Approximate Qualitative Temporal Reasoning

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Abstract

In this paper I define four sets of binary topological relations between one dimensional regions in a one-dimensional space: (1) boundary *in*sensitive relations in a *non*-directed space, (2) boundary *in*sensitive relations in a directed space, (3) boundary sensitive relations in a *non*-directed space, and (4) boundary sensitive relations in a directed space.

For each of these sets of relations between exact regions I define a corresponding set of relations between approximations of regions with respect to an underlying regional partition. I discuss syntactic and semantic generalizations of relations between exact regions to corresponding relations between approximations and show the equivalence of syntactic and semantic generalization.

Introduction

Every temporal object and every spatio-temporal object is located at a unique region of time bounded by the begin and the end of its existence. In every moment of time a spatio-temporal object, o, is exactly located at a single region, x, of space (Casati & Varzi 1995). This region is the exact or precise location of o at the time point t, i.e., $x = r(o)$ at t. Spatio-temporal wholes have temporal parts, which are located at parts of the temporal regions occupied by their wholes. Consider, for example, the region of time, y, where the object, o, is located temporally, while being spatially located at the spatial region x. If y is a maximal connected temporal region, i.e., o was once spatially located at x for a while, left and never came back, then y is bounded by the time instances (points) t_1 and t_2. Since time is a totally ordered set of time points forming a directed one-dimensional space (McTaggart 1927; Geach 1966), we have $t_1 < t_2$.

In knowledge representation we are interested in representing spatio-temporal reality as experienced by human beings. In this context it is essential to represent spatio-temporal location (Galton 1997). In this paper I concentrate on representing temporal location. One way of representing temporal location is to represent qualitative relationships between temporal regions occupied by temporal and spatio-temporal objects and their parts (Allen 1983).

Human knowledge is gained by observations and reasoning about observations. (Bittner 1999) argued[1] that by means of observation and measurement (a precise form of observation) humans *cannot know* the exact spatial and, hence, exact temporal location. Observations and measurement yield knowledge about *approximate spatio-temporal location*, i.e., knowledge about relations between spatio-temporal objects and cells of regional partitions of space and time. Regional partitions are sets of regions (cells) that do not overlap but sum up the whole space. Regional partitions are created by measurement and observation processes. Approximate location can be known by observing (qualitative) relations between objects and the cells of the underlying regional partitions.

Consider, for example, the measurement of temporal location. Measurement of temporal location is based on clocks. A clock creates a regional partition of the time-line. The cells forming this partition are time intervals separated by 'clock ticks'. Measurement of temporal location involves counting time intervals and observing relationships between time intervals and (temporal parts of) temporal or spatio-temporal objects. No matter how fine the resolution of the partition there are always partition cells that are disjoint to (the exact region of) the observed object, there may be partition cells that are completely covered by (the exact region of) the observed object, and there always are partition cells that are partly covered by (the exact region of) the observed object. Consequently, observing spatio-temporal location means observing *relations* between partition cells and regions occupied by spatio-temporal objects, i.e., observing *approximate location* rather than exact location. Other examples of regional partition in which approximate temporal location is observed is the partition of the time line into past and future separated by the present moment, the hours of the day, forenoon and afternoon, the four seasons. Consequently, in the context of knowledge representation reasoning about approximate spatio-temporal location, i.e., approximations of spatial and temporal regions, is more important than reasoning about exact location, i.e., spatial and temporal regions themselves.

In the remainder of this paper I omit the distinction between objects and spatial and temporal regions and the

[1] Based on (Carnap 1966).

(functional) relation of (exact) location between them and concentrate on the approximation of the exact regions (of objects) with respect to an underlying regional partition. Moreover, I concentrate on temporal regions and approximations of temporal regions. This paper builds on (Bittner & Stell 1998) and (Bittner & Stell 2000), in which various ways of providing qualitative approximations of regions with respect to a partition of the plane as well as reasoning about those approximations were described.

(Bittner & Stell 2000) showed that approximate qualitative reasoning is based on: (1) Jointly exhaustive and pairwise disjoint sets of qualitative relations between exact regions. These relations need to be defined in terms of the meet operation of the underlying Boolean algebra structure of the domain of regions. As a set these relations must form a lattice with bottom and top element. (2) Approximations of regions with respect to a regional partition of the underlying space. (3) Pairs of join and meet operations on those approximations, which approximate join and meet operations on exact regions. This this is reflected by the structure of this paper:

I start with the definition of qualitative relations between temporal regions. I distinguish boundary sensitive and boundary insensitive sets of relations and relations between regions in a directed and non-directed underlying one-dimensional space. Based on the definition of approximations of temporal regions with respect to an underlying regional partition I then generalize the definitions of relations between temporal regions to definitions of relations between approximations of regions. This provides the formal basis for qualitative temporal reasoning about approximate location in time. The conclusions are given in the end.

Relations between one dimensional regions

Boundary insensitive relations

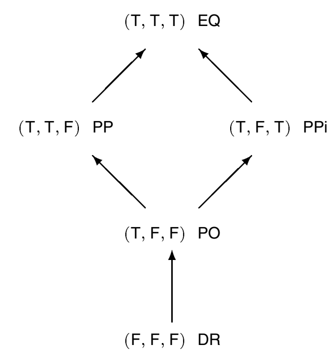
RCC5 relations Given two regions x and y boundary insensitive topological relation (RCC5 relations[2]) between them can be determined by considering the triple of boolean values (Bittner & Stell 2000):

$$(x \wedge y \neq \bot,\ x \wedge y = x,\ x \wedge y = y).$$

The correspondence between such triples and boundary insensitive relations between regions on an undirected line is given in the following table (Bittner & Stell 2000).

$x \wedge y \neq \bot$	$x \wedge y = x$	$x \wedge y = y$	RCC5
F	F	F	DR
T	F	F	PO
T	T	F	PP
T	F	T	PPi
T	T	T	EQ

The set of triples is partially ordered by setting $(a_1, a_2, a_3) \leq (b_1, b_2, b_3)$ iff $a_i \leq b_i$ for $i = 1, 2, 3$, where the Boolean values are ordered by F $<$ T. The Hasse diagram is given in the diagram below. (Bittner & Stell 2000) call this graph the RCC5 lattice to distinguish it from the conceptual neighborhood graph (Goodday & Cohn 1994).



