Qualitative Spatial Reasoning à la Allen: An Algebra for Cyclic Ordering of 2D Orientations

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Abstract

We define an algebra of ternary relations for cyclic ordering of 2D orientations, which is a refinement of the CYCORD theory. The algebra (1) contains 24 atomic relations, hence 2^{24} general relations, of which the usual CYCORD relation is a particular relation; and (2) is NP-complete, which is not surprising since the CYCORD theory is. We then provide the following: (1) a constraint propagation algorithm for the algebra; (2) a proof that the propagation algorithm is polynomial, and complete for a subclass including all atomic relations; (3) a proof that another subclass, expressing only information on parallel orientations, is NP-complete; and (4) a solution search algorithm for a general problem expressed in the algebra. A comparison to related work indicates that the approach is promising.

Introduction

Qualitative spatial reasoning (QSR) has become an important and challenging research area of Artificial Intelligence. An important aspect of it is topological reasoning (e.g. (Cohn 1997)). However, many applications (e.g., robot navigation (Levitt & Lawton 1990), reasoning about shape (Schlieder 1994)) require the representation and processing of orientation knowledge. A variety of approaches to this have been proposed: the theory of CYCORDs for cyclic ordering of 2D orientations (Megiddo 1976; Röhrig 1994; 1997), Frank's (1992) and Hernández's (1991) sector models and Schlieder's (1993) representation of a panorama.

A cyclic ordering problem can be seen as a ternary constraint satisfaction problem of which:

1. the variables range over the points of a circle, for example the circle of centre $(0,0)$ and of unit radius; and
2. the constraints give for triples of variables the order in which they should appear when, say, the circle is scanned clockwise.

In real applications, information expressed by CYCORDs may not be specific enough. For instance, one may want to represent information such as "objects A, B and C are such that B is to the left of A; and C is to the left of both A and B, or to the right of both A and B", which is not representable in the CYCORD theory. This explains the need for refining the theory, which is what we propose in the paper. Before providing the refinement, which is an algebra of ternary relations, we shall define an algebra of binary relations which is much less expressive (it cannot represent the CYCORD relation). Among other things, we shall provide a composition table for the algebra of binary relations. One reason for doing this first is that it will then become easy to understand how the relations of the refinement are obtained.

So far, constraint-based approaches to QSR have mainly used constraint propagation methods achieving path consistency. These methods have been borrowed from qualitative temporal reasoning à la Allen (Allen 1983), and make use of a composition table. It is, for instance, well-known from works of van Beek that path consistency achieves global consistency for CSPs of Allen's convex relations. The proof of this result, given in (van Beek & Cohen 1990; van Beek 1992), shows that it is mainly due to the 1-dimensional nature of the temporal domain. The proof uses the specialisation of the well-known, but unfortunately not much used[1] in QSR, Helly's theorem to $n = 1$: "If S is a set of convex regions of the n-dimensional space $\mathbb{R}^n$ such that every $n+1$ elements in S have a non empty intersection then the intersection of all elements of S is non empty". For the 2-dimensional space ($n = 2$), the theorem gets a bit more complicated, since one has to check non emptiness of the intersection of every three elements, instead of just every two. This suggests that constraint-based approaches to QSR should, if they are to be useful, devise propagation methods achieving more than just path consistency. The constraint propagation algorithm to be given for the algebra of ternary relations achieves indeed strong 4-consistency, and we shall show that it has a similar behaviour for a subclass including all atomic relations as path consistency for Allen's convex relations.

We first provide some background on the CYCORD theory; then the two algebras. Next, we consider CSPs

[1]Except some works by Faltings such as in (Faltings 1995).

on cyclic ordering of 2D orientations. We then provide the following: (1) a constraint propagation algorithm for the algebra of ternary relations; (2) a proof that the propagation algorithm is polynomial, and complete for a subclass including all atomic relations; (3) a proof that another subclass, expressing only information on parallel orientations, is NP-complete; and (4) a solution search algorithm for a general problem expressed in the algebra. Before summarising, we shall discuss some related work.

CYCORDs

Given a circle centred at O, there is a natural isomorphism from the set of 2D orientations to the set of points of the circle: the image of orientation X is the point P_X such that the orientation of the directed straight line (OP_X) is X. A CYCORD X-Y-Z represents the information that the images P_X, P_Y, P_Z of orientations X, Y, Z, respectively, are distinct and encountered in that order when the circle is scanned clockwise starting from P_X.

We now provide a brief background on the CYCORD theory, taken from (Megiddo 1976; Röhrig 1994; 1997). For this purpose, we consider a set $S = \{X_0, \ldots, X_n\}$.

Definition 1 (cyclic equivalence) *Two linear orders* $(X_{i_0}, \ldots, X_{i_n})$ *and* $(X_{j_0}, \ldots, X_{j_n})$ *on* S *are called cyclically equivalent if there exists* $m \in \mathbb{N}$ *such that:* $\forall k \in \{0, \ldots, n\}(j_k = (i_k + m) mod (n + 1))$.

Definition 2 (total cyclic order) *A total cyclic order on* S *is an equivalence class of linear orders on* S *modulo cyclic equivalence;* X_{i_0}-$\ldots$-X_{i_n} *denotes the equivalence class containing* $(X_{i_0}, \ldots, X_{i_n})$.

Definition 3 (partial cyclic order) *A closed partial cyclic order on* S *is a set* T *of cyclically ordered triples such that:*

(1)	$X\text{-}Y\text{-}Z \in T \Rightarrow X \neq Y$	*(irreflexivity)*
(2)	$X\text{-}Y\text{-}Z \in T \Rightarrow Z\text{-}Y\text{-}X \notin T$	*(asymmetry)*
(3)	$\{X\text{-}Y\text{-}Z, X\text{-}Z\text{-}W\} \subseteq T \Rightarrow X\text{-}Y\text{-}W \in T$	*(transitivity)*
(4)	$X\text{-}Y\text{-}Z \notin T \Rightarrow Z\text{-}Y\text{-}X \in T$	*(closure)*
(5)	$X\text{-}Y\text{-}Z \in T \Rightarrow Y\text{-}Z\text{-}X \in T$	*(rotation)*

The algebra of binary relations

The algebra is very similar to Allen's (1983) temporal interval algebra. We describe briefly its relations and its three operations (converse, intersection and composition).

Given an orientation X of the plane, another orientation Y can form with X one of the following qualitative configurations:

1. Y is equal to X (the angle (X, Y) is equal to 0).
2. Y is to the left of X (the angle (X, Y) belongs to $(0, \pi)$).
3. Y is opposite to X (the angle (X, Y) is equal to π).
4. Y is to the right of X (the angle (X, Y) belongs to $(\pi, 2\pi)$).

b	$b^{\smile}$
e	e
l	r
o	o
r	l

$\otimes_2$	e	l	o	r
e	e	l	o	r
l	l	{l,o,r}	r	{e,l,r}
o	o	r	e	l
r	r	{e,l,r}	l	{l,o,r}

Figure 1: The converse $b^{\smile}$ of an atomic relation b (Left), and the composition for atomic relations (Right).

We denote the four configurations by *(Y e X)*, *(Y l X)*, *(Y o X)* and (Y r X), respectively. Clearly, these configurations are Jointly Exaustive and Pairwise Disjoint (JEPD): given any two orientations of the plane, they stand in one and only one of these configurations.

Definition 4 (relations of the algebra) *The algebra contains four atomic relations:* e, l, o, r. *A (general) relation is any subset of the set BIN of all four atomic relations (when a relation is a singleton set (atomic), we omit the braces in its representation). A relation* $B = \{b_1, \ldots, b_n\}, n \leq 4$, *between orientations* X *and* Y, *written* $(Y\ B\ X)$, *is to be interpreted as* $(Y b_1 X) \vee \ldots \vee (Y b_n X)$.

