Analyzing Discontinuities in Physical System Models

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Abstract

Physical systems are by nature continuous. Perceived discontinuities in system models are in reality nonlinear behaviors that are linearized to reduce computational complexity and prevent stiffness in system analysis and behavior generation. Discontinuities arise primarily from component parameter simplification and time-scale abstraction. Discontinuities in models are handled by introducing idealized switching elements controlled by finite state automata to bond graph models. In this paper, we make a systematic study of the nature and effects of discontinuities in physical system models, and present an algorithm that generates consistent and correct behaviors from hybrid models. A primary contribution of this paper is the characterization of discontinuous changes, and a systematic method for validating system models and behavior generation algorithms.

Introduction

Physical systems are by nature continuous, but often exhibit nonlinearities that make behavior generation complex and hard to analyze especially with qualitative reasoning schemes. This complexity is often reduced by linearizing model constraints so that component models (e.g., transistors, oscillators) exhibit *multiple piecewise linear* behavior or by abstracting time so that component models (e.g., switches, valves) exhibit *abrupt discontinuous* behavior changes[2, 11]. In either case, the physical components are modeled to operate in *multiple modes*, and system models exhibit mixed continuous/discontinuous behaviors.

Typically, simulation methods for generating physical system behavior, whether quantitative (e.g., [12]) or qualitative (e.g., [5, 8]) impose the *conservation of energy* principle and *continuity constraints* to ensure that generated behaviors are meaningful. Energy distribution among the elements of the system defines system *state*, and energy distribution over time reflects the history of system behavior.

Bond graphs[12] provide a general methodology for modeling physical systems in a domain independent way. Its primary elements are two energy storage elements or *buffers*, called capacitors and inductors,[1] and a third element, the resistor, which *dissipates* energy to the environment[10, 12]. Two other elements, sources of flow and effort, allow transfer of energy between the system and its environment. In addition, the modeling framework provides *idealized junctions* that connect sets of elements and allow lossless transfer of energy between them. Two other *specialized junctions*, transformers and gyrators, allow conversion of energy from one form to another. The use of the bond graph modeling language in building *compositional models* of complex systems has been described elsewhere[1].

State changes in a system are caused by energy exchange among its components, which is expressed as *power*, the time derivative, or flow, of energy. Power is defined as a product of two conjugate *power variables*: *effort* and *flow*. Given conservation of energy holds for a system, the time integral relation between energy variables and power variables implies *continuity* of power, and, therefore, effort and flow. As discussed earlier, discontinuities in behavior are attributed to time scale abstraction and parameter abstraction.

Consider an example of an ideal non-elastic collision in Fig. 1. A bullet of mass m_1 and velocity v_1 hits an unattached piece of wood of mass m_2, initially at rest. The collision causes the bullet to get implanted in the wood, forming one body of mass m_1+m_2 moving with velocity say v_2. The initial situation is represented as two bond graph fragments. Each fragment has an inertial element (mass) connected by a 1–junction to a zero effort source (S_e) which indicates that there are no forces acting on either the bullet or the wood. Because there are no sources, the momentum after collision is conserved, so $m_1 v_1 = (m_1+m_2)v_2$, i.e., $v_2 = \frac{m_1}{m_1+m_2}v_1$.

[1] Capacitors store energy in the form of *generalized displacement* and inductors store energy as *generalized momentum*

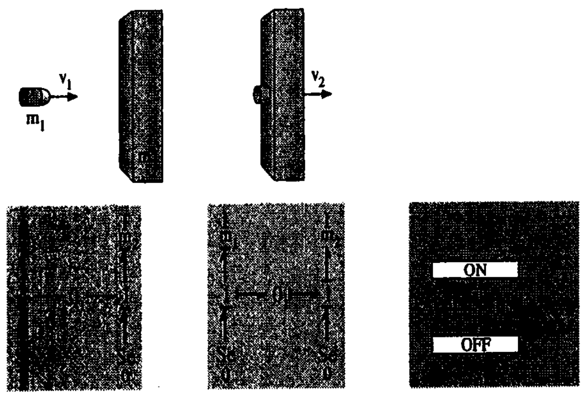

Figure 1: **Hybrid Model: Ideal non-elastic collision of two masses.**

A closer look at the model structure reveals that the initially independent masses become dependent at the instant of collision by virtue of a 0-junction connection, with no dissipation or energy storage elements involved. This represents a discontinuous change in the model, attributed to the modeling assumption that the collision is ideal and non-elastic. A more complete model of the situation would include dissipative and stiffness effects, and the two buffer velocities could not change instantaneously. If the goal of behavior generation is simply to determine the trajectory and final velocity of the bullet-wood system, the more abstract model allowing an abrupt change in velocity is reasonable and computationally efficient.

