Coherent Pricing of Efficient Allocations in Combinatorial Economies

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Abstract

Auctions and exchanges are important coordination mechanisms for multiagent systems. Most multi-good markets are combinatorial in that the agents have preferences over bundles of goods. We study the possibility of determining prices so as to support (efficient) allocations in combinatorial economies where a seller (or arbitrator) wants to implement an efficient allocation. Conditions on the existence of equilibria are presented and a particularly attractive, anonymous pricing scheme is studied in detail. A constructive test for the existence of supporting prices is given. A procedure based on the controlled formation of alliances is suggested that shrinks economies to ensure the existence of prices coherent with the preferred pricing scheme. The relation of equilibrium prices to Vickrey payments is considered, and extensions to two-sided markets are discussed.

Introduction

Auctions and exchanges are important coordination mechanisms for multiagent systems. Most multi-good markets are combinatorial in that the agents have preferences over bundles of goods. Combinatorial auctions and combinatorial exchanges have been subjects of intense study in the last few years due to their importance as a solution mechanism for combinatorial resource and task allocation problems involving self-interested, autonomous agents with private information. While the determination of efficient (or approximately efficient) allocations has been studied extensively, the important role of prices for the practical and theoretical implementability of allocations has drawn less attention (notable exceptions include (Parkes & Ungar 2000a; 2000b; Wurman & Wellman 2000; Bikhchandani *et al.* 2001)).

In this paper we study different schemes for pricing goods and bundles in combinatorial economies where bidders have potentially non-additive preferences on goods, that is, preferences over bundles. Of natural interest are prices that support the computed allocation so that each participating agent will be satisfied with the outcome at the given prices. The different pricing schemes have different impact on the existence of such equilibrium outcomes. We study this in detail for a pricing scheme that minimizes the necessity to enforce the correct implementation of an intended outcome and keeps the prices anonymous. Algorithms are given for testing the existence of equilibrium prices, and for finding them. A procedure is suggested for dealing with non-existence of equilibria due to threshold problems. It is based on controlled formation of alliances among consumers. Finally, we discuss how the results can be extended to two-sided markets (i.e., exchanges).

Pricing schemes

We first study the problem of allocating a finite set, $\Omega = \{1, \dots .m\}$, of m indivisible resources (or *goods*) to a finite set, $N = \{1, \dots, n\}$, of n competing agents (or *consumers*) so as to maximize the economic efficiency of the allocation. The consumers have (integral) utility for bundles of goods, given as a utility function $u_i : 2^{\Omega} \rightarrow \mathbb{N}_0$. All goods belong to a benevolent auctioneer (or *arbitrator*), denoted by 0. A collection $E = (\Omega; u_1, \dots, u_n)$ of the goods and the utility functions will be called an *economy*. It is the task of the arbitrator to implement an efficient allocation by means of a suitably chosen mechanism. The instrument of choice, to enable the elicitation of utility information and the transfer of utility, is pricing. An outcome of a price-based mechanism consists of an allocation and a related vector of payments which determines the amount of money each agent has to pay in order to receive the part of the allocation that is earmarked for him. The arbitrator can only hope to implement a suggested outcome if each agent chooses to implement her part of the outcome. She will do so only if the net utility of doing so is at least as large as the net utility of any other behavioral option (we consider only purchasing decisions as allowed behavioral options). We make the standard assumption that each agent's utility is quasi-linear in money, consequently her net utility can be determined as her utility for the received bundle minus the necessary payment. We will further assume that the option to purchase nothing is available for free (that is, the price of the empty bundle is zero). Additionally, an agent can get rid of any allocated good for free (there is *free disposal*). In this context, it is reasonable to restrict attention to price functions which are monotonously increasing in goods, that is, $p(x) \leq p(y)$ if $x \subset y$.

In a price-based mechanism, there is an intimate relation between announced prices and resulting payments. Furthermore, the chosen pricing scheme determines the set of purchasing options to be considered.

To see this, consider the following setting. Assume that the arbitrator operates a shop.[1] Each evening the arbitrator runs an allocation mechanism on his Web site which collects utility information from his customers for his goods and bundles of goods, and which determines an (efficient) allocation of his goods from this information. Early in the morning he enters his shop and executes one of the following *pricing schemes*:

1. He attaches a price tag to each good.
2. He posts a price list with a price for each possible bundle of goods.
3. He posts a price list with a price for each possible bundle, with the additional rule: "Only one bundle per customer!".
4. He bundles the goods according to the efficient allocation and attaches a price tag to each resulting bundle.[2]

Now his customers visit the shop sequentially in arbitrary order. When a customer visits the shop, the customer makes his individual purchasing decisions, pays, and leaves the shop. Is it possible for the shop clerk to determine his prices so that the implementation of an outcome with an efficient allocation is self-enforcing?