Definition 5 (converse) *The converse of an atomic relation* b *is the atomic relation* $b^{\smile}$ *such that:*

$$\forall X, Y((Y\ b\ X) \Leftrightarrow (X\ b^{\smile}\ Y))$$

The converse $B^{\smile}$ *of a general relation* B *is the union of the converses of its atomic relations:* $B^{\smile} = \bigcup_{b \in B}\{b^{\smile}\}$.

Definition 6 (intersection) *The intersection of two relations* B_1 *and* B_2 *is the relation* B *consisting of the set-theoretic intersection of* B_1 *and* B_2*:* $B = B_1 \cap B_2$.

Definition 7 (composition) *The composition of two relations* B_1 *and* B_2, *written* $B_1 \otimes_2 B_2$, *is the strongest relation* B *such that:*

$$\forall X, Y, Z((Y B_1 X) \wedge (Z B_2 Y) \Rightarrow (Z B X))$$

Figure 1 gives the converse of each of the atomic relations, as well as the composition for atomic relations.

Definition 8 (induced ternary relation) *Given three atomic binary relations* b_1, b_2, b_3, *we define the induced ternary relation* $b_1 b_2 b_3$ *as follows (see Figure 2(I)):*

$$\forall X, Y, Z(b_1 b_2 b_3(X, Y, Z) \Leftrightarrow (Y b_1 X) \wedge (Z b_2 Y) \wedge (Z b_3 X))$$

The composition table of Figure 1(Right) has 12 entries consisting of atomic relations, the remaining four consisting of three-atom relations. Therefore any three 2D orientations stand in one of the following 24 JEPD configurations: *eee, ell, eoo, err, lel, lll, llo, llr, lor, lre, lrl, lrr, oeo, olr, ooe, orl, rer, rle, rll, rlr, rol, rrl, rro, rrr*. According to Definition 8, *rol(X, Y, Z)*, for instance, means

$$(Y\ r\ X) \wedge (Z\ o\ Y) \wedge (Z\ l\ X)$$

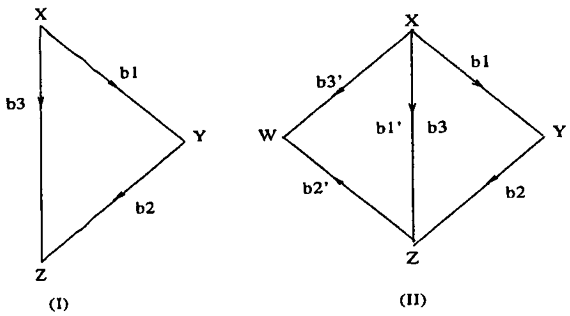

Figure 2: (I) The ternary relation induced from three atomic binary relations: $b_1b_2b_3(X,Y,Z)$ iff $((Y\ b_1\ X) \wedge (Z\ b_2\ Y) \wedge (Z\ b_3\ X))$. (II) The conjunction $b_1b_2b_3(X,Y,Z) \wedge b_1'b_2'b_3'(X,Z,W)$ is inconsistent if $b_3 \neq b_1'$.

The composition table for atomic binary relations rules out the other, $(4\times4\times4)-24$, induced ternary relations $b_1b_2b_3$; these are inconsistent: no triple (z_1,z_2,z_3) of orientations exists such that for such an induced relation one has

$$(z_2\ b_1\ z_1) \wedge (z_3\ b_2\ z_2) \wedge (z_3\ b_3\ z_1)$$

Refining the CYCORD theory: the algebra of ternary relations

The algebra of binary relations introduced above cannot represent a CYCORD. However, if we use the idea of what we have called an *"induced ternary relation"*, we can easily define an algebra of ternary relations of which the CYCORD relation will be a particular relation.

Definition 9 (ternary relation) *An atomic ternary relation is any of the 24 JEPD configurations a triple of 2D orientations can stand in. We denote by TER the set of all atomic ternary relations:*

$$TER = \{eee, ell, eoo, err, lel, lll, llo, llr, lor, lre, lrl, lrr, oeo, olr, ooe, orl, rer, rle, rll, rlr, rol, rrl, rro, rrr\}$$

A (general) ternary relation is any subset T *of TER:*

$$\forall X,Y,Z(T(X,Y,Z) \Leftrightarrow \bigvee_{t\in T} t(X,Y,Z))$$

As an example, a CYCORD X-Y-Z can be represented by the ternary relation $CR = \{lrl, orl, rll, rol, rrl, rro, rrr\}$ (see Figure 3):

$$\forall X,Y,Z(X\text{-}Y\text{-}Z \Leftrightarrow CR(X,Y,Z))$$

Definition 10 (converse) *The converse of an atomic ternary relation* t *is the atomic ternary relation* $t^{\smile}$ *such that:*

$$\forall X,Y,Z(t(X,Y,Z) \Leftrightarrow t^{\smile}(X,Z,Y))$$

The converse $T^{\smile}$ *of a general ternary relation* T *is the union of the converses of its atomic relations:*

$$T^{\smile} = \bigcup_{t\in T}\{t^{\smile}\}$$

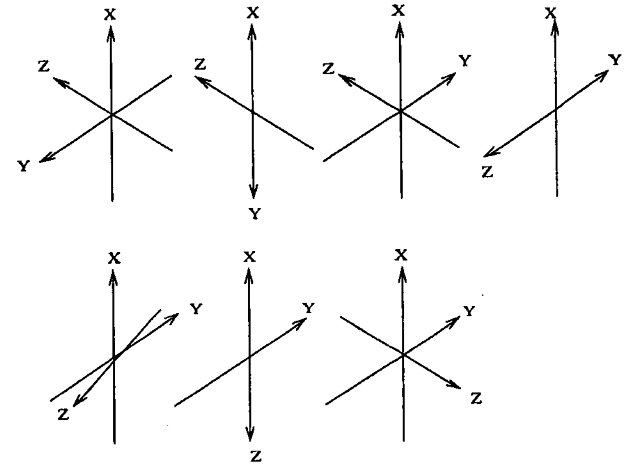

Figure 3: The CYCORD relation is a particular relation of the algebra of ternary relations: X-Y-Z if and only if $\{lrl, orl, rll, rol, rrl, rro, rrr\}(X,Y,Z)$.

t	$t^{\smile}$	$t^{\frown}$	t	$t^{\smile}$	$t^{\frown}$	t	$t^{\smile}$	$t^{\frown}$
eee	eee	eee	lor	rol	olr	rer	rer	ell
ell	lre	lre	lre	ell	rer	rle	err	lel
eoo	ooe	ooe	lrl	lll	rrr	rll	lrr	lrl
err	rle	rle	lrr	rll	rlr	rlr	rrr	lll
lel	lel	err	oeo	oeo	eoo	rol	lor	orl
lll	lrl	lrr	olr	rro	llo	rrl	llr	rrl
llo	orl	lor	ooe	eoo	oeo	rro	olr	rol
llr	rrl	llr	orl	llo	rro	rrr	rlr	rll

Figure 4: The converse $t^{\smile}$ and the rotation $t^{\frown}$ of an atomic ternary relation t.

The converse of an atomic ternary relation $t = \alpha\beta\gamma$ can be expressed in terms of the atomic binary relations α, β, γ and their converses in the following way:

$$(\alpha\beta\gamma)^{\smile} = \gamma\beta^{\smile}\alpha$$

This is so because if $(\alpha\beta\gamma)^{\smile} = \alpha'\beta'\gamma'$ then $\alpha\beta\gamma(X,Y,Z)$ iff $\alpha'\beta'\gamma'(X,Z,Y)$. But $\alpha\beta\gamma(X,Y,Z)$ stands for the conjunction $((Y\alpha X) \wedge (Z\beta Y) \wedge (Z\gamma X))$, and $\alpha'\beta'\gamma'(X,Z,Y)$ for the conjunction $((Z\alpha' X) \wedge (Y\beta' Z) \wedge (Y\gamma' X))$. A simple comparison of the atomic binary relations in the two conjunctions leads to $\alpha' = \gamma, \beta' = \beta^{\smile}, \gamma' = \alpha$.