Our previous work[9, 10] in developing a uniform approach to analyzing mixed continuous/discontinuous system behavior without violating fundamental physical principles such as conservation of energy and momentum, led to the development of a *hybrid* modeling and behavior analysis scheme that combines

1. traditional energy-related bond graph elements to model the physical components of the system, and
2. control flow models based on switching junctions whose *on–off* characteristics are modeled by finite state automata.

The primary characteristics of the hybrid modeling scheme are summarized in the next section. In this paper, we establish a theory for modeling discontinuities in physical systems and then present a formal methodology for verifying consistency of hybrid system models.

The Hybrid Modeling Scheme

A system with n components each with k behavior modes, can assume k^n overall configurations, however, in practical situations, only a small fraction of the configurations are physically realized. In previous work, researchers have defined a number of approaches, such as transition functions (e.g., [8]), finite state automata and switching bonds[2], to handle discrete changes in physical system configuration and behavior. Most of these methods assume that the range of system behaviors are pre-enumerated, but, in general that can be a very difficult task. Recent compositional modeling approaches[1, 4] overcome this problem and build system models *dynamically* by composing model fragments. Our hybrid modeling scheme adopts this methodology, and implements a dynamic model switching methodology in the bond graph modeling framework.

Instead of identifying a global control structure and pre-enumerating bond graph models for each of the modes, the overall physical model is developed as one bond graph model that covers the energy flow relations within the system. Discontinuous mechanisms and components in the system are then identified, and each mechanism is modeled locally as a *controlled junction* which can assume one of two states – *on* and *off*. The local control mechanism for a junction is implemented as a *finite state automaton* and represented as a *state transition graph* or *table*.

Controlled Junctions

The input to the finite state automata associated with controlled junctions are the power variables, *effort* and *flow*, and their output are control signals that determine the *on/off* state of the associated junction. Each controlled junction has an associated control specification function, called CSPEC, which switches the junction *on* and *off*. The state transition diagram from which the CSPEC is derived may have several internal states that map onto the *on* and *off* signals but in every transition sequence *on/off* signals have to alternate. Furthermore, CSPEC conditions have to result in at least one continuous mode of operation for all reachable energy distributions.

The set of local control mechanisms associated with controlled junctions constitute the *signal flow model* of the system. The signal flow model performs no energy transfer, therefore, it is distinct from the bond graph model that deals with the dynamic behavior of the physical system variables. Signal flow models describe the *transitional*, i.e., mode-switching behavior of the system. A *mode* of a system is determined by the combination of the *on/off states* of all the controlled junctions in its bond graph model. Note that the system modes and transitions are dynamically generated, they do not have to be pre-enumerated.

When *on*, a controlled junction behaves like a normal junction, but when *off* it forces either the effort or flow value at all connected bonds to become 0, thus inhibiting energy transfer across the junction. Therefore, controlled junctions exhibit *ideal switch* behavior, and modeling discontinuous behavior in this way is consistent with bond graph theory[12]. Deactivating controlled junctions can affect the behaviors at adjoining junctions, and, therefore, the causal relations among system variables. Controlled junctions are marked by subscripts (e.g., 1_1, 0_2). They define the *interactions* between the signal flow and energy flow models of the system.

The use of controlled junctions is illustrated for the bullet-wood system whose hybrid model is shown in Fig. 1. m_1 and m_2 are the inertias (masses) of the bullet and wood block, respectively. When the bullet attaches itself to the wood block, the model switch occurs because the 0-junction turns on when $x_{bullet} \geq x_{wood}$. Once the bullet is lodged in the wood there is no mechanism to dislodge it, therefore, once turned *on* this junction cannot be turned *off*. This is indicated by the FALSE condition on the *on/off* transition for the junction. This example illustrates a seamless integration of multi-mode behaviors in one model. Other examples of hybrid bond graph models are discussed in [9, 10].

Mode Switching in the Hybrid Model

Discontinuous effects in physical system models occur when energy or power variables cross a certain threshold value. For example, a diode is modeled to come on when the voltage drop across it exceeds 0.6V. These discontinuous effects establish or break energetic connections in the model, and this may cause related signals to change discontinuously. The tank system, Fig. 2, has two pressure valves that open when either pressure e_1 or e_2 crosses a pre-set threshold value. If the specified threshold value for the valve controlled by e_1 is $\geq$ the threshold value of the second valve, the opening of the first valve will cause the second to open. If the resistance, capacity, and flow inertia of the connecting pipe are abstracted out of the model, these two discrete changes occur instantaneously and the mode where only the valve controlled by e_1 is open occurs for a brief instant in time (according to the modeling assumptions it is instantaneous). Therefore, discontinuous changes may cause other signal value changes which result in a sequence of discontinuous changes. Since modeling assumptions require that discontinuous changes occur instantaneously, these transitions are called *mythical.*

In this paper we establish that state variables in the system do not change during mythical changes.