RCC_1^9 relations Given two one dimensional regions x and y. I assume that x and y are maximal connected one dimensional regions, i.e., intervals. Boundary insensitive topological relation between *intervals* x and y on a *directed* line (RCC_1^9 relations) can be determined by considering the triple of truth values:

$$(x \wedge y \not\sim \bot,\ x \wedge y \sim x,\ x \wedge y \sim y)$$

where

$$x \wedge y \not\sim \bot = \begin{cases} FL & \text{if } x \wedge y \neq \bot \text{ and } x \ll y \geq x \gg y \\ FR & \text{if } x \wedge y \neq \bot \text{ and } x \ll y < x \gg y \\ T & \text{if } x \wedge y = \bot \end{cases}$$

where

$$x \wedge y \sim x = \begin{cases} FR & \text{if } x \wedge y \neq x \text{ and } y \ll x \geq y \gg x \\ FL & \text{if } x \wedge y \neq x \text{ and } y \ll x < y \gg x \\ T & \text{if } x \wedge y = x \end{cases}$$

and where

$$x \wedge y \sim y = \begin{cases} FL & \text{if } x \wedge y \neq y \text{ and } x \ll y \geq x \gg y \\ FR & \text{if } x \wedge y \neq y \text{ and } x \ll y < x \gg y \\ T & \text{if } x \wedge y = y \end{cases}$$

with

$$x \ll y = \begin{cases} T & \text{if } L(x) \wedge L(y) = L(x) \\ F & \text{if } L(x) \wedge L(y) \neq L(x) \end{cases} \quad \text{and}$$

$$x \gg y = \begin{cases} T & \text{if } R(x) \wedge R(y) = R(x) \\ F & \text{if } R(x) \wedge R(y) \neq R(x) \end{cases}$$

$L(x)$ $(R(y))$ is the one dimensional region occupying the whole line left (right)[3] of x. The intuition behind FL (FR)

[2] I use the notion RCC in order to stress the correspondence between the relations defined in this paper and relations defined by Cohn and his co-workers in terms of the region connection calculus (RCC) (Cohn *et al.* 1997). Correspondence in this context means that I am talking about regular regions that satisfy the RCC-axioms (Randell, Cui, & Cohn 1992) and that similar relations could be defined or have been defined in terms of RCC, e.g., (Randell, Cui, & Cohn 1992; Cohn, Gooday, & Bennett 1994; Cohn *et al.* 1997). I am going to use sub- and superscripts (e.g., RCC_1^9) where the superscript refers to the number of relations in the denoted set and the subscript refers to the dimension of the regions and the embedding space.

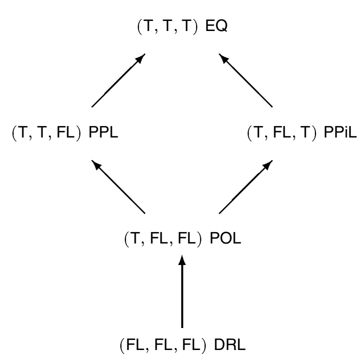
[3] I use the spatial metaphor of a line extending from the left to the right rather than the time-line extending from the past to the future in order to focus on the aspects of the time-line as a one-dimensional directed space. Time itself is much more difficult. For example, it is not clear if the future already exists yet (Broad 1923).

is "false because of parts 'sticking out' to the left (right)"[4]. The triples formally describe jointly exhaustive and pairwise disjoint relations under the assumption that x and y are intervals in a one dimensional directed space. The correspondence between the triples and the boundary insensitive relations between intervals is given in the following table.

$x \wedge y \not\sim \bot$	$x \wedge y \sim x$	$x \wedge y \sim y$	RCC_1^9
FL	FL	FL	DRL
FR	FR	FR	DRR
T	FL	FL	POL
T	FR	FR	POR
T	T	FL	PPL
T	T	FR	PPR
T	FL	T	PPiL
T	FR	T	PPiR
T	T	T	EQ

For example. The intuition behind $\mathsf{DRL}(x, y)$ is that x and y do not overlap and x is left of y. The intuition behind $\mathsf{POL}(x, y)$ is that x and y do overlap without containing each other and the non overlapping parts of x are left of y. The intuition behind $\mathsf{PPL}(x, y)$ is that x is contained in y but x does not cover the very right parts of y. Possible geometric interpretations of the relations defined above are given in Figure 1.

Assuming the ordering $\mathsf{FL} < \mathsf{T} < \mathsf{FR}$ a lattice is formed, which has $(\mathsf{FL}, \mathsf{FL}, \mathsf{FL})$ as minimal element and $(\mathsf{FR}, \mathsf{FR}, \mathsf{FR})$ as maximal element. The lower part of the Hasse diagram of this lattice is given in the diagram below.



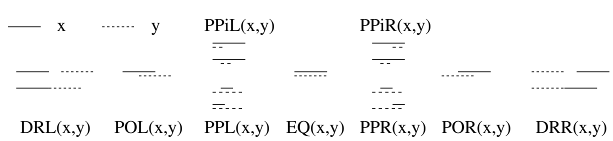


Figure 1: Possible geometric interpretations of the RCC_1^9 relations.