Definition 11 (rotation) *The rotation of an atomic ternary relation* t *is the atomic ternary relation* $t^{\frown}$ *such that:*

$$\forall X,Y,Z(t(X,Y,Z) \Leftrightarrow t^{\frown}(Y,Z,X))$$

The rotation $T^{\frown}$ *of a general ternary relation* T *is the union of the rotations of its atomic relations:*

$$T^{\frown} = \bigcup_{t\in T}\{t^{\frown}\}$$

Similarly to the converse, the rotation of an atomic ternary relation $t = \alpha\beta\gamma$ can be expressed in terms of the atomic binary relations α, β, γ and their converses in the following way:

$$(\alpha\beta\gamma)^{\frown} = \beta\gamma^{\smile}\alpha^{\smile}$$

Figure 4 gives the converse and rotation for each of the 24 atomic ternary relations.

Definition 12 (intersection) *The intersection of two ternary relations* T_1 *and* T_2 *is the ternary relation* T *consisting of those atomic relations belonging to both* T_1 *and* T_2 *(set-theoretic intersection):*

$$\forall X, Y, Z(T(X, Y, Z) \Leftrightarrow T_1(X, Y, Z) \wedge T_2(X, Y, Z))$$

Definition 13 (composition) *The composition of two ternary relations* T_1 *and* T_2*, written* $T_1 \otimes_3 T_2$*, is the most specific ternary relation* T *such that:*

$$\forall X, Y, Z, W(T_1(X, Y, Z) \wedge T_2(X, Z, W) \Rightarrow T(X, Y, W))$$

If we know the composition for atomic ternary relations, we can compute the composition of any two ternary relations T_1 and T_2:

$$T_1 \otimes_3 T_2 = \bigcup_{t_1 \in T_1, t_2 \in T_2} t_1 \otimes_3 t_2$$

In other words, what we need is to give a composition table for atomic ternary relations, similar to Allen's (1983) composition table for temporal interval relations.

Given four 2D orientations X, Y, Z, W and two atomic ternary relations $t_1 = b_1 b_2 b_3$ and $t_2 = b'_1 b'_2 b'_3$, the conjunction $t_1(X, Y, Z) \wedge t_2(X, Z, W)$ is inconsistent if $b_3 \neq b'_1$ (see Figure 2(II) for illustration). Stated otherwise, when $b_3 \neq b'_1$ we have $t_1 \otimes_3 t_2 = \emptyset$. Therefore, in defining composition for atomic ternary relations, we have to consider four cases:

1. Case 1: $b_3 = b'_1 = e$ ($t_1 \in \{eee, lre, ooe, rle\}$ and $t_2 \in \{eee, ell, eoo, err\}$).
2. Case 2: $b_3 = b'_1 = l$ ($t_1 \in \{ell, lel, lll, lrl, orl, rll, rol, rrl\}$ and $t_2 \in \{lel, lll, llo, llr, lor, lre, lrl, lrr\}$).
3. Case 3: $b_3 = b'_1 = o$ ($t_1 \in \{eoo, llo, oeo, rro\}$ and $t_2 \in \{oeo, olr, ooe, orl\}$).
4. Case 4: $b_3 = b'_1 = r$ ($t_1 \in \{err, llr, lor, lrr, olr, rer, rlr, rrr\}$ and $t_2 \in \{rer, rle, rll, rlr, rol, rrl, rro, rrr\}$).

Composition for the CYCORD theory as introduced in (Megiddo 1976; Röhrig 1994; 1997) (see Definition 3, rule (3)) consists of one single rule. Again, this is due to the fact that the theory is not specific enough. The algebra of ternary relations has a much finer level of granularity, and hence is much more specific: composition splits into many more cases, which are grouped together in four composition tables (one composition table for each of the above four cases). See Figure 5 for details: the entries E_1, E_2, E_3, E_4 stand for

$\otimes_3$	*eee*	*ell*	*eoo*	*err*
eee	*eee*	*ell*	*eoo*	*err*
lre	*lre*	E_1	*llo*	E_2
ooe	*ooe*	*orl*	*oeo*	*olr*
rle	*rle*	E_4	*rro*	E_3

$\otimes_3$	*lel*	*lll*	*llo*	*llr*	*lor*	*lre*	*lrl*	*lrr*
ell	*ell*	*ell*	*eoo*	*err*	*err*	*eee*	*ell*	*err*
lel	*lel*	*lll*	*llo*	*llr*	*lor*	*lre*	*lrl*	*lrr*
lll	*lll*	*lll*	*llo*	E_2	*lrr*	*lre*	E_1	*lrr*
lrl	*lrl*	E_1	*llo*	*llr*	*llr*	*lre*	*lrl*	E_2
orl	*orl*	*orl*	*oeo*	*olr*	*olr*	*ooe*	*orl*	*olr*
rll	*rll*	E_4	*rro*	*rrr*	*rrr*	*rle*	*rll*	E_3
rol	*rol*	*rrl*	*rro*	*rrr*	*rer*	*rle*	*rll*	*rlr*
rrl	*rrl*	*rrl*	*rro*	E_3	*rlr*	*rle*	E_4	*rlr*

$\otimes_3$	*oeo*	*olr*	*ooe*	*orl*
eoo	*eoo*	*err*	*eee*	*ell*
llo	*lre*	E_1	*llo*	E_2
oeo	*orl*	*olr*	*oor*	*orr*
rro	*rro*	E_3	*rle*	E_4

$\otimes_3$	*rer*	*rle*	*rll*	*rlr*	*rol*	*rrl*	*rro*	*rrr*
err	*err*	*eee*	*ell*	*err*	*ell*	*ell*	*eoo*	*err*
llr	*llr*	*lre*	*lrl*	E_2	*lrl*	E_1	*llo*	*llr*
lor	*lor*	*lre*	*lrl*	*lrr*	*lel*	*lll*	*llo*	*llr*
lrr	*lrr*	*lre*	E_1	*lrr*	*lll*	*lll*	*llo*	E_2
olr	*olr*	*ooe*	*orl*	*olr*	*orl*	*orl*	*oeo*	*olr*
rer	*rer*	*rle*	*rll*	*rlr*	*rol*	*rrl*	*rro*	*rrr*
rlr	*rlr*	*rle*	E_4	*rlr*	*rrl*	*rrl*	*rro*	E_3
rrr	*rrr*	*rle*	*rll*	E_3	*rll*	E_4	*rro*	*rrr*

Figure 5: The composition tables for ternary relations: case 1, case 2, case 3 and case 4, respectively, from top to bottom.

the relations $\{lel, lll, lrl\}$, $\{llr, lor, lrr\}$, $\{rer, rlr, rrr\}$, $\{rll, rol, rrl\}$, respectively.[2]

The unique CYCORD composition rule can be checked using the four composition tables:

$$\forall X, Y, Z, W$$

$$(CR(X, Y, Z) \wedge CR(X, Z, W) \Rightarrow CR(X, Y, W))$$

CR, as we have already seen, stands for the ternary relation $\{lrl, orl, rll, rol, rrl, rro, rrr\}$.