Before we answer this question, let us study the consequences of the different pricing schemes for the purchasing options that an agent has to consider. Assume that agent 1 enters the shop and that his most preferred bundle, $\{A, B\}$, is still available (in a slight abuse of notation, AB will be used instead of $\{A, B\}$ to denote the bundle if the context allows). Now, in pricing scheme 1, the payment $t_1(AB)$ he has to expect is the sum of prices for good A and good B, $p(A) + p(B)$. His net utility of purchasing the bundle will be $v_1(AB) = u_1(AB) - t_1 = u_1(AB) - (p(A) + p(B))$. He will have to compare this to the net utility of any other possible bundle to make an optimal purchasing decision.

In pricing scheme 2, his calculation will be different: instead of buying the bundle in two transaction (paying $p(A) + p(B)$) he may also choose to buy the bundle directly in one transaction (paying the price $p(AB)$).[3] A utility-maximizing consumer will always look for the best possible combination of transactions to determine the potential payment, e.g. the payment that agent 1 will consider for the bundle AB will be $\min\{p(AB), p(A) + p(B)\}$. In pricing scheme 3, the payment for a bundle that is to be considered is the given price for the bundle, e.g. $t_1(AB) = p(AB)$. Pricing scheme 4 is similar to pricing scheme 1, with the notable exception that neither A and B nor AB might be available for purchasing. This would be the case if they have been packaged into bundles containing other goods as well, say AC and BD, so the payment that agent 1 has to consider is the best obtainable price or sum of prices for a bundle or a collection of bundles that contains the considered bundle.

Coherent prices

To make this more precise, some formalization is necessary. We could continue to study the payments that result from the prices. A different possibility is to consider pricing scheme 3 only (here, the payment to be considered for any bundle is equal to the given price) and to map the other pricing schemes into *coherence* conditions on the structure of the prices of scheme 3. First, some terminology is required.

Definition 1 (Allocation, Outcome, Value). *An* allocation *is a vectorized partition $X = (X_1, \ldots, X_n)$ of the goods in Ω, such that $\bigcup_{i \in N} X_i = \Omega$ and $\bigcap_{i \in N} X_i = \emptyset$ (because of the free disposal assumption, we can safely assume that all goods will always be distributed). A $2n$-ary vector $(X_1, \ldots, X_n; t_1, \ldots, t_n)$ will be called* outcome *if $(X_1, \ldots, X_n)$ is an allocation and $t_i \geq 0$ for all $i \in N$ (note that these values are payments to be made, so, in contrast to their sign, they have a negative effect on a consumer's utility). $\sum_{i \in N} u_i(X_i)$ is the* value *of an allocation respectively an outcome.*

Definition 2 (Net utility of implementation). *Let $(X_1, \ldots, X_n; t_1, \ldots, t_n)$ be an outcome. Then $v_i = u_i(X_i) - t_i$ is the* net utility *of an implementation of the outcome for consumer $i \in N$.*

We assume that each consumer controls her behavior autonomously (cannot be forced to purchase a bundle) and behaves individually rationally (does not pay more for a bundle than it is worth). An outcome is not implementable if $v_i < 0$ for any $i \in N$. The (rational) objective of each consumers is it to maximize her net utility when presented with a choice of options. As has been said above, we will consider purchasing options only (though this will be extended below to a form of controlled collusion). The available options are determined by the pricing scheme. A (monotonous) price function and the coherence conditions for the different pricing schemes can now be defined as follows.

Definition 3 (Price function). *Let $E = (\Omega; u_1, \ldots, u_n)$ be an economy. We call a function $p : 2^\Omega \rightarrow \mathbb{N}_0^+$ a* price function, *if $p(\emptyset) = 0$. (Abusing notation, we sometimes speak of a price vector and write p_x instead of $p(x)$.)*

Definition 4 (Coherent Prices). *A price function $p : 2^\Omega \rightarrow \mathbb{N}_0$, to be used with pricing scheme 3, is* coherent *with scheme 1 respectively 2, if*

$$p(x) = \sum_{z \in x} p(\{z\}) \quad \forall x \subseteq \Omega, x \neq \emptyset \tag{1}$$

respectively

$$p(x) = \min_{Z \in \Pi(x)} \sum_{z \in Z} p(z) \quad \forall x \subseteq \Omega, x \neq \emptyset \tag{2}$$

Above, $\Pi(x)$ is the set of all possible partitions of x. Furthermore, every price function is coherent with scheme 3.

[1] Admittedly a special kind of shop, because his objective is not to maximize his income but economic efficiency ("welfare") among the bidders.

[2] The decision to bundle the goods would allow us to apply each of the pricing schemes 1,2, and 3 to the new situation. We will only consider the analog of pricing scheme 1.

[3] In fact he may also choose to buy bundles A and BC or the bundle ABC–even if C does not add to his utility in the case that this promises a better deal – however, the assumed monotony of prices makes this type of considerations unnecessary, so we will leave this aside from now on.

