's calculus exhibits nice properties: its weak representations relate neatly to the underlying endpoints; it has only one representation up to isomorphism; any weak representation has an interpretation (in particular, atomic path-consistent networks are consistent), and moreover this interpretation is basically unique; path-consistency in Allen's case implies consistency for the well-studied subset of ORD-Horn, or pre-convex relations;

The main conclusions of the discussion in this paper are that although calculi which are directly based on totally ordered interpretations share most of these properties, the case is quite different for formalisms based on partial orderings, or on circular structures, as well as for formalisms describing topology or orientation. In consequence, we raised many questions about the properties of these formalisms.

Concerning further progress in that direction, we suggest that a systematic development of the algebraic point of view should prove quite fruitful, as evidenced by recent work in similar fields (Bulatov, Krokhin, & Jeavons 2000).

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