Definition 14 (projection) *Let* T *be a ternary relation. The 1st, 2nd and 3rd projections of* T*, which we shall refer to as* $proj_1(T), proj_2(T), proj_3(T)$*, respectively, are the binary relations defined as follows:*

$$proj_1(T) = \{b_1 \in BIN | (\exists b_2, b_3 \in BIN | b_1 b_2 b_3 \in T)\};$$

$$proj_2(T) = \{b_2 \in BIN | (\exists b_1, b_3 \in BIN | b_1 b_2 b_3 \in T)\};$$

$$proj_3(T) = \{b_3 \in BIN | (\exists b_1, b_2 \in BIN | b_1 b_2 b_3 \in T)\}.$$

Definition 15 (cross product) *The cross product of three*
binary relations B_1, B_2, B_3*, written* $\Pi(B_1, B_2, B_3)$*, is the ternary relation consisting of those atomic relations* $b_1 b_2 b_3$ *such that* $b_1 \in B_1, b_2 \in B_2, b_3 \in B_3$*:*

$$\Pi(B_1, B_2, B_3) =$$

$$\{b_1 b_2 b_3 | (b_1 \in B_1, b_2 \in B_2, b_3 \in B_3)\} \cap TER$$

[2] Alternatively, one can define a single composition table for the algebra of ternary relations. Such a table would have 24×24 entries, most of which (i.e., $24 \times 24 - (16 + 64 + 16 + 64)$) would be the empty relation.

CSPs of 2D orientations

A CSP of 2D orientations (henceforth 2D-OCSP) consists of

1. a finite number of variables ranging over the set 2DO of 2D orientations[3]; and
2. relations on cyclic ordering of these variables, standing for the constraints of the CSP.

A binary (resp. ternary) 2D-OCSP is a 2D-OCSP of which the constraints are binary (resp. ternary). We shall refer to binary 2D-OCSPs as BOCSPs, and to ternary 2D-OCSPs as TOCSPs.

We now consider a 2D-OCSP P (either binary or ternary) on n variables $X_1, \ldots, X_n$.

Remark 1 (normalised 2D-OCSP) *If P is a BOCSP, we assume that for all i, j, at most one constraint involving X_i and X_j is specified. The network representation of P is the labelled directed graph defined as follows:*

1. *The vertices are the variables of P.*
2. *There exists an edge (X_i, X_j), labelled with B, if and only if a constraint of the form $(X_j B X_i)$ is specified.*

If P is a TOCSP, we assume that for all i, j, k, at most one constraint involving X_i, X_j, X_k is specified.

Definition 16 (matrix representation) *If P is a BOCSP, it is associated with an $n \times n$-matrix, which we shall refer to as P for simplicity, and whose elements will be referred to as $P_{ij}, i, j \in \{1, \ldots, n\}$. The matrix P is constructed as follows:*

1. *Initialise all entries of P to the universal relation BIN: $P_{ij} := BIN, \forall i, j \in \{1, \ldots, n\}$.*
2. *$P_{ii} := e, \forall i = 1 \ldots n$.*
3. *For all $i, j \in \{1, \ldots, n\}$ such that P contains a constraint of the form $(X_j\ B\ X_i)$: $P_{ij} := P_{ij} \cap B$; $P_{ji} := P_{ij}^{\smile}$.*

If P is a TOCSP, it is associated with an $n \times n \times n$-matrix, which we shall refer to as P, and whose elements will be referred to as $P_{ijk}, i, j, k \in \{1, \ldots, n\}$. The matrix P is constructed as follows:

1. *Initialise all entries of P to the universal relation TER: $P_{ijk} := TER, \forall i, j, k \in \{1, \ldots, n\}$.*
2. *$P_{iii} := eee, \forall i = 1 \ldots n$.*
3. *For all $i, j, k \in \{1, \ldots, n\}$ such that P contains a constraint of the form $T(X_i, X_j, X_k)$:*
   - *(a) $P_{ijk} := P_{ijk} \cap T$; $P_{ikj} := P_{ijk}^{\smile}$;*
   - *(b) $P_{jki} := P_{ijk}^{\frown}$; $P_{jik} := P_{jki}^{\smile}$;*
   - *(c) $P_{kij} := P_{jki}^{\frown}$; $P_{kji} := P_{kij}^{\smile}$;*
4. *For all $i, j \in \{1, \ldots, n\}, i < j$:*
   - *(a) $B := \bigcap_{k=1}^{n} proj_1(P_{ijk})$;*
   - *(b) $P_{iij} := \Pi(e, B, B)$; $P_{iji} := P_{iij}^{\smile}$; $P_{jii} := P_{iji}^{\frown}$;*
   - *(c) $P_{jji} := \Pi(e, B^{\smile}, B^{\smile})$; $P_{jij} := P_{jji}^{\smile}$; $P_{ijj} := P_{jij}^{\frown}$;*

[3]The set 2DO is isomorphic to the set $[0, 2\pi)$.

Definition 17 (closure under projection) *The TOCSP P is closed under projection if:*

$$\forall i, j, k, l(proj_1(P_{ijk}) = proj_1(P_{ijl}))$$

A *TOCSP* P can always be transformed into an equivalent *TOCSP* which is closed under projection. This can be achieved using a loop such as the following:

repeat
- (a) consider four variables X_i, X_j, X_k, X_l such that $proj_1(P_{ijk}) \neq proj_1(P_{ijl})$
- (b) $B := proj_1(P_{ijk}) \cap proj_1(P_{ijl})$
- (c) If $B = \emptyset$ then exit (the *TOCSP* is inconsistent)
- (d) $P_{ijk} := \Pi(B, proj_2(P_{ijk}), proj_3(P_{ijk})) \cap P_{ijk}$
- (e) $P_{ijl} := \Pi(B, proj_2(P_{ijl}), proj_3(P_{ijl})) \cap P_{ijl}$

until$(\forall i, j, k, l(proj_1(P_{ijk}) = proj_1(P_{ijl})))$

From now on, we make the assumption that a *TOCSP* is closed under projection.

Definition 18 *(Freuder 1982) An instantiation of P is any n-tuple $(z_1, z_2, \ldots, z_n)$ of $[0, 2\pi)^n$, representing an assignment of an orientation value to each variable. A consistent instantiation, or solution, is an instantiation satisfying all the constraints. A sub-CSP of size k, $k \leq n$, is any restriction of P to k of its variables and the constraints on the k variables. P is k-consistent if every solution to every sub-CSP of size $k-1$ extends to every k-th variable; it is strongly k-consistent if it is j-consistent, for all $j \leq k$.*

1-, 2- and 3-consistency correspond to node-, arc- and path-consistency, respectively (Mackworth 1977; Montanari 1974). Strong n-consistency of P corresponds to global consistency (Dechter 1992). Global consistency facilitates the exhibition of a solution by backtrack-free search (Freuder 1982).

Remark 2 *If we make the assumption that a 2D-OCSP does not include the empty constraint, which indicates a trivial inconsistency, then:*

1. *A BOCSP is strongly 2-consistent:*
   - *(a) A 1-variable BOCSP has no constraint, so its unique variable can be consistently instantiated to any value in $[0, 2\pi)$ (1-consistency).*
   - *(b) On the other hand, a 2-variable BOCSP has two variables, say X_1 and X_2, and one constraint, say $(X_2 B X_1)$. If X_1 is instantiated to any value, say z_1, then that instantiation is a solution of the 1-variable sub-CSP consisting of variable X_1, and we can still find an instantiation to X_2, say z_2, in such a way that the relation $(z_2 B z_1)$ holds. Similarly, any instantiation z_2 to X_2 is solution to the 1-variable sub-CSP consisting of variable X_2; and this can always be extended to an instantiation z_1 of X_1 such that $(X_1, X_2) = (z_1, z_2)$ satisfies the constraint $(X_2 B X_1)$.*
2. *A TOCSP is strongly 3-consistent:*

Since a BOCSP is strongly 2-consistent, it follows that if it is path-consistent (3-consistent) then it is strongly 3-consistent. The atomic relations of the algebra of ternary relations are obtained from the composition table of the algebra of binary relations, which records all possible 24 3-variable BOCSPs of atomic relations which are strongly 3-consistent. Strong 4-consistency of a TOCSP follows.

We now assume that to the plane is associated a reference system (O, x, y); and refer to the circle centred at O and of unit radius as $\mathcal{C}_{O,1}$. Given an orientation z, we denote by $rad(z)$ the radius $(O, P_z]$ of $\mathcal{C}_{O,1}$, excluding the centre O, such that the orientation of the directed straight line (OP_z) is z. An orientation z can be assimilated to $rad(z)$.