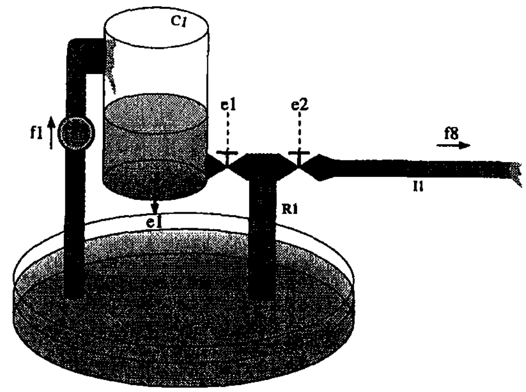

Figure 2: **Tank with two pressure controlled valves.**

The system does undergo *discrete state* changes, determined by the finite state automata associated with the *controlled junctions.* Eventually, a sequence of discrete switches terminates in a real mode (system behavior again evolves as a function of time), and the continuous state vector for this new mode has to be derived. This is illustrated in Fig. 3. Mythical modes are depicted in a white background and real modes in a dark background. In real mode M_0 a signal value crosses a threshold at time t_s^-, which causes a discontinuous change to model configuration M_1, represented by the discrete state vector σ^1. The power variable values (E_i, F_i) in this new configuration are calculated from the original energy distribution (P_s, Q_s) values. If the new values cause another instantaneous mode change, a new mode M_2 is reached, where the new power variables values, (E_i,F_i), are calculated from the original energy distribution (P_s, Q_s). Further mythical mode changes may occur till a real mode, M_N, is reached. The final step involves mapping the energy distribution, or continuous state variable values, of the departed real mode to the new real mode. This issue is non-trivial, and discussed in detail in a later section. Real time continuous simulation resumes at t_s^+ so system behavior in real time implies mode M_N follows M_0. The formal *Mythical Mode Algorithm* (MMA) is outlined below.

1. Calculate the energy values (Q_s, P_s) and signal values (E_s, F_s) for bond graph model M_0 using (Q_0, P_0), values at the previous simulation step as initial values.

2. Use *CSPEC* to infer a possible new mode given (E_s, F_s).

3. If one or more controlled junctions switch states then:

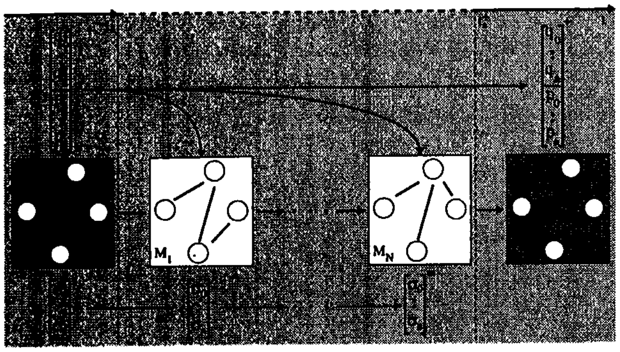

Figure 3: **State Evolution: Mythical + Real Modes.**

(a) Derive the bond graph for this configuration.

(b) Assign causal strokes using the SCAP algorithm[12].

(c) Calculate the signals (E_i, F_i) for the new mode, M_i, based on the initial values (Q_s, P_s).

(d) Use *CSPEC* again to infer a possible new mode based on (E_i, F_i) for the new mode, M_i.

(e) Repeat step 3 till no more mode changes occur.

4. Establish the final mode, M_N, as the new system configuration.

5. Map (Q_0, P_0) to the energy distribution for M_N, (Q_N, P_N).

A complete simulation system that incorporates continuous simulation and the MMA algorithm been implemented and tested on a number of physical system examples[9].

Analyzing Model Discontinuities

Since one discontinuous change may cause a sequence of mythical changes, what is the guarantee that simulation with a system model will not produce an infinite loop of mythical mode changes? This can happen if a mythical mode loops back to a mythical mode that has already appeared in the sequence of mode changes. Discontinuous changes are modeled to be instantaneous, and the occurrence of a loop during mythical mode changes would imply that system behavior does not progress over time. This violates physical reality because all dynamic physical systems have to satisfy *divergence of time*[6]. In such situations, the system model is incorrect and needs to be modified to generate correct system behaviors.

Our methodology for analyzing the consistency of a sequence of switches is based on applying the principle of *invariance of state* to the MMA. This defines a *multiple energy phase space analysis* scheme to determine the physical consistency of hybrid models. The initial values of energy storing elements in the new real mode are determined by the principle of *conservation of state*.