Boundary sensitive relations

RCC8 relations In order to describe boundary sensitive relations between regions x and y (Bittner & Stell 2000) use a triple, where the three entries may take one of three truth values rather than the two Boolean ones. The scheme has the form:

$$(x \wedge y \not\equiv \bot, x \wedge y \equiv x, x \wedge y \equiv y)$$

where

$$x \wedge y \not\equiv \bot = \begin{cases} \mathsf{T} & \text{if the interiors of } x \text{ and } y \text{ overlap,} \\ & i.e., x \wedge y \neq \bot \\ \mathsf{M} & \text{if only the boundaries } x \text{ and } y \text{ overlap,} \\ & i.e., x \wedge y = \bot \text{ and } \delta(x) \wedge \delta(y) \neq \bot \\ \mathsf{F} & \text{if there is no overlap between } x \text{ and } y, \\ & i.e., x \wedge y = \bot \text{ and } \delta(x) \wedge \delta(y) = \bot \end{cases}$$

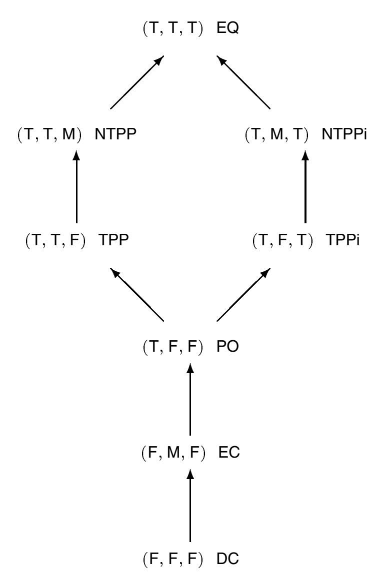
and where

$$x \wedge y \equiv x = \begin{cases} \mathsf{T} & \text{if } x \text{ is contained in the interior of } y, \\ & i.e., x \wedge y = x \text{ and } \delta(x) \wedge \delta(y) = \bot \\ \mathsf{M} & \text{if } x \text{ is contained in } y \text{ but not its interior,} \\ & i.e., x \wedge y = x \text{ and } \delta(x) \wedge \delta(y) \neq \bot \\ \mathsf{F} & \text{if } x \text{ is not contained within } y, \\ & i.e., x \wedge y \neq x \end{cases}$$

and similarly for $x \wedge y \equiv y$. The correspondence between such triples and boundary sensitive topological relations is given in the following table (Bittner & Stell 2000).

$x \wedge y \not\equiv \bot$	$x \wedge y \equiv x$	$x \wedge y \equiv y$	RCC8
F	F	F	DC
M	F	F	EC
T	F	F	PO
T	M	F	TPP
T	T	F	NTPP
T	F	M	TPPi
T	F	T	NTPPi
T	T	T	EQ

(Bittner & Stell 2000) define $\mathsf{F} < \mathsf{M} < \mathsf{T}$ and call the corresponding Hasse diagram (diagram below) RCC8 *lattice* to distinguish it from the conceptual neighborhood graph (Goodday & Cohn 1994).



[4]Notice that this dose not exclude that there are also 'parts sticking out' to the opposite side.

In this paper I concentrate on regions of one-dimensional space and relations between them. In order to distinguish sets of relations between one dimensional regions from relations between regions of higher dimension I use the notion RCC_1^8 rather than RCC8. Possible geometric interpretations of their RCC_1^8 relations are given in Figure 2.

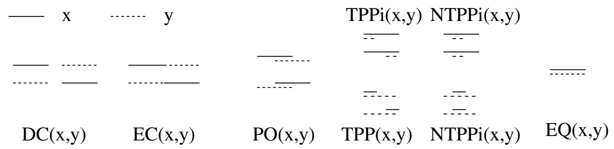


Figure 2: Geometric interpretations of RCC_1^8 relations between one-dimensional regions of a non-directed line.

RCC_1^{15} relations In order to describe boundary sensitive relations between *intervals* on a *directed* line (RCC_1^{15})[5] we define the relationship between x and y by using a triple, where the three entries may take one of four truth values. The scheme has the form

$$(x \wedge y \not\approx \bot, x \wedge y \approx x, x \wedge y \approx y)$$

where

$$x \wedge y \not\approx \bot = \begin{cases} \mathsf{T} & x \wedge y \not\equiv \bot = \mathsf{T} \\ \mathsf{ML} & x \wedge y \not\equiv \bot = \mathsf{M} \text{ and } x \ll y \geq x \gg y \\ \mathsf{MR} & x \wedge y \not\equiv \bot = \mathsf{M} \text{ and } x \ll y < x \gg y \\ \mathsf{FL} & x \wedge y \not\equiv \bot = \mathsf{F} \text{ and } x \ll y \geq x \gg y \\ \mathsf{FR} & x \wedge y \not\equiv \bot = \mathsf{F} \text{ and } x \ll y < x \gg y \end{cases}$$

and where

$$x \wedge y \approx x = \begin{cases} \mathsf{T} & x \wedge y \equiv x = \mathsf{T} \\ \mathsf{MR} & x \wedge y \equiv x = M \text{ and } y \ll x \geq y \gg x \\ \mathsf{ML} & x \wedge y \equiv x = M \text{ and } y \ll x < y \gg x \\ \mathsf{FR} & x \wedge y \equiv x = F \text{ and } y \ll x \geq y \gg x \\ \mathsf{FL} & x \wedge y \equiv x = F \text{ and } y \ll x < y \gg x \end{cases}$$

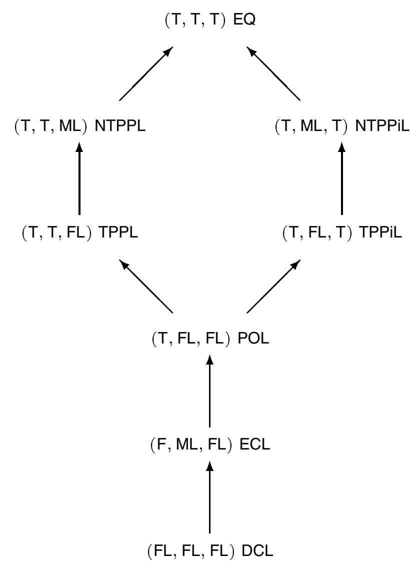
and similarly for $x \wedge y \approx y$.

The correspondence between such triples, boundary sensitive topological relations between intervals on a directed line, and the 13 relations defined by (Allen 1983) is given in the table below.