For scheme 4, the coherence is relative to an allocation, that is, $p(\cdot)$ is coherent with pricing scheme 4, if an allocation X exists such that[4]

$$p(x) = \sum_{z \in X, z \subseteq x} p(z) \quad \forall x \in 2^X, x \neq \emptyset \tag{3}$$

and

$$p(x) = \min_{z \supset x, z \in 2^X} p(z) \quad \forall x \notin 2^X, x \neq \emptyset \tag{4}$$

We also say that the prices are coherent *with the allocation X.*

Further on, we will assume that the setting is that of scheme 3 (at most one bundle per consumer is allowed). Note that the following observation holds

Proposition 5. *Let $p(\cdot)$ be a price function. If $p(\cdot)$ is coherent with scheme 1, it is also coherent with scheme 4. If $p(\cdot)$ is coherent with scheme 4, it is also coherent with scheme 2.*

Proof. Please, see the appendix for left-out proofs. □

Pricing schemes 1 and 3 have been studied extensively in the literature (see (Kelso & Crawford 1982; Gul & Stacchetti 1999) for scheme 1 and (Wurman & Wellman 2000) for scheme 3). Both may have show deficiencies. Prices in scheme 1 that self-enforce (or, a little bit weaker: support) efficient allocations are only guaranteed to exist under rather weak conditions (gross-substitutes). While supporting prices in scheme 3 do always exist, they require strict means of enforcement to ensure the "correctness" of an implementation—in the above example, enforcing the rule on the sign would require to register the customers (to prevent them from making multiple purchases throughout the day). It would also be good to prevent them from sending in a friend that acts as a buyer and hands his purchase to the original agent. Such enforcement is certainly not viable or desirable in all settings, especially on the Internet where pseudonyms tend to be cheap. We will therefore focus on pricing scheme 4 (we will sometimes write *coherent prices* instead of *prices coherent with scheme 4*).

Now, as the dependency of the purchasing decision on the pricing scheme is hidden in the coherence condition, we can use the prices directly in the definition of the net utilities that are to be considered.

Definition 6 (Net utility function). *Let Ω be a set of goods, $p(\cdot)$ a suitable price function, and i a consumer with a quasi-linear utility function based on $u_i(\cdot)$. Then the function $v_i^p : 2^\Omega \to \mathbb{N}$, defined as $v_i^p(x) = u_i(x) - p(x)$ will be called the* net utility function *of i with respect to Ω and $p(\cdot)$.*

[4]To understand the following notation note that *power set* and *element-of operator* are used here on a partitioning sequence X in a canonical extension of their usual meaning. The *elements* of X are the sets $X_i \subseteq \Omega$. The *power set* 2^X consists of all combinations of the *elements* of X. We write $x \in 2^X$ for a $x \subseteq \Omega$ if a partition of x exists such that every element of the partition (itself a subset of Ω) is an element of X (in other words: the partition is an element of 2^X).

Let us return to the above example and consider the first customer, say i, entering the store. She is faced with the whole range of purchasing options. Let us restrict our attention to prices that are coherent with scheme 4. To make her purchasing decision, she will have to pick the optimal way to purchase each bundle. Because of the coherence conditions, it is now not anymore necessary to consider multiple transactions – the one-transaction price given for a bundle is already minimal. She will have to compare the obtainable net utility with the net utility related to any other bundle, that is, she has to solve the problem $\arg\max_{x \subseteq \Omega} v_i^p(x)$ at the given prices p.

Equilibria

To ease the treatment of race conditions and indifference, we will switch now from a shop environment to a distribution environment where goods and bundles are presented in a catalog. Once the arbitrator has determined an efficient allocation from certain valuation information, he will determine prices coherent with the chosen pricing scheme. He will then send the price list (and the additional condition of scheme 3) to the participating consumers. Each consumer will determine a set B containing all bundles that maximize her net utility at the given prices. She will then submit a list of mutually exclusive orders of individual bundles, containing all bundles from B. Once the arbitrator has received the orders, he will distribute the goods to the customers so as to maximize efficiency. If a customer receives one of the requested bundles, she will be satisfied with the outcome. If every customer receives a requested bundle and if the objective of the arbitrator is fulfilled by the resulting allocation, the outcome determines an equilibrium.

Definition 7 (Satisfied, Supports). *A consumer i is* satisfied *with an allocation X at given prices $p(\cdot)$, iff the bundle X_i he receives maximizes his net utility, that is*

$$v_i^p(X_i) \geq v_i^p(z) \quad \forall z \subseteq \Omega.$$

The price function supports *an allocation X if every actor $i \in N$ is satisfied with X.*

Definition 8 (Equilibrium). *Let E be an economy, X an allocation, and $p(\cdot)$ a price function. The pair $(X; p(\cdot))$ is an* equilibrium *(of interests), if every participant is individually satisfied with the induced outcome. In the considered situation, this corresponds to*

Arbitrator $(X_1, \ldots, X_n)$ *is an efficient allocation.*

Consumer *Every consumer $i \in N$ is satisfied with X_i.*

If such an outcome exists for an economy and a given price function, the price function will be called an equilibrium price function.