The Nature and Effects of Discontinuities

As a first step, we analyze discontinuous changes by considering the bullet-wood non-elastic collision illustrated in Fig. 1. Conservation of momentum determines that after collision both masses have a common velocity, $v_2 = \frac{m_1}{m_1+m_2}v_1$. The amount of energy contained in the system before the collision is $\frac{1}{2}m_1{v_1}^2$, and, the amount of energy in the system after the collision is $\frac{1}{2}(m_1+m_2)(\frac{m_1}{m_1+m_2}v_1)^2 = \frac{1}{2}\frac{{m_1}^2}{m_1+m_2}{v_1}^2$. This implies that the collision causes a loss of energy equal to $\frac{1}{2}m_1{v_1}^2 - \frac{1}{2}\frac{{m_1}^2}{m_1+m_2}{v_1}^2 = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}{v_1}^2$. Imposition of conservation of momentum results in a discontinuous loss of energy to the environment, and is represented by a *Dirac pulse*[2] whose area is determined by the model. The explanation for this energy loss is that an instantaneous change in system configuration results in two independent buffers (the masses) becoming dependent, which then causes dissipation of energy to the environment as heat. In case the environment is considered isothermal, this thermal energy flow need not be explicitly modeled by a heat sink between the system and the environment (like dissipation of resistive elements). Note that the instantaneous loss of energy would not occur if a resistive element modeled material deformation between the buffers (i.e., their connection was non-ideal).

From a physical perspective, buffers that become dependent have to be analyzed carefully. Bond graph modeling theory assumes that physical system behaviors change slowly enough to satisfy the *lumped parameter* assumption where distribution of energy within energy buffers is considered homogeneous. Therefore, phenomena like dynamic turbulence effects caused by sudden large pressure differences in a tank cannot be easily taken into account in system models. The observation of an instantaneous energy redistribution when buffers become dependent is basically an artifact of an abstract model operating on a coarse time scale.

To study the lumped parameter assumption in detail, consider the free expansion experiment conducted by Gay-Lussac and Joule shown in Fig. 4 [3]. A chamber is made up of two connected bulbs with an on-off valve that switches the connection on or off. Initially, only the left bulb contains a gas and the valve is closed. When the valve connecting the two volumes is opened,

[2]This is a pulse during an infinitely short interval of time with a specific area.

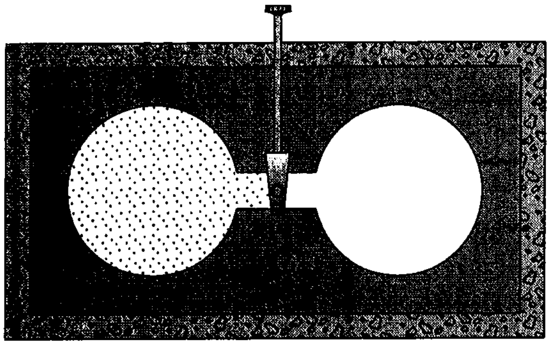

Figure 4: **Instantaneous free expansion of a gas by diffusion.**

the gas in the left bulb expands freely and starts diffusing into the right bulb. Even if the connecting orifice is non-resistive this diffusion introduces turbulence effects that result in non-homogeneities, which violate the lumped parameter assumption. From a thermodynamics perspective, when the goal is to compute equilibrium temperatures and pressures, using energy balance on a homogeneous model is adequate.

For the dynamic effects to be negligible in the time scale of interest, they have to occur in a mode of continuous operation. In case of the above experiment, if the valve were closed quickly enough after opening, i.e., we have two instantaneous changes, the gas cannot diffuse and the homogeneous distribution over both of the volumes is never actually established. For CSPEC conditions based on the energy variables involved, the time scale is too small for the lumped parameter assumption to hold, and bond graphs cannot be used to model such processes. In effect, an immediate closing of the opening leads to no redistribution of energy which substantiates the observation that there never was a connection in real time.

In conclusion, switching conditions based on energy stored in buffers that are between dependent and independent without an intervening mode of continuous operation are inconsistent with bond graph assumptions and, therefore, prohibited. In this case, either model refinement or another modeling approach has to be chosen.

Evolution of System Behavior: Invariance of State

A consequence of the absence of a physical manifestation of mythical modes is that the system cannot exchange energy with its environment during a sequence of discontinuous changes, in other words, it is *isolated.* This is compatible with the fact that there is no redistribution of stored energy within the system during such a sequence.

The principle that energy is not redistributed during discontinuous, instantaneous changes, but only after a new mode is reached is termed the principle of *invariance of state.* The mythical mode algorithm for deriving discontinuous changes, and the corresponding model verification technique are based on this principle.

Consider an ideal elastic collision, Fig. 5. When the ball hits the floor with a velocity v, it induces a force which results in a reaction force by the floor. In case of an ideal elastic collision this causes the ball to instantaneously reverse its velocity and start traveling upwards. In the corresponding hybrid bond graph model, the change of velocity is shown as a modulated flow source which delivers its power when the controlled junction 0_1 comes on. The CSPEC specifies that this occurs at the point when the ball hits the floor. The moment the ball starts to move upwards, the controlled junction turns off and the ball is represented by a separate model fragment again. The floor is represented as a source to indicate it is part of the system environment and the amount of energy transferred back to the ball from the floor becomes a modeling decision. The change in momentum of the ball can only occur when a real mode is reached after mythical transitions. In general, any transfer of energy into or out of the system can only be applied in the real mode. However, during mythical transitions, a Dirac pulse may be generated and cause further CSPEC transitions. Once generated, their effect on all CSPECs have to be analyzed immediately. If the Dirac pulses can be propagated without causing another CSPEC transition, the current mode is established as a real one and the effect of the Dirac pulse is taken into account when computing the new state vector in this mode. Otherwise, this mode is labeled as mythical, and the active CSPEC is used to generate the next mode, and the analysis continues as before without actually propagating the Dirac pulses. Thus invariance of state holds across a series of mythical mode changes and the continuous state vector is invariant to mythical switches.