Definition 19 (sector of a binary relation) *The sector determined by an orientation z and a binary relation B, written $sect(z, B)$, is the sector of circle $\mathcal{C}_{O,1}$, excluding the centre O, representing the set of orientations z' related to z by relation B:*

$$sect(z, B) = \{rad(z') | z' \ B \ z\}$$

Remark 3 *The sector determined by an orientation and a binary relation does not include the centre O of circle $\mathcal{C}_{O,1}$. Therefore, given n orientations $z_1, \ldots, z_n$ and n binary relations $B_1, \ldots, B_n$, the intersection $\bigcap_{i=1}^{n} sect(z_i, B_i)$ is either the empty set or a set of raddi: this cannot be equal to the centre O, which would be possible if the sector determined by an orientation and a binary relation included O.*

Definition 20 *The projection, $proj(P)$, of a TOCSP P is the BOCSP P' having the same set of variables and such that:*

$$\forall i, j, k (P'_{ij} = proj_1(P_{ijk}))$$

A ternary relation, T, is projectable if $T = \Pi(proj_1(T), proj_2(T), proj_3(T))$. A TOCSP is projectable if for all i, j, k, P_{ijk} is a projectable relation.

Definition 21 *The dimension of a binary relation is the dimension of its sector. A binary relation, B, is convex if the sector determined by B and any orientation is a convex part of the plane; it is holed if*

1. *it is equal to BIN; or*
2. *the difference $BIN \setminus B$ is a binary relation of dimension 1 (is equal to e, o or $\{e, o\}$).*

The subclass of all binary relations which are either convex or holed will be referred to as BCH. There are:

1. *eight convex binary relations: $e, l, o, r, \{e, l\}, \{e, r\}, \{l, o\}, \{o, r\}$; and*
2. *four holed binary relations: $\{l, r\}, \{e, l, r\}, \{l, o, r\}, \{e, l, o, r\}$.*

A ternary relation is {convex,holed} if

1. *it is projectable; and*
2. *each of its projections belongs to BCH.*

The subclass of all {convex,holed} ternary relations will be referred to as TCH.

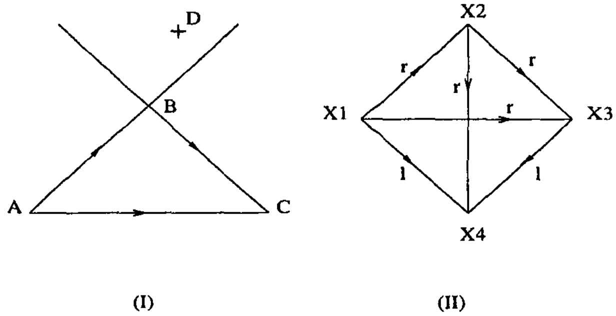

Figure 6: (I) The 'Indian tent'; and (II) its associated *BOCSP*: the *BOCSP* is path consistent but not consistent (path consistency does not detect inconsistency even for BOCSPs with atomic labels).

Example 1 (the 'Indian tent') *The 'Indian tent' consists of a clockwise triangle (ABC), together with a fourth point D which is to the left of each of the directed lines (AB) and (BC) (see Figure 6(I)).*

The knowledge about the 'Indian tent' can be represented as a BOCSP on four variables, X_1, X_2, X_3 and X_4, representing the orientations of the directed lines (AB), (AC), (BC) and (BD), respectively. From (ABC) being a clockwise triangle, we get a first set of constraints: $\{(X_2 r X_1), (X_3 r X_1), (X_3 r X_2)\}$. From D being to the left of each of the directed lines (AB) and (BC), we get a second set of constraints: $\{(X_4 l X_1), (X_4 l X_3)\}$.

If we add the constraint $(X_4 r X_2)$ to the BOCSP, this clearly leads to an inconsistency. Röhrig (1997) has shown that using the CYCORD theory one can detect such an inconsistency, whereas this cannot be detected using classical constraint-based approaches such as those in (Frank 1992; Hernández 1991).

The BOCSP is represented graphically in Figure 6(II). The CSP is path-consistent; i.e.: $\forall i, j, k (P_{ij} \subseteq (P_{ik} \otimes_2 P_{kj}))$.[4] However, as mentioned above, the CSP is inconsistent. Therefore:

Theorem 1 *Path-consistency does not detect inconsistency even for BOCSPs of atomic relations.*

The algebra of ternary relations is NP-complete:

Theorem 2 *Solving a TOCSP is NP-complete.*

Proof: Solving a TOCSP of atomic relations will be shown to be polynomial. Hence, all we need to show is that there exists a deterministic polynomial transformation of an NP-complete problem to a TOCSP.

The CYCORD theory is NP-complete (Galil & Megiddo 1977). The transformation of a problem ex-

[4] This can be easily checked using the composition table for atomic binary relations.

```
procedure s4c(P);
repeat{
  get next triple (X_i, X_j, X_k) from Queue;
  for m := 1 to n{
    Temp := P_ijm ∩ (P_ijk ⊗3 P_ikm);
    if Temp ≠ P_ijm
      {add-to-queue(X_i, X_j, X_m);change(i, j, m, Temp);}
    Temp := P_ikm ∩ (P_ikj ⊗3 P_ijm);
    if Temp ≠ P_ikm
      {add-to-queue(X_i, X_k, X_m);change(i, k, m, Temp);}
    Temp := P_jkm ∩ (P_jki ⊗3 P_jim);
    if Temp ≠ P_jkm
      {add-to-queue(X_j, X_k, X_m);change(j, k, m, Temp);}
    }
  }
until Queue is empty;
procedure change(i, j, k, T);
  P_ijk := T; P_jki := T⌒; P_kij := P_jki⌒;
  P_ikj := T⌣; P_kji := P_ikj⌒; P_jik := P_kji⌒;
```

Figure 7: A constraint propagation algorithm.

pressed in the CYCORD theory (a conjunction of CYCORD relations) into a problem expressed in the algebra of ternary relations (i.e., into a TOCSP) is immediate from the rule illustrated in Figure 3 transforming a CYCORD relation into a relation of the ternary algebra. ■

A constraint propagation algorithm

A constraint propagation procedure, $s4c(P)$, for TOCSPs is given in Figure 7. The input is a TOCSP P on n variables $X_1, \ldots, X_n$, given by its $n \times n \times n$-matrix. When the algorithm completes, P verifies the following:

$$\forall i, j, k, l \in \{1, \ldots, n\}(P_{ijk} \subseteq P_{ijl} \otimes_3 P_{ilk})$$

The algorithm makes use of a queue *Queue*. Initially, we can assume that all variable triples (X_i, X_j, X_k) such that $1 \leq i < j < k \leq n$ are entered into *Queue*. The algorithm removes one variable triple from *Queue* at a time. When a triple (X_i, X_j, X_k) is removed from *Queue*, the algorithm eventually updates the relations on the neighbouring triples (triples sharing two variables with (X_i, X_j, X_k)). If such a relation is successfully updated, the corresponding triple is sorted, in such a way to have the variable with smallest index first and the variable with greatest index last, and the sorted triple is placed in *Queue* (if it is not already there) since it may in turn constrain the relations on neighbouring triples: this is done by add-to-queue(). The process terminates when *Queue* becomes empty.