Consequence 9. *Note that the following are immediate consequences and hold for any equilibrium outcome: (1) Demand equals supply–in other words: the outcome is implementable, and (2) the supported allocation is efficient.*[5]

[5]Without considering incentive compatibility, this is only true with respect to reported utilities.

It might be surprising that one of the standard results, the first theorem of Welfare economics, has been turned into a definitional consequence–namely that every equilibrium is efficient. This is due to the situation under study: the key property of an equilibrium is that all actors are individually satisfied with the result. Here, one of the actors (the arbitrator), has preferences for complete allocations, which explains the fact that a global social criterion (efficiency) coincides with a criterion for individual satisfaction. In the classic setting, with a set of sellers and no central and self-interested arbitrator, all actors have preferences only for their part of the allocation–in that situation, it becomes important to analyze if a global criterion (efficiency) is an emergent consequence of satisfied individual criteria. In our setting, on the other hand, this is immediate.

Existence of supporting prices

A key question for the arbitrator now is if, for every resource allocation problem and a given pricing scheme, a price function exists that supports an efficient allocation. This is the case for pricing scheme 3:

Proposition 10. *For any economy E and pricing scheme 3, an outcome with an efficient allocation and a supporting price function exists.*

This has been shown in (Wurman & Wellman 2000) as a consequence of results presented in (Leonard 1983)). Such a result does not hold for scheme 1, as the following simple example demonstrates:

		A	B	AB
Utility	Agent 1	0	0	3
	Agent 2	2	2	2
Prices	Scheme 1	≥ 2	≥ 2	$p(A)+p(B) \leq 3$
	Schemes 2,3	2.1	2.1	2.5
	Scheme 4	2.5	2.5	2.5

From the conditions that follow from the necessity to satisfy both agents, a contradiction follows immediately. The prices given for the other schemes are, in contrast, equilibrium prices.

As we stated above, we consider pricing scheme 4 the scheme that combines a significant design flexibility (it allows us to solve an extended set of allocation problems compared to scheme 1) with a reduced necessity for enforcement (in contrast to scheme 3). However, scheme 4 does not solve all existence problems that are due to combinatorial (that is, non-additive) preferences immediately, as the following example of a threshold problem demonstrates:[6]

[6]Existence can be guaranteed if the gross-substitutes condition from (Kelso & Crawford 1982) holds for utility functions that are restricted to the bundles in the efficient allocation and their super bundles. This is an immediate consequence of Theorem 3 in (Kelso & Crawford 1982). The proposition follows below, once the notion of restricted utility functions is formalized.

		A	B	AB
Utility	Agent 1	**5**	5	5
	Agent 2	0	**3**	3
	Agent 3	0	0	7
Prices	Schemes 1,2,4	$\leq p_B$	≤ 3	$p_A+p_B \geq 7$
	Scheme 3	1	2	7.1

We will, however, demonstrate below that the initial economy can be modified without an impact on efficiency such that equilibrium prices coherent with scheme 4 exist. The basic idea is to shrink the economy by creating alliances of agents that submit joint bids. Reconsider the above example with an alliance of agents 1 and 2.

Utility	A	B	AB
Agent (1+2)	5	5	**8**
Agent 3	0	0	7
Scheme 4 prices	7	7	7

Computing the prices

Before we study this in more detail, we present a constructive test for the existence of prices that are coherent with scheme 4. We assume that an (efficient) allocation $X = (X_1, \ldots, X_n)$ has been determined and the task at hand is to compute prices, coherent with scheme 4, that support this allocation.

We first study a reduced economy $E^r = (\Omega^r = \{g_1, \ldots, g_n\}^7; u_1^r, \ldots, u_n^r)$ which results from the original economy E as follows: $g_i = X_i$ and $u_i^r : 2^{\Omega^r} \to \mathbb{N}$ defined as $u_i^r(x) = u_i(x)$ for all $i \in N$ and $x \in 2^{\Omega^r}$. Obviously, the following holds[8]

Proposition 12. *The value $\sum_{i \in N} u_i(X_i)$ of the efficient allocation X in E is equal to the value of an efficient allocation in the reduced economy E^r. In particular, the allocation $X^r = (g_1, \ldots, g_n)$ is efficient.*

Consider now this efficient allocation X^r for a reduced economy E^r. The following algorithm will compute prices supporting X^r if the consideration of utilities is further restricted to utilities for the goods only. Based on these prices (which always exist, see below), extended equilibrium prices for the reduced economy and complete prices for E can be determined if they exist. Below, J collects the agents i which neither request the earmarked bundle X_i nor a good that represents an empty bundle. If this set is empty, either $X_i \in Y_i$ for all $i \in N$ (supporting prices have been determined) or no such price vector could be determined (this cannot happen as we will show below). The vector Δ measures the attractiveness of the goods in Y_i rel-

[7]Some of the goods may represent empty bundles.