To correctly infer model configuration changes, all buffers that become dependent have to be analyzed in derivative causality, i.e., they generate Dirac pulses. Changes in configuration and independence of buffers may lead to further model switches.

The Initial Value Problem

After mythical changes, when a new continuous mode of system operation is established, the initial state vector in the new mode has to be derived from the last system state in the previous continuous mode. The

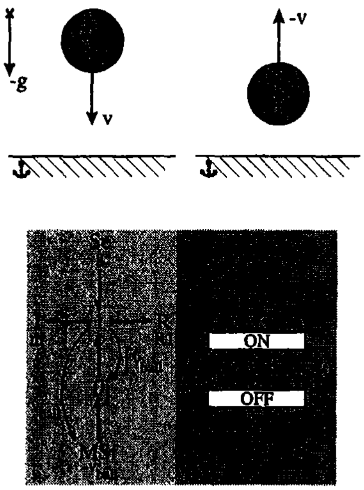

Figure 5: **Ideal elastic collision.**

energy stored in buffers that remain independent does not change in the new configuration because of the integral relation they impose on their energy variables. However, the energy content of dependent buffers can change as determined by the conservation of state principle[3, 13] which states that state variable (charge and momentum) values are conserved. The dependence causes redistribution of the total energy in the buffers, in the ratio of their parameter values as demonstrated by the bullet-wood system.

In case buffers become dependent upon sources, their energy changes according to the signal enforced by the source. An ideal non-elastic collision, depicted in Fig. 6, shows the discontinuous change causing a source-buffer dependency with no dissipative effects (ideal collision). In the hybrid bond graph model this is depicted as a flow source of 0 value that gets switched in at the time of collision. Notice that the momentum in the system before collision, mv_{ball}, becomes 0 instantaneously. Conservation of state cannot be applied to derive the new value of the stored energy in the dependent buffer. Instead, this value is completely determined by the value of the connected source.

In actuality, the ball and floor could be modeled as one system. For the elastic collision, the floor is modeled as a buffer with a large stiffness coefficient, producing the behavior shown in Fig. 7 (x is the displacement, and v the velocity of the object). For the non-elastic collision, the floor is modeled as a large resistance, which dissipates energy, and this brings the ball to a halt with a very fast time constant. For these models, source-buffer dependencies and modeled Dirac sources do not occur, and conservation of state and energy is not violated. Therefore, the apparent violation

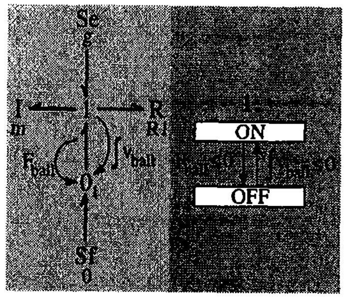

Figure 6: **Ideal non-elastic collision.**

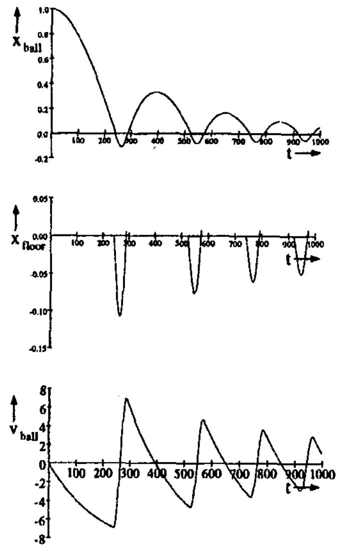

Figure 7: **Ideal elastic collision with the floor modeled as having a relatively large stiffness.**

of these principles can be directly attributed to buffer dependencies in the model.

The above examples illustrate that three situations can be associated with the initial value problem that results from discontinuities in a model:

- Switching modes causes no buffer-buffer or source-buffer dependencies. In this case the continuous state vector is unchanged between real modes.
- Mode transitions cause two or more buffers to become dependent, and this causes the size of the state vector to change between real modes. As discussed, individual energy variable values change, but their total remains the same so conservation of state holds. The new energy distribution is determined by the total value of stored energy in buffers that are involved and the ratios of their parameters. Mode transitions may result in loss of energy to the environment as a

result of changes in the system configuration.

