Qualitative Reasoning with Arbitrary Angular Directions

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Abstract: Agents operating in open fields, like robots, soldiers, rescue workers, or ships, may need to do qualitative reasoning with relative directions. We have introduced here a generalized framework for that purpose, with a parameterized angular zoning scheme (e.g., zoning with 60-degree or 90-degree). Such a parameterization of qualitative reasoning is a new direction, for an area that typically engages in strict propositional reasoning otherwise. We have introduced some important properties of this ontology, the general complexity results for reasoning in it, and provided a maximal tractable subclass. We have also discussed some of the special cases for some constant values of the parameter, showing when this type of reasoning is not possible, and subsuming some of the previously reported interesting results.

Key Words: Geometrical and spatio-temporal Reasoning, Constraint satisfaction problems, Qualitative reasoning

1. Introduction

There are many real life applications where qualitative reasoning with space and time are important. Consider a situation where a group of commandos are trying to communicate with each other in a field of operations, with the objective of identifying the relative positions of themselves, and those of the enemy operatives and their instruments. They are communicating in a relative directional terminology like 'East', 'Southeast' etc. as opposed to the absolute quantitative angular values. For the lack of precise measurements (other than the use of gyros for finding the absolute angular frame of reference), they are communicating with disjunctive information like "the hill is at the {South or Southwest} of me." Checking consistency of such information is a classical spatio-temporal reasoning situation (in Cardinal-directions ontology in this particular case, first proposed in Ligozat, 1998, see Figure 1), and if possible for finding a possible solution. In this article we have proposed a generalized framework for reasoning with such angular directions, when the zoning scheme is with a parameterized angle, not necessarily with 90-degree (as it is in the Cardinal-directions ontology). A more sophisticated reasoning application may need a similar zoning scheme with angles other than 90-degree. For example, a set of autonomous mobile agents (instead of human commandos) in the previous example may not be stuck with the natural language expressions like 'East or Northeast' and may need to do a somewhat more precise reasoning with a 60-degree angular zoning scheme.

Starting from the early studies of simple point-based ontology in linear time, spatio-temporal constraint-based reasoning has matured into a discipline with its own agenda and methodology (Chittaro and Montanari, 2000). The study of such an *ontology* starts with an underlying "space" and develops a set of jointly exhaustive and pair-wise disjoint (JEPD) "basic relations" with respect to a reference object located in that space. Basic relations correspond to the equivalent regions in the space, for the purpose of placing a second object there with respect to the first one. For example, a second point y can be at '<' with respect to a reference point x on a time-line (in the *point-based temporal* ontology, Vilain and Kautz, 1984), where '<' indicates a distinct region, a semi-infinite line left of x. The underlying space and such a relative zoning scheme of the space with respect to a reference object - forms an "ontology" in the context of spatio-temporal knowledge representation.

Qualitative reasoning with such a spatio-temporal ontology involves a given set of objects (e.g., points or time-intervals) located in the corresponding "space" and binary disjunctive relations between some of those objects. Each disjunctive binary relation is a subset of the set of JEPD basic relations. The satisfiability question in the reasoning problem is - whether those relational constraints are consistent with respect to each other or not. The power set of the set of basic relations is closed with respect to the primary reasoning operators like composition,

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inversion, set union and set intersection, thus, forming an algebra. Typically these algebras are Relational Algebras in the Tarskian sense (Jonsson and Tarski, 1952). In the literature on spatio-temporal reasoning area, the term “algebra” is more frequently used while referring to the concept of “ontology” as mentioned in the last paragraph. Thus, “reasoning in the point-

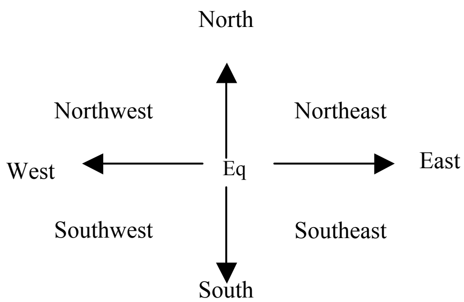


Figure 1: 2D-Cardinal directional ontology

algebra” often means “reasoning in the point-based temporal ontology” (or calculus).