[8]We can now also formulate the missing *existence* proposition:

Proposition 11 (Individualistic Existence Condition). *Let E be an economy and X an efficient allocation. Let E^r be the reduced economy obtained from the construction above. If the reduced utility functions satisfy the gross substitutes condition of Kelso and Crawford (see (Kelso & Crawford 1982) or (Gul & Stacchetti 1999) for alternative formulations), an equilibrium coherent with scheme 4 exists.*

ative to the attractiveness of the earmarked good g_i for each agent i.[9]

Algorithm *Min-Pricing*
(1) $p = (0, \ldots, 0)$; Compute Y; Compute Δ; Compute J;
(2) **while** $J \neq \emptyset$ **do**
(3) $\quad i = \arg\max_{j \in J} \Delta_j$
(4) $\quad$ **Forall** $y \in Y_i$ **do**
(5) $\quad\quad p_y = p_y + \Delta_i$
(6) $\quad$ Compute Y; Compute Δ; Compute J;

Theorem 13. *The algorithm* Min-Pricing *determines, for a given reduced economy E^r and a corresponding efficient allocation $X^r = (g_1, \ldots, g_n)$, prices p, so that*

$$u_i(g_i) - p_{g_i} \geq u_i(g_j) - p_{g_j} \quad \forall j \in \{g_1, \ldots, g_j\}, \forall i \in N$$

Furthermore, this price vector is minimal.

A similar algorithm, *Max-Pricing*, can be given for computing maximal prices, that is, prices which cannot be increased without making one of the efficiently allocated "goods" unattractive to the prospective buyer.

The determined prices are solutions to dual linear programs (compare (Koopmans & Beckmann 1957; Gale 1960; Leonard 1983)) that give minimal and maximal equilibrium prices for economies in which the bundling of goods does not increase the utility of the individual consumers:

Definition 14 (Assignment Economy). *An economy E will be called* assignment economy, *if $u_i(x) = \max_{z \in x} u_i(\{z\})$ for all $i \in N$ and $x \subseteq \Omega$.*

Every price vector determines a different distribution of the value of an allocation to the consumers and the arbitrator. Minimal prices maximize the surplus of the consumers, while maximal prices maximize the surplus of the arbitrator. In the above, we restricted our attention to a certain part of the utility functions. Once we extend this to include the bundles that can be formed from goods in Ω^r (which may introduce complementarities), the resulting impact on the existence conditions for equilibria may exclude that prices coherent with scheme 4 exist.

Theorem 15. *Let E be an economy, X an efficient allocation, and E^a the reduced assignment economy. Let $p^a_{\max}$ be maximal prices supporting the reduced allocation X^a for E^a. Let p^e be the result of extending*[10] *$p^a_{\max}$ to a complete price function. If (X, p^e) is not an equilibrium (coherent with scheme 4 and X), no such equilibrium exists.*

In some cases, the minimal prices for the reduced assignment economy can also be used to determine Vickrey payments. Consider the following example taken from (Gul & Stacchetti 1999): There are three identical objects and two consumers with the same preferences. $u_i(A)$ is 0 for $\#(A) = 0$, 10 for $\#(A) = 1$, 18 for $\#(A) = 2$ and 20 for $\#(A) = 3$. It is an efficient allocation to give one goods to one buyer and two goods to the other. The Vickrey payments are 2 for the one-good buyer and 10 for the other. The (minimal) Walrasian price according to the definition in (Gul & Stacchetti 1999) (which coincides with our scheme 1) is 8 for each good, minimal coherent prices for scheme 4 are 2 for the singleton and 10 for the two-good bundle (with respect to any chosen efficient allocation, necessarily consisting of a singleton and a bundle of two of the goods.).
The above algorithms give means to test for the existence of equilibrium prices for scheme 4 and to determine if equilibrium prices exist that equal Vickrey payments. If no such prices exist (see the "threshold" example above), the arbitrator may choose to ask selected agents to cooperate ("collude") if this promises to be beneficial. This will be formalized and analyzed in the next section.

[9]Formally: Empty $= \{j | X_j = \emptyset\}$, $Y_i = \{x \in \Omega^r | u_i(x) - p_x \geq u_i(y) - p_y \ \forall y \in \Omega^r\}$, $J = \{i \in N : X_i \notin Y_i \ \wedge \nexists x \in Y_i$ with $x \in Empty\}$, and $\Delta_i = (u_i(y) - p_y) - (u_i(X_i) - p_{X_i}) \ \forall i$ and some $y \in Y_i$.

[10]Extending prices follows the coherence conditions for scheme 4: All bundles of g_i's are priced additively, then all other bundles will receive the price of the smallest bundle containing it. As all goods are distributed in X^a, such a bundle always exists.

Shrinking the economy

Definition 16 (Alliance). *Let $K = \{1, \ldots, k\} \subseteq N$ be a set of agents and $\bar{k}$ an additional agent. Then $\bar{k}$ represents an* alliance *respectively an* efficient alliance *of the agents in K, if*

$$u_{\bar{k}}(x) \leq \max_X \sum_{i=k}^{n} u_i(X_i)$$

respectively

$$u_{\bar{k}}(x) = \max_X \sum_{i=k}^{n} u_i(X_i)$$

for all $x \subseteq \Omega$ and all k-ary sequences X partitioning x.