$x \wedge y \not\approx \bot$	$x \wedge y \approx x$	$x \wedge y \approx y$	RCC_1^{15}	Allen
FL	FL	FL	DCL	before
FR	FR	FR	DCR	after
ML	FL	FL	ECL	meets
MR	FR	FR	ECR	meets$_i$
T	FL	FL	POL	overlaps
T	FR	FR	POR	overlaps$_i$
T	ML	FL	TPPL	starts
T	MR	FR	TPPR	finishes
T	T	FL	NTPPL	during
T	T	FR	NTPPR	during
T	FL	ML	TPPiL	starts$_i$
T	FR	MR	TPPiR	finishes$_i$
T	FL	T	NTPPiL	during$_i$
T	FR	T	NTPPiR	during$_i$
T	T	T	EQ	equal

[5]To be distinguished from RCC15 relations (Cohn *et al.* 1997) between concave regions of higher dimension.

We define $\mathsf{FL} < \mathsf{ML} < \mathsf{T} < \mathsf{MR} < \mathsf{FR}$ and call the corresponding Hasse diagram (the diagram below shows the lower part) RCC_1^{15} *lattice* to distinguish it from the conceptual neighborhood graph (Freksa 1992). Possible geometric interpretations of the lower RCC_1^{15} relations are given in Figure 3.



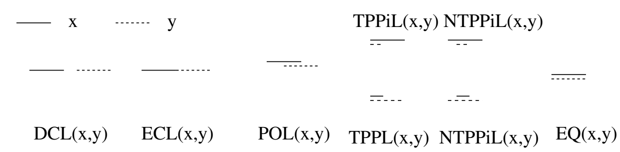


Figure 3: Geometric interpretations of the lower RCC_1^{15} relations between connected intervals.

Approximations

Approximating regions

Boundary insensitive approximation Consider the set of regions, R, of a one-dimensional space. By imposing a partition, G, on R we can approximate elements of R by elements of Ω_3^G (Bittner & Stell 1998). That is, we approximate regions in R by functions from G to the set $\Omega_3 = \{\mathsf{fo}, \mathsf{po}, \mathsf{no}\}$. The function which assigns to each region $r \in R$ its approximation will be denoted $\alpha_3 : R \to \Omega_3^G$. The value of $(\alpha_3 r)g$ is fo if r covers all the of the cell g, it is po if r covers some but not all of the interior of g, and it is no if there is no overlap between r and g. (Bittner & Stell 1998) call the elements of Ω_3^G the overlap & containment sensitive approximations of regions $r \in R$ with respect to the underlying regional partition G.

Boundary sensitive approximation Consider one dimensional non-directed space. We can further refine the approximation of regions R with respect to the partition G by taking boundary points shared by neighboring partition regions into account. That is, we approximate regions in R by functions from $G \times G$ to the set $\Omega_4 = \{\mathsf{fo}, \mathsf{bo}, \mathsf{nbo}, \mathsf{no}\}$. The function which assigns to each region $r \in R$ its boundary sensitive approximation will be denoted $\alpha_4 : R \to \Omega_4^{G\times G}$. The value of $(\alpha_4 r)(g_i, g_j)$ is fo if r covers all of the cell g_i, it is bo if r covers the boundary point, (g_i, g_j), shared by the cell g_i and g_j and some but not all of the interior of g_i, it is nbo if r does not cover the boundary point (g_i, g_j) and covers some but not all of the interior of g_i, and it is no if there is no overlap between r and g_i.

The Semantic of approximate regions Each approximate region $X \in \Omega_3^G$ ($X \in \Omega_4^{G\times G}$) stands for a set of precise regions, i.e., all those precise regions having the approximation X. This set which will be denoted $[\![X]\!]^3$ ($[\![X]\!]^4$) provides a semantic for approximate regions.

$$[\![X]\!]^3 = \{r \in R \mid \alpha_3 r = X\},\ [\![X]\!]^4 = \{r \in R \mid \alpha_4 r = X\}$$

Where ever the context is clear the superscript is omitted.

Approximate operations

The domain of regions is equipped with join and meet operations, $\vee$ and $\wedge$. (Bittner & Stell 1998) showed that join meet operations on regions can be approximated by pairs of greatest minimal and least maximal operations on approximations. In this paper I discuss the operations $\overline{\wedge}$ and $\underline{\wedge}$ on boundary insensitive approximations and boundary sensitive approximations. A detailed discussion can be found in (Bittner & Stell 1998).

Boundary insensitive operations Firstly we define operations $\underline{\wedge}$ and $\overline{\wedge}$ on the set $\Omega_3 = \{\mathsf{fo}, \mathsf{po}, \mathsf{no}\}$.

$\underline{\wedge}$	no	po	fo
no	no	no	no
po	no	no	po
fo	no	po	fo

$\overline{\wedge}$	no	po	fo
no	no	no	no
po	no	po	po
fo	no	po	fo

These operations extend to elements of Ω_3^G (i.e. the set of functions from G to Ω_3) by

$$(X \,\underline{\wedge}\, Y)g = (Xg) \,\underline{\wedge}\, (Yg)$$

and similarly for $\overline{\wedge}$.

Boundary sensitive operations We define the operations $\overline{\wedge}$ on the set $\Omega_4 = \{\mathsf{fo}, \mathsf{bo}, \mathsf{nbo}, \mathsf{no}\}$ as:

$\overline{\wedge}$	no	nbo	bo	fo
no	no	no	no	no
nbo	no	nbo	nbo	nbo
bo	no	nbo	bo	bo
fo	no	nbo	bo	fo

These operations extend to elements of $\Omega_4^{G\times G}$ (i.e. the set of functions from $G \times G$ to Ω_4) by $(X \,\overline{\wedge}\, Y)(g_i, g_j) = (X(g_i, g_j)) \,\overline{\wedge}\, (Y(g_i, g_j))$.