In the last few decades many such spatio-temporal ontologies/calculi have been invented. In this work we have proposed a new one for reasoning with qualitative angular directions between point-objects in a two-dimensional space. We call this ontology as Star-ontology. (It is debatable whether we should continue to use the overused term “ontology” or should call it a class of calculi.) A uniqueness of this one is that it provides a generalized scheme (Star-ontolgy(□), for an integer □) for a class of similar ontologies. Such a parameterized ontology has not been studied before. From this perspective this work is different from just developing another new ontology. The generalization encompasses the 2D-Cardinal directions ontology studied previously (Figure 1, for □=4, 90-degree zoning, Ligozat, 1998). It is also related to compass calculi of Maddux (1994). The generalized scheme provides directions to many new and interesting ontologies for different values of □ and further works on them. Our main results presented here comprise of studies of the ontology and some complexity issues of doing reasoning in it.

2. Generalized framework for the Star-ontology

Star-ontology(□), where □ stands for any even integer, is a generalized angular zoning scheme with respect to a point in the 2D space, with (360/□)-degree angle between any consecutive pair of lines. The set B of (2*□ +1) number of basic relations is {0, 1, 2, 3, …, 2*□}, where 0 indicates ‘Equality’ with respect to the reference point, every odd-numbered relation corresponds to a semi-infinite line away from the origin (the reference point, not inclusive), and the even-numbered relations indicate a pie-slice

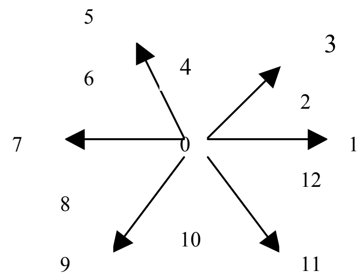


Figure 2. Star-ontology(6)

or conic-sectional region bounded between two such consecutive lines. Figure 2 corresponds to the Star-ontology(6) with 60-degree angle between the lines. Note that there are two types of basic relations in B depending on their dimensionality, other than the relation ‘0’ that has zero dimensionality. We will refer to them as r^e of even type corresponding to a 2D-region (conic-section), and r^o of odd type corresponding to a 1D-region (semi-infinite line).

Figure 3 is a graphical representation (G(□)) of the zoning scheme in the Star-ontology(□). Each circle represents an equivalent region, the dark ones are the 2D-regions, the white ones are the 1D-regions, the central dot represents the ‘Equality’ region 0, and the arcs represent the adjacency between the regions.

[*Proposition 1*] The following formula expresses the *inverse* $r^{\square}$ of any basic relation r □ B: $r^{\square} = (r + \square) \bmod 2*\square$. [*Proposition 2*] The *composition* operations over a pair of basic relations in B are defined as follows:
When the two operands are not inverse to each other,
$r^o . r^o = (r^o - r^o)$; $r^e . r^o = [r^e - r^o)$; $r^o . r^e = (r^o - r^e]$; $r^e . r^e = [r^e - r^e]$;
where ‘.’ indicates the binary composition operator. [The usual semantics for the parenthesis or the bracket applies for an open (exclusive) or a closed (inclusive) interval respectively.] The resulting ranges are the shortest intervals on the corresponding graphical representation of the basic relations G(□) (see

Figure 3), or *convex colures* of the two (see Ligozat, 1996).

However, when a basic relation is composed with its own inverse: either

[*Proposition 3*] $r^e . r^{e\square} = r^{e\square} . r^e$ = T (*tautology*, disjunction of all basic relations), or

[*Proposition 4*] $r^o . r^{o\square} = r^{o\square} . r^o = \{ r^o, 0, r^{o\square} \}$.

[*Proposition 5*] The composition is a commutative operator, r.l = l.r. Also, [*Proposition 6*] r.r = r, and

[*Proposition 7*] r.l = inverse($r^{\square} . l^{\square}$).