Definition 17 (Shrunken Economy). *Let $E = (\Omega; u_1, \ldots, u_n)$ be an economy and $P = \{P_1, \ldots, P_k\}$ be a k-ary partition of N, with $P_i \neq \emptyset$ for all $i \in \{1, \ldots, k\}$. Additionally, let $\bar{K} = \{\bar{1}, \ldots, \bar{k}\}$ be a set of agents with k elements, such that for all $i \in \{1, \ldots, k\}$, $\bar{i}$ is an alliance of the agents in P_i. Then, $E^{\bar{K}} = (\Omega; u_{\bar{1}}, \ldots, u_{\bar{k}})$ will be called a* shrunken economy *with respect to E and P. If all $\bar{i}$s are efficient alliances, $E^{\bar{K}}$ is an* efficiently shrunken economy.

We will only consider efficiently shrunken economies below. First, note the following:

Proposition 18. *Let E be an economy, $\bar{K}$ a set of efficient alliances, $E^{\bar{K}}$ the corresponding efficiently shrunken economy and X an efficient allocation of Ω with respect to N. Then $X^{\bar{K}} = (\bigcup_{i_1 \in P_1} X_{i_1}, \ldots, \bigcup_{i_k \in P_k} X_{i_k})$ is an efficient allocation with respect to $E^{\bar{K}}$. In turn, an efficient allocation $X^{\bar{K}}$ for a shrunken economy determines one or more efficient allocations for E: for every $i \in \{1, \ldots, k\}$, a $|P_i|$-ary allocation Y_i of the goods in $X_i^{\bar{K}}$ to the agents in P_i can be found, such that $\sum_{j \in P_k} u_i(Y_i)$ will be maximized for all possible $|P_i|$-ary allocations (Y_i is not necessarily unique). A suitable renumbering of Y_i will lead to an n-ary allocation X_i which is efficient with respect to E.*

These results are immediate consequences of the definition of efficient alliances. Now let E be an economy. The set of all possible partitions consisting of non-empty subsets of N will be denoted with $\mathcal{P}^N$. For every partition $P \in \mathcal{P}^N$, a set K^P of agents can be constructed such that every agent represents an efficient alliance of the corresponding part of P–the existence of such a set follows immediately from the existence of an efficient allocation and the definition of efficient alliances. Let $\mathcal{A}$ be the set of all such pairs (P, K^P). It determines the set of all possible efficiently shrunken economies $\mathcal{E}^{\mathcal{A}}$ with respect to E, which starts from considering all agents as a singular alliance and ends with considering the grand alliance which represents all agents in N.

Proposition 19. *The set $\mathcal{L} \subseteq \mathcal{E}^{\mathcal{A}}$ of efficient allocations, for which equilibrium prices exist that are coherent with scheme 4, is not empty (this extends to schemes 2 and 1).*

This follows immediately by taking the reduced economy with one agent only who represents the grand alliance. Here (with reservation values of 0 of the arbitrator, as assumed), the minimal equilibrium prices are 0 for each bundle.

We will now construct a procedure that picks one of the economies from $\mathcal{L}$ and determines an equilibrium. This will be done iteratively by determining in each round an alliance to form in a way that ensures that the alliance is attractive for the participating agents (relative to a certain policy of the arbitrator that determines how to proceed if the agents would object against the alliance, see below).

Procedure *Shrink*(Economy E, Efficient allocation X)

(1) Determine the maximal prices p_{max} for the assignment economy E^a that corresponds to E and X. Extend the obtained prices to prices coherent with scheme 4.

(2) If the equilibrium conditions are violated for one or more bundles, choose one such bundle y (Note: $u_i(y) - p(y) > u_i(X_i) - p(X_i)$ for at least one i). If no condition is violated, terminate with the shrunken economy E as a result.

(3) Create an alliance a from the agents that receive a part of the chosen bundle in the efficient allocation–a represents $I = \{i \in N | X_i \subset y\}$. Create a new good X_a representing the set $\{X_i \in y\}$, $i \in I$. Remove all goods $\{X_i \in y\}$ from Ω and insert X_a to create Ω^-. Determine the utility of a for all bundles $z \subseteq \Omega^-$ so as to maximize the utility of distributing z among the agents in I.

(4) Remove now all agents I from N and add a to create N^-. Remove all goods $\{X_i \in y\}$ from X and add X_a to create X^-. Choose a suitable index of the agents in N^- and order the elements of X^- so that each agent i will receive the bundle X_i^-. Rename X^-,N^-, and Ω^- to X,N, and Ω. Continue with (1).