- Mode transitions cause source-buffer dependency or explicitly modeled Dirac sources to become active. In this case the switching causes a source (i.e., the environment) to instantaneously transfer energy into or out of the system. The new values of stored energy in the buffers involved is primed to the source enforced values after the new real mode is established.

Step 5 of the MMA handles all three cases appropriately, and solves the initial value problem correctly after mode transitions.

Analyzing Correctness of System Models

Having established the invariance of state principle, the next step is to establish systematic methods for verifying system models that incorporate discontinuous changes. Given the existence of instantaneous mode changes, it is important to establish that mythical transitions do not result in loops and violate the principle of divergence of time.

As a case of interacting discontinuous changes, consider the two ball examples of the last section. Assume that

1. if the ball momentum is greater than a given threshold, (p_{th}), an *ideal elastic* collision occurs,
2. if the ball momentum is below the threshold, the collision is *ideal* and *non-elastic.*

The ball bounce is modeled to be ideally elastic, and the amplitude of the bounce decreases with time because of air resistance. Thus its momentum decreases with every bounce, and once it falls below a certain threshold value, p_{th}, the ball is observed to come to rest.[3] This part of the behavior is modeled as an ideal non-elastic collision. The hybrid model for this system is shown in Fig. 8. The system model has two interacting switches, and this raises the issue of divergence of time. To verify that the system model satisfies this condition, a *multiple energy phase space analysis* technique has been developed.

Phase space analysis is based on a qualitative representation of model behavior. The energy phase space is k−dimensional, where k is the number of independent buffers or state variables in the system. These energy variables cannot change discontinuously across a sequence of mythical modes, therefore, mythical mode points in this space are invariant and correspond to continuous state vectors of the model. To establish

[3]This example is abstracted from the cam-axis subsystem of an automobile engine.

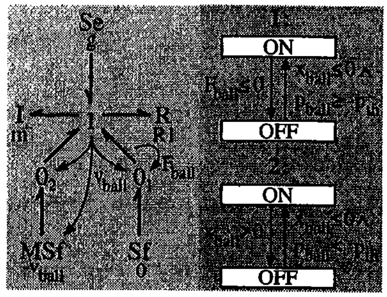

Figure 8: **A combined ideal elastic and non-elastic collision.**

phase space behavior of a hybrid model in terms of its energy distribution, its switching conditions have to be expressed in terms of stored energy (i.e., generalized momentum and generalized displacement). However, switching conditions in the model CSPECs are based on power variables which can be composed of different state variables, depending on the mode of operation. Therefore, an energy phase space has to be constructed for each mode of operation. The rest of this section studies the energy phase space for both the source-buffer and the buffer-buffer dependency conditions that may arise in system models.

Source-Buffer Dependency For the bouncing ball, all CSPEC transitions are specified in terms of switching invariant state variables (Fig. 8), except for the force F_{ball} which is $F_{ball} = F_m + F_{R1} - F_g$, where g is the gravitational force. To derive the conditions under which the flow source, i.e., junction 0_1 turns *off*, F_{ball} has to be expressed in terms of stored energy variables x_{ball} and p_{ball}. When the flow source is active, buffer dependency causes F_m to be derivative, i.e.,

$$F_m = \frac{dp}{dt}, \qquad (1)$$

and, discontinuous changes induce a Dirac pulse, δ. From the CSPEC for 0_1, the condition for switching is $F_{ball} \leq 0$. When this junction is in its *off* state, $p_{ball} = mv_{ball}$. When it switches *on*, the velocity and momentum of the ball become 0 instantaneously. From equation (1), this implies that F_m becomes a Dirac pulse whose magnitude approaches positive or negative infinity, depending on whether the stored momentum was negative or positive, respectively. If the momentum was 0, F_m equals 0. Let the function *sign* be defined as