Surprisingly, the Star-ontology(□), when □ is an odd integer, is not useful for any reasoning purpose. See Figure 4 for the Star-

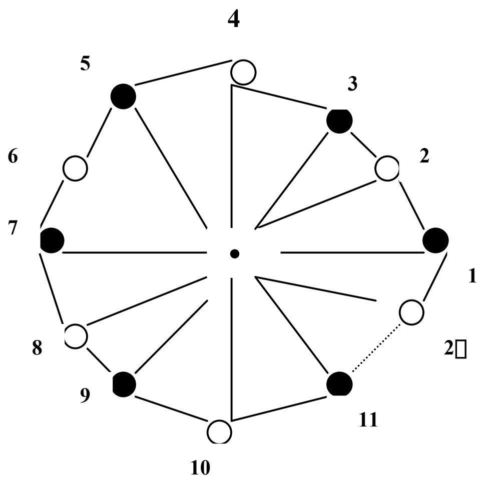


Figure 3: Graphical representation G(□) of the Star-ontology(□)

ontology(3) with basic relations {0, 1, 2, 3, 4, 5, 6}. A major problem here is that the results of the composition operations are not unique. For example, composition operation (2.4), between three points x, y, and z, with [y {2} x], and [z {4} y], would result in both [z {2, 3, 4} x], and [z {0, 1, 2, 3, 4, 5, 6} x], depending on where the point y is located in the supposedly equivalent region '2' with respect to the point x (two figures in 4). A composition table for basic relations cannot be computed for this ontology, and thus, doing any reasoning in the Star-ontology(3), is logically impossible. [*Proposition 8*] This observation is true for any Star-ontology(□), where □ is an odd integer.

Note that a complex or a *disjunctive composition* operation involving two disjunctive relations R and M is done by taking the set union of the corresponding basic composition operations: □ r.m, for every pair of basic relations r □ R and m □ M.

[*Definition 1*] The set of all such disjunctive relations or the power set 2^B of the set of the basic relations B={0, 1, 2, 3, …, 2*□} is closed under disjunctive composition, inverse, set union and intersection operations, forming the *Star-algebra(□)*. A reasoning problem instance in any subset □ of this set 2^B is expressed as (V, E), where V is a set of points situated in the 2D-space, and E is a set of binary constraints R_{ij} between some of the pairs of points i, j in V such that any R_{ij} is an element of □. The satisfiability question in the *reasoning problem*, as mentioned before, is to check if it is feasible to assign the points in the space following all the constraints in E.

Theorem 1: Reasoning with full Star-algebra(□) is NP-complete.

Proof sketch: Construction of a problem instance in the Star-algebra from an arbitrary 3-SAT problem instance would be as follows. (1) For every literal l_{ij} (in the 3-SAT source problem), create two points P_{ij} and R_{ij} such that P_{ij} [2 - (□+2)] R_{ij}, and (2) for every clause C_i we have P_{i1} [(□+2) - (2*□)] R_{i2} and P_{i2} [(□+2) - (2*□)] R_{i3} and P_{i3} [(□+2) - (2*□)] R_{i1}. Also, (3) for every literal l_{ij} that has a complementary literal l_{gh} we have two relations between their corresponding points: P_{ij} [□ - (2*□)] R_{gh} and P_{gh}, P_{ij} [□ - (2*□)] R_{ij}. (A relation P [x – y] Q means, the point P has relation {x, x+1, x+2, …y} with the point Q.) Hence, the problem is NP-hard.

Given some binary constraints between a set of points in any Star-ontology(□), and a set of assignment for those points in the 2D-space (e.g., by their Cartesian coordinates), it could be easily verified whether the assignment follows the constraints or not, in O(|E|) time, for |E| number of binary constraints. Hence, the problem is NP-complete. *End Proof sketch.*

Note that a reasoning problem has binary constraints, each of which is a disjunctive set of basic relations $R_{ij}=\{r_1, r_2, …\}$.

[*Definition 2*] A *convex relation* is defined as a disjunctive set of basic relations, which is expressed as the shortest range [r_1 – r_2, [0]] over the graphical representation G(□), such that r_2 □ (r_1 + □) mod 2*□. When r_2 < (r_1 + □) mod 2*□, then the relation 0 is optionally included (including 0 and without - both are convex relations), but when r_2 = (r_1 + □) mod 2*□, i.e., r_2 is inverse of r_1, then the relation 0 must be present within a convex relation. For all odd basic relations r, {r, 0} and {r, 0, $r^{\square}$} are also convex relations. [*Definition 3*] A *preconvex relation* is either a convex relation or a convex relation c *without* any number of lower dimensional regions in c.