Proposition 20. *Starting from economy E and efficient allocation X, procedure* Shrink *determines a shrunken economy E^-, an allocation X^- and a price vector p such that*

1. *X^- is an efficient allocation of the goods in E^- and the value of X^- is equal to the value of X in E*
2. *p is an equilibrium price vector for the shrunken economy coherent with scheme 4 and the computed allocation X^-.*

If it is the policy of the arbitrator to choose prices that support an inefficient but implementable allocation in the case that a suggested alliance does not form (that is, at least one agent objects against its formation), it becomes attractive for the agents in the alliance to accept its formation: in the above example, an arbitrator with this policy would sell AB to agent 3 for a price of 5 (which can immediately be extended to prices coherent with scheme 4: $p(A) = p(B) = p(AB) = 5$), making it attractive for agents 1 and 2 to accept the suggested alliance and any distribution of the remaining surplus of 1. As this seems to be the case generally (due to the efficiency of the underlying allocation, possibly with a restriction on the comparative surplus that is distributed because this has to be at least as good as the surplus that each agent in the alliance would receive if he objects against its formation), the arbitrator could as well distribute the goods and determine the (now non-anonymous) payments beforehand *as if the suggested alliance would form anyway* (because of the rationality and no-externalities assumptions that are implicit in the quasi-linear utility assumption). The surplus distribution could follow a fixed rule like "random distribution" or "equal amount". This completes the determination of coherent prices for combinatorial economies for which no coherent equilibrium prices exist for their original size.

Conclusions and future research

We suggested a pricing scheme for which enforcement-free, anonymous equilibrium prices exist in a wider range of situation than in the classic prices-for-goods-only scheme. The problematic complementarities can be neglected for sub-bundles of the bundles in the efficient allocation. We also showed how the existence of equilibrium prices can be checked and how such prices can be computed easily. If threshold problems foreclose the existence of equilibrium prices, a procedure that shrinks the economy by forming alliances can be applied. This procedure may iterate and eventually produces an economy for which coherent prices exist.

Some of the results can be extended directly to a setting with income-maximizing sellers and an arbitrator interested in implementing an efficient allocation. Especially the process of forming alliances can symmetrically be applied to the seller side (which can be used to split the surplus from selling a bundle that consists of goods from different sellers). We only give a rough sketch of the basic idea and restrict our attention to pricing scheme 4. Assume again that the arbitrator has determined an (efficient) allocation by suitable means (e.g., by choosing an elicitation policy from the framework suggested in (Conen & Sandholm 2001) that is not price-based (see also (Conen & Sandholm 2002b; Hudson & Sandholm 2002; Smith, Sandholm, & Simmons 2002; Conen & Sandholm 2002a))). Now, the prices for these bundles will have to be determined so as to satisfy buyers and sellers. Let us assume for simplicity that a (buyer-)coherent price vector exists. Now a second price vector that will be (seller-)coherent (the bundle prices will be the maximum of the aggregated prices of its partitions) can be determined so that each side of the market is satisfied with its specific price vector and both vectors will coincide on the prices for the bundles in the efficient allocation and the bundles that can be formed from them. Of course, the problems that are due to non-additive valuations of bundles can now occur on both sides of the market. The formation of alliances helps here as well.

We have only briefly mentioned the (interesting) relation of minimal coherent equilibrium prices to Vickrey payments. More work is required to study the incentive implications of allocation determination and pricing. Note, however, that even if the utilities have not been reported truthfully, the goal to determine supporting prices for a chosen allocation remains important. In situations that call for anonymous, enforcement-free prices for outcomes to be implementable, prices that are coherent with scheme 4 are especially attractive compared to the traditional scheme 1 prices or the more recently suggested scheme 3 prices. Prices of scheme 3 are not truly anonymous. If anonymity is not an issue, the results presented in (Parkes & Ungar 2000a; Bikhchandani & Ostroy 2001) (which may require enforcement) become relevant. A controlled shrinking of the economy, and the suggested partial differentiation of prices between sellers and buyers, may have an interesting impact on their results if enforcement is not an option.

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Appendix

Proof of Proposition 5:

(Scheme 1 $\rightarrow$ Scheme 4) Let X be an arbitrary allocation. Let $p(\cdot)$ be a price function coherent with scheme 1. Consider the price for a bundle x. There are two cases: if $x \in 2^X$, $p(x)$ have to be the sum of prices of the bundles in X that are contained in x. As each such price is the sum of the prices of the goods contained in the bundle (ie., $p(X_i) = \sum_{y \in X_i} p(\{y\})$) it follows that $p(x) = \sum_{z \in x} p(\{z\}) = \sum_{z \in X, z \subseteq x} \sum_{y \in z} p(\{z\}) = \sum_{z \in X, z \subseteq x} p(z)$. The other case is equally straightforward, as is the other part of the proposition (Scheme 4 $\rightarrow$ *Scheme* 2). □

Proof of Theorem 13:

We assume that all quantities are integral. Now, the following holds:

(a) *The algorithm terminates.*

(a1) *For the chosen* Δ_i, $\Delta_i > 0$ *holds*. Immediate. Note also that no price of a good that represents an empty bundle will ever be increased.