$$sign(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases} \qquad (2)$$

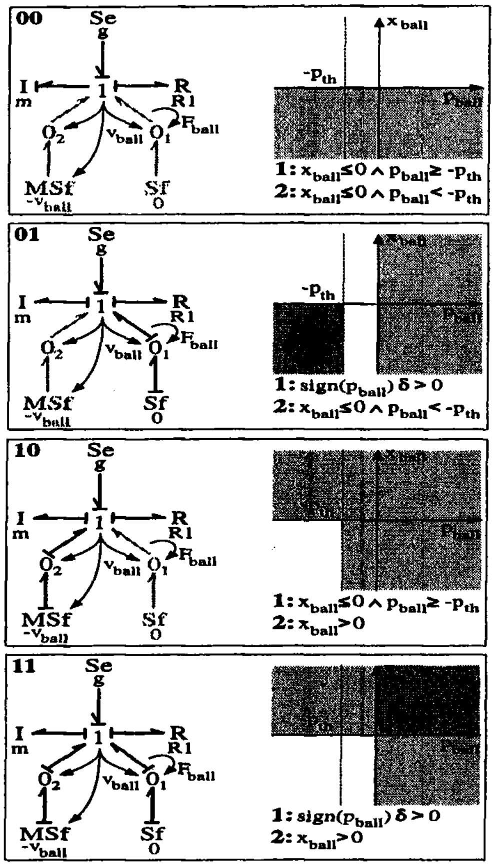

Figure 9: **Energy phase space: Bouncing Ball.**

If $p \neq 0$ the condition for switching becomes $F_m = -sign(p)\delta \leq 0$. Because of the magnitude of the Dirac pulse, the effects of friction and the gravitational force can be neglected at switching. Therefore, the condition for the *on-off* state transition for 0_1 is $sign(p)\delta \leq 0$. This inequality holds for all values of $p > 0$. If $p = 0$ then $F_m = 0$ and $F_{R1} = 0$. The condition becomes $-F_g \leq 0$ which is never true for $g = -9.81$. The area for which transition occurs is represented by $p > 0$ which is grayed out in the phase space, shown in Fig. 9.

Phase spaces are established (Fig. 9) for each of the four modes of the combined elastic and non-elastic collision and labeled 00, 01, 10, and 11, where the first digit indicates whether the controlled junction 2 is *on* (1) or *off* (0), and the second digit indicates the same for controlled junction 1. In the phase spaces the areas that are instantaneously departed are grayed out and the conjunction of the four energy phase spaces is shown in Fig. 10. This phase space shows that there is an energy distribution which does not correspond to a real mode of operation. Since the dimensions of the energy phase space are invariant across switches, this

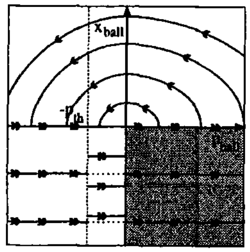

Figure 10: **Conjunction of the multiple energy phase spaces.**

energy distribution cannot reach a real mode of operation during a sequence of switches, thus violating the divergence of time condition.

In this area, when the ball hits the floor, it has positive momentum. For the bouncing ball, this mode is unreachable, and, therefore, the model is physically consistent: The system always moves towards a negative momentum and it instantaneously reverses (depicted by double arrows in Fig. 10) when the displacement becomes zero. Analytically the displacement never becomes negative. However, due to numerical disturbances, or initial conditions, the model may arrive in the physically inconsistent area of operation, especially, in case the floor is another moving body. Therefore, in such situations, when simulating the system, the CSPEC conditions model the desired physical scenario inadequately.

To establish a physically correct system, the CSPEC switching conditions have to be modified. From the physical system it is clear that additional constraints can be imposed based on the momentum of the ball. Since the switching conditions of the controlled junction 1 are not mutually exclusive, the conditions $p_{ball} > 0$ and $p_{ball} \leq 0$ can be added to the *off/on* and *on/off* transitions, respectively. This results in the energy phase spaces shown in Fig. 11. Now, the conjunction of the energy phase spaces results in a real mode of operation for each energy distribution. Because of the combinatorial switching logic, this real mode of operation is reachable in one switching step. A simulation of the physically consistent system is shown in Fig. 12. The air resistance ($R1$), causes the bounce of the ball to dampen until the momentum of the ball falls below the threshold value p_{th} and the ball comes to rest on the floor.

Buffer-Buffer Dependency When two or more buffers become dependent, redistribution of the state variable values according to the buffer ratios is required. For the energy phase space analysis this means that a number of dimensions collapse into one. This