The definition of the operations $\underline{\wedge}$ is slightly more complicated. In this case we need to take the approximation values referring to *both* boundary points (g_i, g_{i-1}) and (g_i, g_{i+1}) into account. Let

$$N(g_i) = \{((X\ (g_i, g_{i-1})), (Y\ (g_i, g_{i-1}))), \\ ((X\ (g_i, g_{i+1})), (Y\ (g_i, g_{i+1})))\}$$

be the set of pairs of approximation values of X and Y with respect to g_i. We define the operation $X \,\underline{\wedge}\, Y$ as follows:

$$(X \,\underline{\wedge}\, Y)(g_i, g_{i+1}) = (X(g_i, g_{i+1}))\ (\underline{\wedge}^{N(g_i)})\ (Y(g_i, g_{i+1}))$$

where $(\underline{\wedge}^{N(g_i)})$ is defined as

$\underline{\wedge}^N$	no	nbo	bo	fo
no	no	no	no	no
nbo	no	$\gamma(N)$	$\gamma(N)$	nbo
bo	no	$\gamma(N)$	bo	bo
fo	no	nbo	bo	fo

and $\gamma(N)$ is defined as

$$\gamma(N) = \begin{cases} \mathsf{no} & \text{if } (\mathsf{bo}, \mathsf{bo}) \notin N \\ \mathsf{nbo} & \text{if } (\mathsf{bo}, \mathsf{bo}) \in N \end{cases}.$$

This definition corresponds to the definitions of operations on boundary sensitive approximations of two-dimensional regions in the plane discussed in (Bittner & Stell 1998).

Semantic and Syntactic Generalizations of RCC*

(Bittner & Stell 2000) showed that there are two approaches to generalizing RCC relations between precise regions to approximate ones: the semantic and the syntactic.

Semantic We can define the RCC relationship between approximate regions X and Y to be the set of relationships which occur between any pair of precise regions approximated by X and Y. That is, we can define

$$\mathcal{SEM}(X, Y) = \{RCC(x, y) \mid x \in [\![X]\!] \text{ and } y \in [\![Y]\!]\}.$$

Syntactic We can take a formal definition of RCC in the precise case, which uses operations on R, and generalize this to work with approximate regions by replacing the operations on R by analogous ones for Ω^G or $\Omega^{G\times G}$.

In the remainder of this section I discuss syntactic and semantic generalizations for RCC5 , RCC_1^8 , RCC_1^9 , and RCC_1^{15} .

Generalization of RCC5 relations

Syntactic generalization If X and Y are approximate regions (i.e. functions from G to Ω_3) we can consider the two triples of Boolean values (Bittner & Stell 2000):

$$(X \,\underline{\wedge}\, Y \neq \bot,\ X \,\underline{\wedge}\, Y = X,\ X \,\underline{\wedge}\, Y = Y),$$
$$(X \,\overline{\wedge}\, Y \neq \bot,\ X \,\overline{\wedge}\, Y = X,\ X \,\overline{\wedge}\, Y = Y).$$

In the context of approximate regions, the bottom element, $\bot$, is the function from G to Ω_3 which takes the value no for every element of G. Each of the above triples defines an RCC5 relation, so the relation between X and Y can be measured by a pair of RCC5 relations. These relations will be denoted by $\underline{R}(X, Y)$ and $\overline{R}(X, Y)$.

Theorem 1 ((Bittner & Stell 2000)) *The pairs* $(\underline{R}(X, Y), \overline{R}(X, Y))$ *which can occur are all pairs* (a, b) *where* $a \leq b$ *with the exception of* (PP, EQ) *and* (PPi, EQ).

Correspondence of semantic and syntactic generalization Let the syntactic generalization of RCC5 be defined by

$$\mathcal{SYN}(X,Y) = (\underline{R}(X,Y), \overline{R}(X,Y)),$$

where $\underline{R}$ and $\overline{R}$ are as defined above.

Theorem 2 ((Bittner & Stell 2000)) *For any approximate regions X and Y syntactic and semantic generalization of RCC5 are equivalent in the sense that*

$$\mathcal{SEM}(X,Y) = \{\rho \in RCC5 \mid \underline{R}(X,Y) \leq \rho \leq \overline{R}(X,Y)\},$$

where RCC5 is the set $\{\mathsf{EQ}, \mathsf{PP}, \mathsf{PPi}, \mathsf{PO}, \mathsf{DR}\}$*, and $\leq$ is the ordering in the RCC5 lattice.*

Generalization of RCC_1^8 relations

Syntactic generalization Let X and Y be boundary sensitive approximations of regions x and y. The generalized scheme has the form

$$\begin{array}{c}((X \underline{\wedge} Y \not\equiv \bot, X \underline{\wedge} Y \equiv X, X \underline{\wedge} Y \equiv Y),\\ (X \overline{\wedge} Y \not\equiv \bot, X \overline{\wedge} Y \equiv X, X \overline{\wedge} Y \equiv Y))\end{array}$$

where

$$X \underline{\wedge} Y \not\equiv \bot = \begin{cases} \mathsf{T} & X \underline{\wedge} Y \neq \bot \\ \mathsf{M} & X \underline{\wedge} Y = \bot \text{ and } \delta(X) \underline{\wedge} \delta(Y) \neq \bot \\ \mathsf{F} & X \underline{\wedge} Y = \bot \text{ and } \delta(X) \underline{\wedge} \delta(Y) = \bot \end{cases}$$

and where

$$X \underline{\wedge} Y \equiv X = \begin{cases} \mathsf{T} & X \underline{\wedge} Y = X \text{ and } \delta(X) \underline{\wedge} \delta(Y) = \bot \\ \mathsf{M} & X \underline{\wedge} Y = X \text{ and } \delta(X) \underline{\wedge} \delta(Y) \neq \bot \\ \mathsf{F} & X \underline{\wedge} Y \neq X \end{cases}$$

and similarly for $X \underline{\wedge} Y \approx Y$, $X \overline{\wedge} Y \not\approx \bot$, $X \overline{\wedge} Y \approx X$,and $X \overline{\wedge} Y \approx Y$. In this context the bottom element, $\bot$, is either the value no or the function from $G \times G$ to Ω_4 which takes the value no for every element of $G \times G$.