Theorem 3 *When applied to a TOCSP of size (number of variables) n, the constraint propagation algorithm runs into completion in $O(n^4)$ time.*

Proof (sketch). The number of variable triples (X_i, X_j, X_k) is $O(n^3)$. A triple may be placed in Queue at most a constant number of times (24, which is the total number of atomic relations). Every time a triple is removed from Queue for propagation, the algorithm performs $O(n)$ operations. ■

Complexity classes

Theorem 4 *The propagation procedure $s4c(P)$ achieves strong 4-consistency for the input TOCSP P.*

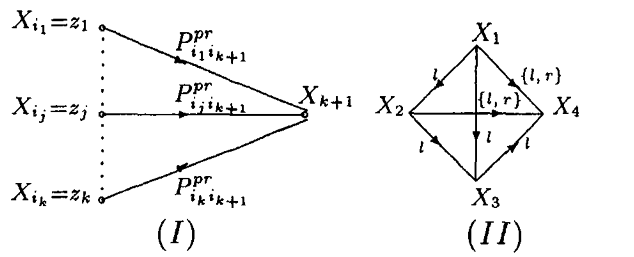

Figure 8: (I) Illustration of the proof of Theorem 5. (II) Illustration of non closure of *TCH* under strong 4-consistency.

Proof. A *TOCSP* is strongly 3-consistent (Remark 2). The algorithm clearly ensures 4-consistency, hence it ensures strong 4-consistency. ■

We refer to the subclass of all 28 entries of the four composition tables of the algebra of ternary relations as *CT*. We show that the closure under strong 4-consistency, CT^c, of CT is tractable. We then show that the subclass $PAR = \{\{oeo, ooe\}, \{eee, oeo, ooe\}, \{eee, eoo, ooe\}, \{eee, eoo, oeo, ooe\}\}$, which expresses only information on parallel orientations, is NP-complete.

Definition 22 ($s4c$-closure) *Let S denote a subclass of the algebra of ternary relations. The closure of S under strong 4-consistency, or s4c-closure of S, is the smallest subclass S^c of the algebra such that:*

1. $S \subseteq S^c$;
2. $\forall T_1, T_2 \in S^c(T_1^{\smile}, T_1^{\frown}, T_1 \cap T_2 \in S^c)$; *and*
3. $\forall T_1, T_2, T_3 \in S^c(proj_3(T_1) = proj_1(T_2) \wedge proj_1(T_1) = proj_1(T_3) \wedge proj_3(T_2) = proj_3(T_3) \Rightarrow T_3 \cap (T_1 \otimes_3 T_2) \in S^c)$.

Theorem 5 *Let P be a TOCSP expressed in TCH. If P is strongly 4-consistent then it is globally consistent.*

Van Beek (1992) used the specialisation to $n = 1$ of Helly's convexity theorem to prove a similar result for a path consistent *CSP* of Allen's convex relations. The proof of Theorem 5 will use the specialisation to $n = 2$:

Theorem 6 (Helly's Theorem (Chvátal 1983)) *Let S be a set of convex regions of the n-dimensional space $\mathbb{R}^n$. If every $n + 1$ elements in S have a non empty intersection then the intersection of all elements of S is non empty.*

Proof of Theorem 5. Since P lies in the *TCH* subclass and is strongly 4-consistent, we have the following:

1. P is equivalent to its projection, say P^{pr}, which is a *BOCSP* expressed in *BCH*.
2. The projection P^{pr} is strongly 4-consistent.

So the problem becomes that of showing that P^{pr} is globally consistent. For this purpose, we suppose that the instantiation $(X_{i_1}, X_{i_2}, \ldots, X_{i_k}) = (z_1, z_2, \ldots, z_k), k \geq 4$, is a solution to a k-variable sub-CSP, say S, of P^{pr} whose variables are $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$. We need to prove that the partial solution can be extended to any $(k+1)$st variable, say $X_{i_{k+1}}$, of P^{pr}.[5] This is equivalent to showing that the following sectors have a non empty intersection (see Figure 8(I) for illustration): $sect(z_1, P^{pr}_{i_1 i_{k+1}}), sect(z_2, P^{pr}_{i_2 i_{k+1}}), \ldots, sect(z_k, P^{pr}_{i_k i_{k+1}})$.

Since the $P^{pr}_{i_j i_{k+1}}, j = 1 \ldots k$, belong to BCH, each of these sectors is:

1. a convex subset of the plane; or
2. almost equal to the surface of circle $C_{O,1}$ (its topological closure is equal to that surface).

We split these sectors into those verifying condition (1) and those verifying condition (2). We assume, without loss of generality, that the first m verify condition (1), and the last $k-m$ verify condition (2). We write the intersection of the sectors as $I = I_1 \cap I_2$, with $I_1 = \bigcap_{j=1}^{m} sect(z_j, P^{pr}_{i_j i_{k+1}}), I_2 = \bigcap_{j=m+1}^{k} sect(z_j, P^{pr}_{i_j i_{k+1}})$.

Due to strong 4-consistency, every three of these sectors have a non empty intersection. If any of the sectors is a radius (the corresponding relation is either e or o) then the whole intersection must be equal to that radius since the sector intersects with every other two.

We now need to show that when no sector reduces to a radius, the intersection is still non empty:

Case 1: m=k. This means that all sectors are convex. Since every three of them have a non empty intersection, Helly's theorem immediately implies that the intersection of all sectors is non empty.

Case 2: m=0. This means that no sector is convex; which in turn implies that each sector is such that its topological closure covers the whole surface of $C_{O,1}$. Hence, for all $j = 1 \ldots k$:

1. the sector $sect(z_j, P^{pr}_{i_j i_{k+1}})$ is equal to the whole surface of $C_{O,1}$ minus the centre (the relation $P_{i_j i_{k+1}}$ is equal to BIN); or
2. the sector $sect(z_j, P^{pr}_{i_j i_{k+1}})$ is equal to the whole surface of $C_{O,1}$ minus one or two radii (the relation $P_{i_j i_{k+1}}$ is equal to $\{e,l,r\}$, $\{l,o,r\}$ or $\{l,r\}$).

So the intersection of all sectors is equal to the whole surface of $C_{O,1}$ minus a finite number (at most $2k$) of radii. Since the surface is of dimension 2 and a radius is of dimension 1, the intersection must be non empty (of dimension 2).

Case 3: $0 < m < k$. This means that some sectors (at least one) are convex, the others (at least one) are such that their topological closures cover the whole surface of $C_{O,1}$. The intersection I_1 is non empty due to Helly's theorem, since every three sectors appearing in it have a non empty intersection:

Subcase 3.1: I_1 is a single radius, say r. Since the sectors appearing in I_1 are less than π, there must exist two sectors, say s_1 and s_2, appearing in I_1 such that their intersection is r. Since, due to strong 4-consistency, s_1 and s_2 together with any sector appearing in I_2 form a non empty intersection, the whole intersection, i.e., I, must be equal to r.

Subcase 3.2: I_1 is a 2-dimensional (convex) sector. It should be clear that the intersection I_2 is the whole surface of $C_{O,1}$ minus a finite number of radii (at most $2(k-m)$ radii). Since a finite union of radii is of dimension 0 or 1, and that the intersection I_1 is of dimension 2, the whole intersection I must be non empty (of dimension 2).

The intersection of all sectors is non empty in all cases. The partial solution can hence be extended to variable $X_{i_{k+1}}$ (which can be instantiated with any orientation in the intersection of the k sectors). ■

It follows from Theorems 3, 4 and 5 that if the TCH subclass is closed under strong 4-consistency, it must be tractable. Unfortunately, as illustrated by the following example, TCH is not so closed.

Example 2 *The BOCSP depicted in Figure 2(II) can be represented as the projectable TOCSP P whose matrix representation verifies:* $P_{123} = lll, P_{124} = \Pi(l,\{l,r\},\{l,r\}), P_{134} = P_{234} = \Pi(l,l,\{l,r\})$. *Applying the propagation algorithm to P leaves unchanged $P_{123}, P_{134}, P_{234}$, but transforms P_{124} into the relation $\{lll, llr, lrr\}$, which is not projectable: this is done by the operation $P_{124} := P_{124} \cap (P_{123} \otimes_3 P_{134})$.*

Enumerating CT^c leads to 49 relations (including the empty relation), all of which are {convex,holed} relations. Therefore:

Corollary 1 (tractability of CT^c) *The subclass CT^c is tractable.*

Proof. Immediate from Theorems 3, 4 and 5. ■

The enumeration of CT^c is given in Figure 9.