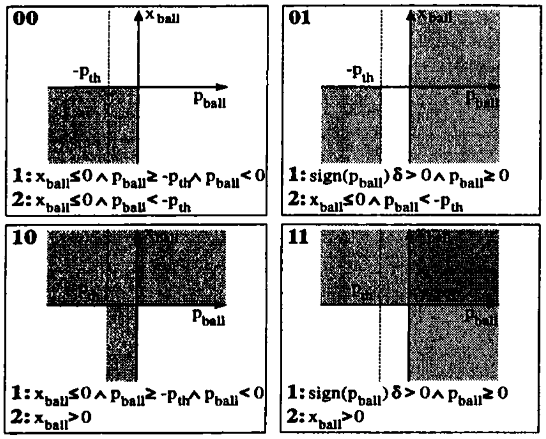

Figure 11: **Modified multiple energy phase spaces.**

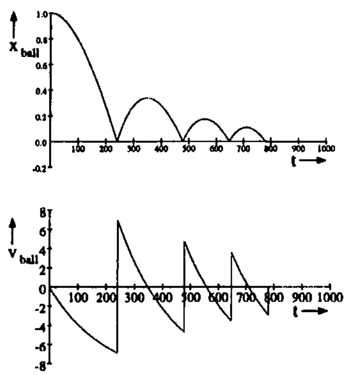

Figure 12: **Bouncing ball: Physically consistent simulation.**

is illustrated by the two capacitor system in Fig. 13. When the relay is open, the capacitors are independent and charge individually. When the voltage difference between the two crosses a threshold value, say 0, the relay *closes* and the capacitors become dependent, causing an immediate redistribution in charge (state variable) among the capacitors so both capacitors have the same effort value, $e = \frac{q_1+q_2}{C_1+C_2} = \frac{q_s}{C_s}$. Substitution of these values for V_1 and V_2 in the relay's CSPEC yields the switching condition $\frac{q_s}{C_s} \leq \frac{q_s}{C_s}$. This is true in the entire energy phase space (Fig. 14). Notice that a degree of freedom is lost because the buffer dependency causes the two-dimensional phase space to collapse into a one-dimensional phase space. When the two energy phase spaces are analyzed, it is apparent that divergence of time is not guaranteed. The inconsistency results from the equal sign in the switching *off* condition for the relay. When the relay is *on*, and both voltages

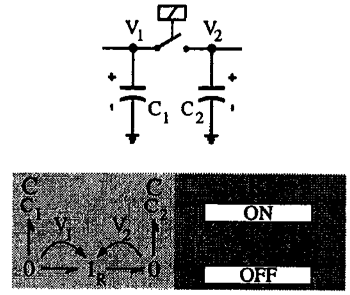

Figure 13: **Dependent capacitors.**

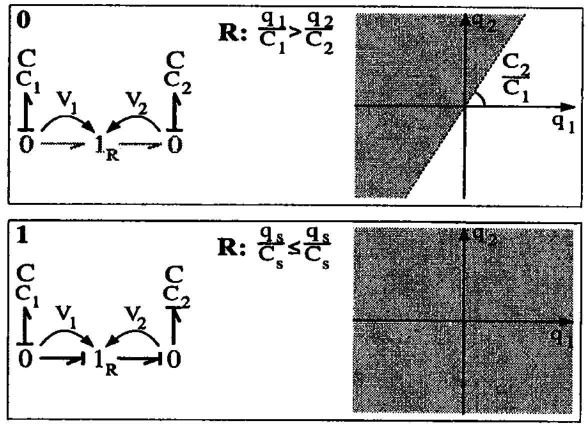

Figure 14: **Dependent Capacitors: Energy phase spaces.**

become equal it will be switched *off* instantaneously and vice versa. To solve this problem, the condition for switching needs to be changed to $V_1 < V_2$. Then the relay does not switch out of its *on* state and divergence of time is guaranteed. This example illustrates that the energy phase space analysis applies even when dependency relations among the buffers cause changes in the system model.

Conclusions and Discussion

Physical systems are by nature continuous. Discontinuities introduced in system models are in reality nonlinear behaviors that are linearized to: (i) prevent stiffness when performing numerical simulation, (ii) reduce computational complexity in system configurations and behavior analysis, and (iii) apply qualitative reasoning methods to gain a better understanding of overall system behavior. Our hybrid modeling framework combines bond graphs, controlled junctions, and finite state automata to model and analyze discontinuities in physical system behavior[9, 10]. In some situations, the hybrid models seem to violate the basic physical principles of conservation of energy and conservation of momentum. A primary contribution of this paper is the development of a systematic approach to analyze the correctness of the mode transition algorithm (MMA), and also verify the consistency of physical system models.

This approach is based on characterizing discontinuous transitions as those that involve (i) no change in the independence of buffers in the system, (ii) two or more buffers within the system become dependent, and (iii) explicitly modeled Dirac source and source-buffer dependencies occur, where energy exchange takes place with the environment. It is shown that these effects are correctly handled by the mythical mode algorithm.

Complex hybrid models may contain multiple controlled junctions, and, in general, there is no guarantee that a switching process, once initiated, will ever terminate. We have discussed that these situations violate the principle of divergence of time and, therefore, indicate modeling inconsistencies. A multiple energy phase space analysis technique is developed to analyze the divergence of time in individual model modes, and validated against the principle of invariance of state. Therefore, this research extends previous work by Alur, Henzinger, and Nicollin[6] who show divergence of time only for models that have a constant rate of change of variable values.

Iwasaki *et al.* introduce the concept of *hypertime* to represent the instantaneous switching time stamp as an infinitely short interval of time[7]. Switches are then said to occur in hypertime, where there is an infinitely small interval of hypertime between each. Effectively this introduces small time constants of parasitic effects in the simulation engine that were abstracted away from the model. This renders it possible to simulate physically inconsistent models, only valid by merit of execution semantics and the legitimacy of the results is questionable since energy exchange takes place during switching. Moreover, this does not eliminate the problem of numerical stiffness and disregards the alternatives of tightening switching conditions or modifying landmarks to ensure consistency.

In summary, we have investigated the nature, effects, and consistency of discontinuities in physical system models, and developed algorithms and systematic evaluation methods for validating our algorithm and model. Future work will focus on more general analysis of hybrid models in terms of reachability as well as the issue of time complexity by partitioning the model in instantaneously independent parts.

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