Assume the partial order of the RCC_1^8 -lattice. $\delta(X) \underline{\wedge} \delta(Y) \neq \bot$ is true if and only if the least relation RCC_1^8 -relation that can hold between $x \in [\![X]\!]$ and $y \in [\![Y]\!]$ involves boundary intersection of $\delta(x)$ and $\delta(y)$ at a boundary point, (g_i, g_j), of the underlying partition G. $\delta(X) \overline{\wedge} \delta(Y) \neq \bot$ is true if and only if the greatest RCC_1^8 -relation that can hold between $x \in [\![X]\!]$ and $y \in [\![Y]\!]$ involves boundary intersection at a boundary point in G. For a detailed discussion of the 2D case see (Bittner & Stell 2000).

Each of the above triples defines a RCC_1^8 relation, so the relation between X and Y can be measured by a pair of RCC_1^8 relations. These relations will be denoted by $\underline{R^8}(X,Y)$ and $\overline{R^8}(X,Y)$. Let X and Y be approximations of one dimensional regions in one dimensional space. Then the following holds:

Theorem 3 *The pairs $(\underline{R^8}(X,Y), \overline{R^8}(X,Y))$ which can occur are all pairs (a,b) where $a \leq b$ with the exception of* $(\mathsf{TPP},\mathsf{EQ})$, $(\mathsf{TPPi},\mathsf{EQ})$,$(\mathsf{NTPP},\mathsf{EQ})$, $(\mathsf{NTPPi},\mathsf{EQ})$, $(\mathsf{EC},\mathsf{TPP})$, $(\mathsf{EC},\mathsf{TPPi})$, $(\mathsf{EC},\mathsf{EQ})$, $(\mathsf{DC},\mathsf{EC})$, $(\mathsf{DC},\mathsf{TPP})$, $(\mathsf{DC},\mathsf{TPPi})$, $\mathsf{EC},\mathsf{NTPP})$, $(\mathsf{EC},\mathsf{NTPPi})$, $(\mathsf{TPP},\mathsf{NTPP})$, $(\mathsf{TPP},\mathsf{NTPPi})$[6].

[6] This is an application of theorem 5 in (Bittner & Stell 2000) to the one-dimensional case.

Correspondence of syntactic and semantic generalization Let $SEM(X,Y)$ be a set of RCC_1^8 relations defined as $SEM(X,Y) = \{\rho \in \mathsf{RCC}_1^8 \mid \rho(x,y), x \in [\![X]\!], y \in [\![Y]\!]\}$.

Theorem 4 *If there are $g_i, g_j \in G$ such that $(X(g_i,g_j)) = (Y(g_i,g_j)) = \mathsf{bo}$ then $\min(SEM(X,Y)) = \mathsf{PO}$*[7].

Assume $(X(g_i,g_j)) = \mathsf{bo}$ and $(Y(g_i,g_j)) = \mathsf{bo}$. Since $\mathsf{bo} \wedge \mathsf{bo} = \mathsf{bo}$ we have $X \underline{\wedge} Y \neq \bot$ and possibly $X \underline{\wedge} Y = X$, i.e., $\underline{R^8}(X,Y) \geq \mathsf{PO}$. This conflicts with $\min(SEM(X,Y)) = \mathsf{PO}$. We define the semantically corrected syntactic generalization of RCC_1^8 as:

$$\mathcal{SYN}(X,Y) = (\underline{R_c^8}(X,Y), \overline{R^8}(X,Y))$$

where $\underline{R_c^8}(X,Y) = \mathsf{PO}$ if there are $g_i, g_j \in G$ such that $(X(g_i,g_j)) = (Y(g_i,g_j)) = \mathsf{bo}$ and $\underline{R_c^8}(X,Y) = \underline{R^8}(X,Y)$ otherwise. The semantic generalization of RCC_1^8 relations is defined as $\mathcal{SEM}(X,Y) = \{\rho \in \mathsf{RCC}_1^8 \mid \underline{R_c^8}(X,Y) \leq \rho \leq \overline{R^8}(X,Y)\}$.

Theorem 5 *For any boundary sensitive approximations X and Y of regular one dimensional regions, the syntactic and semantic generalization of RCC_1^8 are equivalent in the sense that $\mathcal{SYN}(X,Y) = \mathcal{SEM}(X,Y)$*[8].

Generalization of RCC_1^9 relations

Syntactic generalization If X and Y are approximate regions then we can consider the two triples of Boolean values:

$$\begin{array}{c}(X \underline{\wedge} Y \not\sim \bot,\ X \underline{\wedge} Y \sim X,\ X \underline{\wedge} Y \sim Y),\\ (X \overline{\wedge} Y \not\sim \bot,\ X \overline{\wedge} Y \sim X,\ X \overline{\wedge} Y \sim Y).\end{array}$$

where

$$X \underline{\wedge} Y \not\sim \bot = \begin{cases} FL & \text{if } X \underline{\wedge} Y \neq \bot \text{ and} \\ & \quad (X \ll Y) \geq (X \gg Y) \\ FR & \text{if } X \underline{\wedge} Y \neq \bot \text{ and} \\ & \quad (X \ll Y) < (X \gg Y) \\ T & \text{if } X \underline{\wedge} Y = \bot \end{cases}$$

and where

$$X \overline{\wedge} Y \not\sim \bot = \begin{cases} FL & \text{if } X \overline{\wedge} Y \neq \bot \text{ and} \\ & \quad (X \ll Y) \geq (X \gg Y) \\ FR & \text{if } X \overline{\wedge} Y \neq \bot \text{ and} \\ & \quad (X \ll Y) < (X \gg Y) \\ T & \text{if } X \overline{\wedge} Y = \bot \end{cases}$$

and similarly for $X \underline{\wedge} Y \sim X$, $X \overline{\wedge} Y \sim X$, $X \underline{\wedge} Y \sim Y$, and $X \overline{\wedge} Y \sim Y$. We define $X \ll Y$ as

$$X \ll Y = \begin{cases} \mathsf{T} & \text{if } L(X) \underline{\wedge} L(Y) = L(X) \\ \mathsf{M} & \text{if } L(X) \underline{\wedge} L(Y) \neq L(X) \text{ and} \\ & \quad L(X) \overline{\wedge} L(Y) = L(X) \\ \mathsf{F} & \text{if } L(X) \overline{\wedge} L(Y) \neq L(X) \end{cases}$$

and similarly $X \gg Y$ using $R(X)$ and $R(Y)$, where L and R are defined as follows.