With (a1), it follows that in each round at least one price will be increased. Let k be the valuation of an agent j for good g_j. After at most k incrementations of the price of good g_j, the good can not be in Y_j anymore without having an empty bundle in Y_j as well. In consequence, J would be empty after a finite number of iterations.

(b) It is known that the sought-after minimal price vector exists (compare, for example, Gale (Gale 1960) and his proof for the existence of integral dual prices in optimal assignment problems). Let p^* denote this price vector. $u_i(g_i) - u_i(g_j) \geq p^*_{g_i} - p^*_{g_j}$ holds for all $j \in \{g_1, \ldots, g_j\}$ and all $i \in N$. In other words, there is a specific distance between the prices for each pair g_i, g_j of goods. This distance is limited from above by the distance of the corresponding valuations (compare Fig. 1).

From the fact that the price for goods representing empty bundles is 0, $p^*_{g_i}$ is bound from above by $u_i(g_i)$. We will

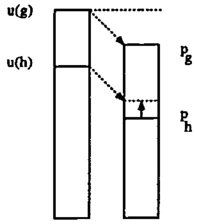

Figure 1: The valuation of agent g for the goods g and h bounds the difference of the prices of good g and good h. Furthermore, the valuation for good g bounds the price of g from above. The algorithm modifies non-equilibrium prices to ensure that the necessary equilibrium condition is satisfied.

now show that $p_{g_i} \leq p^*_{g_i} \leq u_i g_i$ holds for all $i \in N$ prior to each round.

(Induction base: Round 1) From free disposal follows that $u_i(x) \geq 0$ for all $i \in N$, $x \in \Omega^r$. If the algorithms terminates in step (3), we are done.

(Induction step) Assume that $p_{g_i} \leq p^*_{g_i} \leq u_i g_i$ holds for all $i \in N$ before round n. If the algorithm terminates in (3), we are done. Otherwise, we assume that i is the agent selected in (4) and j is the index of the good selected from Y_i. Now, p_{g_j} will be adjusted as follows: $p^+_{g_j} = p_{g_j} + ((u_i(g_j) - p_{g_j}) - (u_i(g_i) - p_{g_i} = u_i(g_j) - u_i(g_i) + p_{g_i}$, that is $u_i(g_i) - u_i(g_j) = p_{g_i} - p^+_{g_j} \geq p^*_{g_i} - p^*_{g_j}$. With the induction assumption $p_{g_i} \leq p^*_{g_i}$ follows $p^+_{g_j} \leq p^*_{g_j}$. □

Proof of Theorem 15:

Assume that (X, p^e) is not an equilibrium, ie., there is a bundle $x \subseteq \Omega$ and an agent i such that $u_i(x) - p^e(x) > u_i(X_i) - p^e(X_i)$. It follows from the monotony of the utility functions and the equilibrium property of $(X^a, p^a_{\max})$ that no bundle that is covered by one of the X_i can violate the equilibrium condition. Furthermore, if x would be a bundle that can not be split into elements of X (ie. bundles of the efficient allocation), monotony and the coherence of p^e would require that another bundle x' that can be split and is a minimal cover of x would also violate the condition. We can therefore safely assume that $x \in 2^X$. Now, to make the bundle x inattractive for agent i, the prices of one or more of elements of X that are covered by x could be increased. This would, however, immediately violate one of the equilibrium conditions that hold for the elements of X (because p^e is based on the maximal prices for the reduced assignment economy, and thus, an increase in a price for one of the X_i's would necessarily violate one of the conditions $u_i(X_i) - p^a_{\max}(X_i) \geq u_i(X_j) - p^a_{\max}(X_j)$ $\forall j \in \{1, \ldots, n\}$). Similarly, lowering prices is not possible without violating the equilibrium assumption on $(X^a, p^a_{\max})$ (namely the efficiency assumption). □

Proof of Proposition 20:

(ad 2: Termination) In each iteration, either coherent equilibrium prices are found or the set of agents is shrunken. The aggregation of the agents to alliances may continue until only one agent remains in the reduced economy. In this situation, prices that coherently support the efficient allocation (assuming validity of assumption 1) exist necessarily (0 is such a price, a consequence of the efficiency of the supported allocation). Termination follows.

(ad 1) Assume that the allocation X, used as an input to a new iteration of the algorithm. is efficient (this is satisfied for the first iteration). Let y be the bundle that is selected in step. From the efficiency of X follows that it is an efficient distribution of the X_i-"goods" in y to assign each X_i-good to agent $i \in I$. The valuation for the bundle y, which will be determined for the aggregated agents a', is the sum of the valuation of the agents in I, that is, $u_{a'}(y) = \sum_{i \in I} u_i(X_i)$. Furthermore, $u_{a'}(z) = \max_Z \sum_{i \in N} u_i(Z_i)$, where Z iterates over the $|I|$-ary partitions of z, that is, the valuation corresponds to the best possible use of the "goods" in z by the agents in I prior to the aggregation—therefore, assuming that the allocation that will be determined would violate the efficiency criteria would immediately contradict the assumption of the efficiency of X. □