$\emptyset$	$\Pi(o,r,l)$	$\Pi(\{l,r\},r,r)$
$\Pi(e,e,e)$	$\Pi(r,e,r)$	$\Pi(\{l,o,r\},r,l)$
$\Pi(e,l,l)$	$\Pi(r,l,e)$	$\Pi(\{l,r\},r,l)$
$\Pi(e,o,o)$	$\Pi(r,l,l)$	$\Pi(\{l,r\},l,l)$
$\Pi(e,r,r)$	$\Pi(r,l,r)$	$\Pi(r,l,\{e,l,r\})$
$\Pi(l,e,l)$	$\Pi(r,o,l)$	$\Pi(r,r,\{l,o,r\})$
$\Pi(l,l,l)$	$\Pi(r,r,l)$	$\Pi(l,l,\{l,o,r\})$
$\Pi(l,l,o)$	$\Pi(r,r,o)$	$\Pi(r,\{l,r\},r)$
$\Pi(l,l,r)$	$\Pi(r,r,r)$	$\Pi(l,l,\{l,r\})$
$\Pi(l,o,r)$	$\Pi(l,\{e,l,r\},l)$	$\Pi(l,r,\{e,l,r\})$
$\Pi(l,r,e)$	$\Pi(l,\{l,o,r\},r)$	$\Pi(r,\{l,r\},l)$
$\Pi(l,r,l)$	$\Pi(r,\{e,l,r\},r)$	$\Pi(l,\{l,r\},l)$
$\Pi(l,r,r)$	$\Pi(r,\{l,o,r\},l)$	$\Pi(r,r,\{l,r\})$
$\Pi(o,e,o)$	$\Pi(\{e,l,r\},r,r)$	$\Pi(l,\{l,r\},r)$
$\Pi(o,l,r)$	$\Pi(\{l,o,r\},l,r)$	$\Pi(r,l,\{l,r\})$
$\Pi(o,o,e)$	$\Pi(\{e,l,r\},l,l)$	$\Pi(l,r,\{l,r\})$
	$\Pi(\{l,r\},l,r)$	

Figure 9: Enumeration of the CT^c subclass.

[5] Since the $TOCSP$ P is projectable, any solution to any sub-CSP of the projection P^{pr} is solution to the corresponding sub-CSP of P. This would not be necessarily the case if P were not projectable.

Example 3 *Transforming the BOCSP of the 'Indian tent' into a TOCSP, say P', leads to $P'_{123} = rrr, P'_{124} = rrl, P'_{134} = rll, P'_{234} = rlr$. P' lies in CT^c, hence the propagation algorithm must detect its inconsistency. Indeed, the operation $P'_{124} := P'_{124} \cap (P'_{123} \otimes_3 P'_{134})$ leads to the empty relation, since $rrr \otimes_3 rll = rll$.*

We now show NP-completeness of *PAR*.

Theorem 7 (NP-completeness of *PAR*) *The subclass PAR is NP-complete.*

Proof. The subclass *PAR* belongs to NP, since solving a TOCSP of atomic relations is polynomial (Corollary 1). We need to prove that there exists a (deterministic) polynomial transformation of an NP-complete problem (we consider 3-SAT: a SAT problem of which every clause contains exactly three literals) into a TOCSP expressed in *PAR* in such a way that the former is satisfiable if and only if the latter is consistent.

Suppose that S is a 3-SAT problem, and denote by:

1. $Lit(S) = \{\ell_1, \ldots, \ell_n\}$ the set of literals appearing in S;
2. $Cl(S)$ the set of clauses of S; and
3. $BinCl(S)$ the set of binary clauses which are subclauses of clauses in $Cl(S)$.

The TOCSP, P_S, we associate with S is as follows. Its set of variables is $V = \{X(c) | c \in Lit(S) \cup BinCl(S)\} \cup \{X_0\}$. X_0 is a truth determining variable: all orientations which are equal to X_0 correspond to elements of $Lit(S) \cup BinCl(S)$ which are true, the others (those which are opposite to X_0) to elements of $Lit(S) \cup BinCl(S)$ which are false. The constraints of P_S are constructed as follows:

1. for all pair $(X(p), X(\overline{p}))$ of variables such that $\{p, \overline{p}\} \subseteq Lit(S)$, p and $\overline{p}$ should have complementary truth values; hence $X(p)$ and $X(\overline{p})$ should be opposite to each other in P_S:
$$\{oeo, ooe\}(X(p), X(\overline{p}), X_0)$$
2. for all variables $X(c_1), X(c_2)$ such that $c_1 \vee c_2$ is a clause of S, c_1 and c_2 cannot be simultaneously false; translated into P_S, $X(c_1)$ and $X(c_2)$ should not be both opposite to X_0:
$$\{eee, oeo, ooe\}(X(c_1), X(c_2), X_0)$$
3. for all variables $X(\ell_1 \vee \ell_2), X(\ell_1)$, if ℓ_1 is true then so is $(\ell_1 \vee \ell_2)$:
$$\{eee, eoo, ooe\}(X(\ell_1 \vee \ell_2), X(\ell_1), X_0)$$
4. for all other triple $(X, Y, Z) \in V^3$ of variables, add to P_S the constraint
$$\{eee, eoo, oeo, ooe\}(X, Y, Z)$$

The transformation is deterministic and polynomial. If M is a model of S, it is mapped to a solution of P_S as follows. X_0 is assigned any value of $[0, 2\pi)$. For all $\ell \in Lit(S)$, $X(\ell)$ is assigned the same value as X_0 if M assigns the value true to literal ℓ, the value opposite to that of X_0 otherwise. For all $(\ell_1 \vee \ell_2) \in BinCl(S)$, $X(\ell_1 \vee \ell_2)$ is assigned the same value as X_0 if either $X(\ell_1)$ or $X(\ell_2)$ is assigned the same value as X_0, the opposite value otherwise. On the other hand, any solution to P_S can be mapped to a model of S by assigning to every literal ℓ the value true if and only if the variable $X(\ell)$ is assigned the same value as X_0. ■

```
Input: the matrix representation of a TOCSP P;
Output: true if and only if P is consistent;
function consistent(P);
   s4c(P);
   if(P contains the empty relation)return false;
   else
     if(P contains triples labelled with non atomic relations){
       choose such a triple, say (X_i, X_j, X_k);
       T := P_ijk;
       for each atomic relation t in T{
         instantiate triple (X_i, X_j, X_k) with t (P_ijk := t);
         if(consistent(P))return true;
         }
       P_ijk := T;
       return false;
       }
     else return true;
```

Figure 10: A solution search algorithm for *TOCSPs*.

A solution search algorithm

Since the constraint propagation procedure *s4c* of Figure 7is complete for the subclass of atomic ternary relations (Corollary 1), it is immediate that a general *TOCSP* can be solved using a solution search algorithm such as the one in Figure 10, which is similar to the one provided by Ladkin and Reinefeld (1992) for temporal interval networks, except that:

1. it instantiates triples of variables at each node of the search tree, instead of pairs of variables; and
2. it makes use of the procedure *s4c*, which achieves strong 4-consistency, in the preprocessing step and as the filtering method during the search, instead of a path consistency procedure.

The other details are similar to those of Ladkin and Reinefeld's algorithm.

Related work

Representing a panorama

In (Levitt & Lawton 1990), Levitt and Lawton discussed QUALNAV, a qualitative landmark navigation system for mobile robots. One feature of the system is the representation of the information about the order of landmarks as seen by the visual sensor of a mobile robot. Such information provides the panorama of the robot with respect to the visible landmarks.