[7] This is an application of theorem 6 in (Bittner & Stell 2000) to the one-dimensional case.

[8] This is an application of theorem 7 in (Bittner & Stell 2000) to the one-dimensional case.

Firstly, we define the complement operation $X' g_i = (X\, g_i)'$ with $\mathsf{no}' = \mathsf{fo}$, $\mathsf{po}' = \mathsf{po}$, and $\mathsf{fo}' = \mathsf{no}$. Assuming that partition cells g_i are numbered in increasing order in direction of the underlying space, we secondly define $L(Y)$ and $R(Y)$ as

$$(L(Y)\, g_i) = \begin{cases} (Y\, g_i)' & \text{if } i \leq \min\{k \mid (Y\, g_k) \neq \mathsf{no}\} \\ \mathsf{no} & \textit{otherwise} \end{cases}$$

and

$$(R(Y)\, g_i) = \begin{cases} (Y\, g_i)' & \text{if } i \geq \max\{k \mid (Y\, g_k) \neq \mathsf{no}\} \\ \mathsf{no} & \textit{otherwise} \end{cases} .$$

Each of the above triples defines an RCC_1^9 relation, so the relation between X and Y can be measured by a pair of RCC_1^9 relations. These relations will be denoted by $\underline{R^9}(X, Y)$ and $\overline{R^9}(X, Y)$.

Theorem 6 *The pairs*

$$(\min\{\underline{R^9}(X, Y), \overline{R^9}(X, Y)\}, \max\{\underline{R^9}(X, Y), \overline{R^9}(X, Y)\})$$

that can occur are all pairs (a, b) where $a \leq b \leq \mathsf{EQ}$ and $\mathsf{EQ} \leq a \leq b$ with the exception of ($\mathsf{PPL}, \mathsf{EQ}$), ($\mathsf{PPR}, \mathsf{EQ}$), ($\mathsf{PPiL}, \mathsf{EQ}$), ($\mathsf{PPiR}, \mathsf{EQ}$)*, and* ($\mathsf{EQ}, \mathsf{DRR}$).

The pairs ($\mathsf{PPL}, \mathsf{EQ}$), ($\mathsf{PPR}, \mathsf{EQ}$), ($\mathsf{PPiL}, \mathsf{EQ}$), ($\mathsf{PPiR}, \mathsf{EQ}$) cannot occur, since they are refinements of the relations (PP, EQ), ($\mathsf{PPi}, \mathsf{EQ}$), which cannot occur in the undirected case. The pair ($\mathsf{EQ}, \mathsf{DRR}$) cannot occur due to the non-symmetric definition of FL and FR.

The pair ($\mathsf{DRL}, \mathsf{EQ}$) represents the most indeterminate case. Since ($\mathsf{DRL}, \mathsf{EQ}$) is consistent with ($\mathsf{EQ}, \mathsf{DRR}$) and ($\mathsf{DRL}, \mathsf{EQ}$) was chosen arbitrarily, ($\mathsf{DRL}, \mathsf{EQ}$) is corrected syntactically to ($\mathsf{DRL}, \mathsf{DRR}$). The corrected relation will be denoted by $\overline{R^9_c}(X, Y)$.

Correspondence of semantic and syntactic generalization Let the syntactic generalization of RCC_1^9 be defined by

$$\mathcal{SYN}(X, Y) = (\min\{\underline{R^9}(X, Y), \overline{R^9_c}(X, Y)\}, \max\{\underline{R^9}(X, Y), \overline{R^9_c}(X, Y)\}),$$

where $\underline{R^9}$ and $\overline{R^9_c}$ are defined as discussed above.

Proposition 1 *For approximations X and Y syntactic and semantic generalization of RCC_1^9 relations are equivalent in the sense that*

$$\mathcal{SEM}(X, Y) = \{\rho \in \mathsf{RCC}_1^9 \mid \min\{\underline{R^9}(X, Y), \overline{R^9_c}(X, Y)\} \leq \rho \leq \max\{\underline{R^9}(X, Y), \overline{R^9_c}(X, Y)\}\},$$

where RCC_1^9 is the set RCC_1^9 relations and $\leq$ is the ordering in the RCC_1^9 lattice.

Generalization of RCC_1^{15} relations

Syntactic generalization If X and Y are boundary sensitive approximations of intervals x and y in a directed one-dimensional space then we can consider the two triples of Boolean values:

$$(X \underline{\wedge} Y \not\approx \bot,\ X \underline{\wedge} Y \approx X,\ X \underline{\wedge} Y \approx Y),$$
$$(X \overline{\wedge} Y \not\approx \bot,\ X \overline{\wedge} Y \approx X,\ X \overline{\wedge} Y \approx Y).$$

where

$$X \wedge Y \not\approx \bot = \begin{cases} \mathsf{T} & X \underline{\wedge} Y \not\equiv \bot = \mathsf{T} \\ \mathsf{ML} & X \underline{\wedge} Y \not\equiv \bot = \mathsf{M} \text{ and } (X \ll Y) \geq (X \gg Y) \\ \mathsf{MR} & X \underline{\wedge} Y \not\equiv \bot = \mathsf{M} \text{ and } (X \ll Y) < (X \gg Y) \\ \mathsf{FL} & X \underline{\wedge} Y \not\equiv \bot = \mathsf{F} \text{ and } (X \ll Y) \geq (X \gg Y) \\ \mathsf{FR} & X \underline{\wedge} Y \not\equiv \bot = \mathsf{F} \text{ and } (X \ll Y) < (X \gg Y) \end{cases}$$

and similarly for $X \wedge Y \not\approx \bot, X \underline{\wedge} Y \approx X, X \overline{\wedge} Y \approx X, X \underline{\wedge} Y \approx Y$, and $X \overline{\wedge} Y \approx Y$. In order to define L and R, we define the complement operation $X' (g_i, g_j) = (X\, (g_i, g_j))'$ with