Figure 11 illustrates the panorama of an object S with respect to five reference objects (landmarks) A, B, C, D, E in Schlieder's system (Schlieder 1993) (page 527). The panorama is described by the total cyclic order of the five directed straight lines $(SA), (SB), (SC), (SD), (SE)$, and the lines which are opposite to them, namely $(Sa), (Sb), (Sc), (Sd), (Se)$:

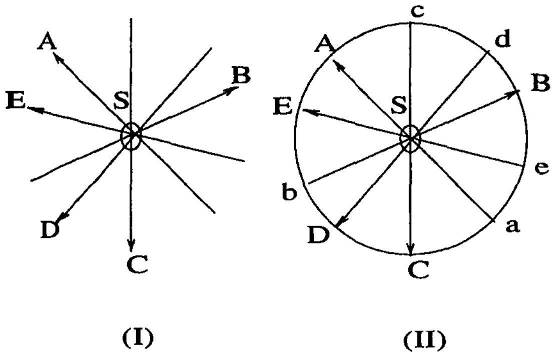

Figure 11: The panorama of a location.

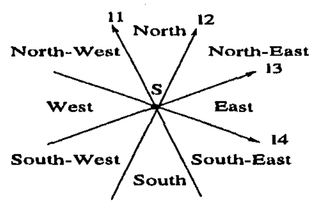

Figure 12: Frank's system of cardinal directions.

(SA)-(Sc)-(Sd)-(SB)-(Se)-(Sa)-(SC)-(SD)-(Sb)-(SE). By using the algebra of binary relations, only the five straight lines joining S to the landmarks are needed to describe the panorama: $\{(SB)r(SA), (SC)r(SB), (SD)r(SB), (SD)r(SC), (SE)l(SB), (SE)l(SA)\}$; using the algebra of ternary relations, the description can be given as a 2-relation set: $\{rll((SA), (SB), (SE)), rrr((SB), (SC), (SD))\}$.

Schlieder's system seems to make an implicit assumption, which is that the object to be located (i.e., S) is supposed not to be on any of the straight lines joining pairs of the reference objects. The use of the algebra of binary relations rules out the assumption (the relations e(qual) and o(pposite) can be used to describe object S being on a strainght line joining two reference objects). Note that Schlieder does not describe how to reason about a panorama description.

Sector models for reasoning about orientations

These models use a partition of the plane into sectors determined by straight lines passing through the reference object, say S. The sectors are generally equal, and the granularity of a sector model is determined by the number of sectors, therefore by the number of straight lines (n straight lines determine $2n$ sectors). Determining the relation of another object relative to the reference object becomes then the matter of giving the sector to which the object belongs.

Suppose that we consider a model with $2n$ sectors, determined by n (directed) straight lines $\ell_1, \ldots, \ell_n$ which we shall refer to as reference lines. We can assume without loss of generality that (the orientations of) the reference lines verify: ℓ_i is to right of ℓ_j (i.e., $(\ell_i r \ell_j)$), for all $i \in \{2, \ldots, n\}$, for all $j \in \{1, \ldots, i-1\}$. We refer to the sector determined by ℓ_i and ℓ_{i+1}, $i = 1 \ldots n-1$, as s_i, to the sector determined by ℓ_n and the directed line opposite to ℓ_1 as s_n. For each sector $s_i, i = 1 \ldots n$, the opposite sector will be referred to as s_{n+i}. Figure 12 illustrates these notions for the system of cardinal directions presented in (Frank 1992), for which $n = 4$:

1. The reference lines $\ell_1, \ldots, \ell_4$ are as indicated in the figure.
2. The sectors $s_1, \ldots, s_{2\times 4}$ are North,North-East,East,South-East,South,South-West,West and North-West, respectively.

Hernández's (Hernández 1991) sector models can also benefit from this representation.

Suppose that a description is provided, consisting of qualitative positions of objects relative to the reference object S. S may be a robot for which the current panorama has to be given; the description may consist of sentences such as "landmark 1 is North-East, and landmark 2 South of the robot". We refer to such a description as a sector description of a configuration.

A sector description can be translated into a *BOCSP* P in the following natural way. P includes all the relations described above on pairs of the reference lines. For each sentence such as the one above, the relations $(X_{(rob,1)}\ r\ \ell_2)$, $(X_{(rob,1)}\ l\ \ell_3)$, $(X_{(rob,2)}\ l\ \ell_1)$ and $(X_{(rob,2)}\ r\ \ell_2)$ are added to P. $X_{(rob,1)}$, for instance, stands for the orientation of the directed straight line joining the reference object 'robot' to landmark 1.

An important point to notice, which is not hard to show, is that a sector description is consistent if and only if the corresponding translation into a *BOCSP* is consistent. The 'only-if' is trivial. The 'if' can be shown by exploiting the fact that if the *BOCSP* is consistent then any solution to it can be mapped into a solution of the sector description by appropriate rotations of the values assigned to the variables which bring the reference lines to the desired positions.

Reasoning about 2D segments

In his paper "Reasoning About Ordering", Schlieder (1995) presented a set of line segment relations. These relations are based on the cyclic ordering of endpoints of the segments involved. We believe that reasoning about 2D segements should combine at least orientational and topological information. Orientational information would be information about cyclic ordering of the orientations of the directed lines supporting the segments; on the other hand, topological information would be the description of the relative positions of the segments' endpoints. For instance, using the algebra of binary relations on 2D orientations, as defined in this work, we could define an algebra of 2D segments, which would have the following segment relations (given a segment s, we denote by s_l and s_r its left and right endpoints, respectively, i.e., s is the segment $[s_l s_r]$; by z_s

the orientation of the directed line $(s_l s_r)$ supporting segment s):

1. If the orientations z_{s_1} and z_{s_2} are equal, the endpoints of s_2 are:
   (a) both to the left of the directed line supporting segment s_1 (one relation);
   (b) both on the line supporting segment s_1 (13 relations: see Allen's (1983) temporal interval algebra); or
   (c) both to the right of the directed line supporting segment s_1 (one relation).
2. If z_{s_2} is to the left of z_{s_1}, this leads to 25 segment relations, which are obtained as follows:
   (a) The endpoints of s_1 partition the directed line supporting the segment into five regions: the region strictly to the left of the left endpoint, the region consisting of the left endpoint, the region strictly between the two endpoints, the region consisting of the right endpoint, and the region strictly to the right of the right endpoint. Similarly, the endpoints of s_2 partition the directed line supporting the segment into five regions.
   (b) The lines supporting the segments s_1 and s_2 are intersecting, and the intersecting point is in either of the five regions of the first line, and in either of the five regions of the second line. This gives the 25 segment relations.
3. if z_{s_1} and z_{s_2} are opposite to each other, we get 15 relations in a similar manner as in point 1. above.
4. if z_{s_2} is to the right of z_{s_1}, we get 25 relations in a similar manner as in point 2. above.

Therefore the total number of segment relations would be 80.

Summary and future work

We have provided a refinement of the theory of cyclic ordering of 2D orientations, known as CYCORD theory (Megiddo 1976; Röhrig 1994; 1997). The refinement has led to an algebra of ternary relations, for which we have given a constraint propagation algorithm and shown several complexity results.

We have discussed briefly how this work relates to some others in the literature; in particular, the discussion has highlighted the following: (1) Existing systems for reasoning about 2D orientations are covered by the presented approach (CYCORDs (Megiddo 1976; Röhrig 1994; 1997) and sector models (Frank 1992; Hernández 1991)); (2) The presented approach seems more adequate than the one in (Schlieder 1993) for the representation of a panorama.

There has been much work on Allen's interval algebra (Allen 1983). For instance, Nebel and Bürckert (1995) have shown that the ORD-Horn subclass of the algebra was the unique maximal tractable subclass containing all 13 atomic relations. Most of this work could be adapted for the two algebras of 2D orientations we have defined.

Finally, a calculus of 3D orientations, similar to what we have presented for 2D orientations, might be developed.

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