ω	no	nbo	bo	fo
ω'	fo	bo	nbo	no

Assuming that partition cells g_i are numbered in increasing order in direction of the underlying space, $L(Y)$ is defined as

$$(L(Y)\, (g_i, g_j)) = \begin{cases} (Y\, (g_i, g_j))' & \text{if } i \leq \min\{k \mid (Y\, (g_k, g_l)) \neq \mathsf{no}\} \\ \mathsf{no} & \textit{otherwise} \end{cases} ,$$

and $R(Y)$ is defined as

$$(R(Y)\, (g_i, g_j)) = \begin{cases} (Y\, (g_i, g_j))' & \text{if } i \geq \max\{k \mid (Y\, (g_k, g_l)) \neq \mathsf{no}\} \\ \mathsf{no} & \textit{otherwise} \end{cases} .$$

Each of the above triples provides a RCC_1^{15} relation, so the relation between X and Y can be measured by a pair of RCC_1^{15} relations. These relations will be denoted by $\underline{R^{15}}$ and $\overline{R^{15}}(X, Y)$. The pairs of relations $(\min\{\underline{R^{15}}(X, Y), \overline{R^{15}}(X, Y)\}, \max\{\underline{R^{15}}(X, Y), \overline{R^{15}}(X, Y)\})$ that can occur are all pairs (a, b) where $a \leq b \leq \mathsf{EQ}$ and $\mathsf{EQ} \leq a \leq b$ with the exception of pairs of relations that are refinements of pairs of relations that cannot occur in the undirected case (RCC_1^9 theorem 6) or that cannot occur in the boundary insensitive case (RCC_1^8 theorem 3).

Correspondence of semantic and syntactic generalization Corresponding to the generalization of the RCC_1^8 and the RCC_1^9 relations syntactic corrections are needed in order to generalize RCC_1^{15} relations between intervals, x and y, to pairs of RCC_1^{15} relations between approximations X and Y:

Firstly. Corresponding to the RCC_1^8 case we define $\underline{R^{15}_c}(X, Y) = \mathsf{PO}(L/R)$[9] if there are $g_i, g_j \in G$ such that $(X(g_i, g_j)) = (Y(g_i, g_j)) = \mathsf{bo}$ and $\underline{R^{15}_c}(X, Y) = \underline{R^{15}}(X, Y)$ otherwise. Secondly. Corresponding to the RCC_1^9 case the pair ($\mathsf{DCL}, \mathsf{EQ}$) represents the most indeterminate case. Since ($\mathsf{DCL}, \mathsf{EQ}$) is consistent with ($\mathsf{EQ}, \mathsf{DCR}$) and ($\mathsf{DCL}, \mathsf{EQ}$) was chosen arbitrarily, ($\mathsf{DCL}, \mathsf{EQ}$) is corrected syntactically to ($\mathsf{DCL}, \mathsf{DCR}$). The corrected relation will be denoted by $\overline{R^{15}_c}(X, Y)$.

[9] POL if $\underline{R^{15}}(X, Y) < \mathsf{EQ}$ and POR otherwise.

Let the syntactic generalization of RCC_1^{15} be defined by

$$\mathcal{SYN}(X,Y) = (\min\{\underline{R_c^{15}}(X,Y), \overline{R_c^{15}}(X,Y)\}, \max\{\underline{R_c^{15}}(X,Y), \overline{R_c^{15}}(X,Y)\}),$$

where $\underline{R_c^{15}}$ and $\overline{R_c^{15}}$ are defined as discussed above.

Proposition 2 *For approximations X and Y syntactic and semantic generalization of RCC_1^{15} relations are equivalent in the sense that*

$$\mathcal{SEM}(X,Y) = \{\rho \in \mathsf{RCC}_1^{15} \mid \min\{\underline{R_c^{15}}(X,Y), \overline{R_c^{15}}(X,Y)\} \leq \rho \leq \max\{\underline{R_c^{15}}(X,Y), \overline{R_c^{15}}(X,Y)\}\},$$

where RCC_1^{15} is the set RCC_1^{15} relations and $\leq$ is the ordering in the RCC_1^{15} lattice.

Conclusions

In this paper I defined methods of approximate qualitative temporal reasoning. Approximate qualitative temporal reasoning is based on:

1. Jointly exhaustive and pair-wise disjoint sets of qualitative relations between exact regions, which are defined in terms of the meet operation of the underlying Boolean algebra structure of the domain of regions. As a set these relations must form a lattice with bottom and top element.
2. Approximations of regions with respect to a regional partition of the underlying space. Semantically, an approximation corresponds to the set of regions it approximates.
3. Pairs of meet operations on those approximations, which approximate the meet operation on exact regions.

Based on those 'ingredients' syntactic and semantic generalizations of jointly exhaustive and pair-wise disjoint relations between exact one-dimensional regions were defined. Generalized relations hold between approximations of regions rather than between (exact) regions themselves. Syntactic generalization is based on replacing the meet operation defining relations between exact regions by its minimal and maximal counterparts on approximations. Semantically, syntactic generalizations yield upper and lower bounds (within the underlying lattice structure) on relations that can hold between the corresponding approximated exact regions.

In the temporal domain I defined four sets of topological relations between one dimensional regions:

RCC5 Boundary *in*sensitive binary topological relations between regions in a *non*-directed one-dimensional space.

RCC_1^9 Boundary *in*sensitive binary topological relations between maximally connected regions (intervals) in a directed one-dimensional space.

RCC_1^8 Boundary sensitive binary topological relations between regions in a *non*-directed one-dimensional space.

RCC_1^{15} Boundary sensitive binary topological relations between maximally connected regions (intervals) in a directed one-dimensional space.

For each of these sets of relations between exact regions I discussed the syntactic and semantic generalization for the corresponding approximations and showed the equivalence of syntactic and semantic generalization. This provides the formal basis for qualitative temporal reasoning about approximate location in time.

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