[*Proposition 9*] The set C of all convex relations is closed under the disjunctive composition. This fact can be trivially proved since the basic-composition operation of a pair of basic relations results in a shortest continuous range over G(□) by Proposition 2, since disjunctive-composition is a union of the results from the corresponding basic-compositions, and since convex relations are themselves shortest ranges over G(□). Tautology appears in a special case for composition between $[r_1 - r_2]$ and $[r_3 - r_4]$, when r_1 and r_4 are separated by more than half circumference on G(□). [*Proposition 10*] The set of all convex relations are closed under the inverse, and the set intersection operations: also trivial to show.

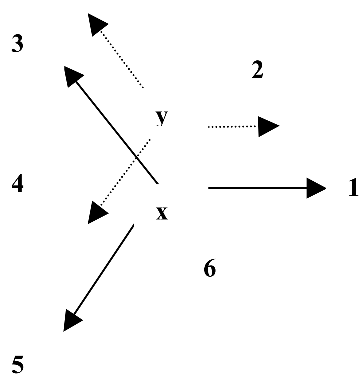
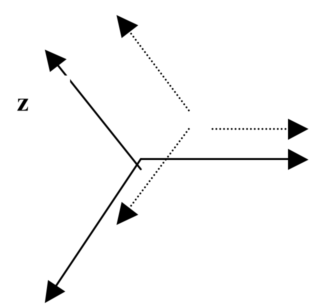


Figure 4. Star-ontology(3), basic relations do not have unique composition

[*Proposition 11*] The set of all preconvex relations P are closed under disjunctive composition operation. Observing this fact is little complicated. Note that a preconvex relation also spans a range (shortest) on G(□) like a convex relation, except that some lower dimensional regions (r^o, or 0) may be absent from the range. The fact that disjunctive-composition of relations spanning over the two ranges will compose to another range - remains true here as well as in the case of convex relations. However, the absence of an r^o from one of the operands could have made the resulting range discontinuous. This situation will not arise: for any absent internal r^o from an operand the two adjacent r^e relations would be present in the same operand and they will compensate for the absent r^o. For example, compose two preconvex relations {2, 4, 5} and {3, 4, 6} in the Star-ontology(6) (Figure 2). Although, 1D-regions (r^o) 3 and 5 are absent in the two operands respectively, their adjacent 2D-regions (r^e) are present. The result of the composition operation is 2.3 □ 2.4 □ 2.6 □ 4.3 □ 4.4 □ 4.6 □ 5.3 □ 5.4 □ 5.6. The adjacent 2D-regions compensate for the corresponding absent 1D-region in any operand and the result of the composition remains unaffected for that absence. However, if two absent r^o relations in the two operands are on the same line (e.g., {2, 4, 5} and {1, 2, 4}, absent are 3 and 3), then their composition results (3.3 = 3) may not be reproduced by the adjacent 2D-regions. Since such a resulting absent region could be only of 1D-type, the result remains a preconvex relation. Hence, the set of all preconvex relations remains closed under disjunctive composition.

[*Proposition 12*] Preconvex set P is closed under the set intersection and the inverse



operations. Since the possible absence of 1D-regions from the two operands (ranges otherwise) of the set intersection operation could not cause any 2D-region to be absent from the result of the operation - the preconvex property is preserved by the latter. Inverse operation will trivially preserve the preconvex property of its operand. [*Proposition 13*] Thus, the sets C and P (C □ P) form two sub-classes of the Star-algebra(□).

Conjecture 2: *4-consistency is sufficient to imply global consistency for the preconvex sub-class P.*

The proof of the above conjecture-2 could be developed by using an extension of the Helly's theorem for *convex sets* as stated in Chvátal (1983, Theorem 17.2): "Let F be a finite family of at least n+1 convex sets in R^n such that every n+1 sets in F have a point in common. Then all the sets in F have a point in common."

One could define a corresponding notion of a *pre-convex set*, where from a convex set c, some strictly-lower dimensional-convex subsets of c may be absent. A circle is a convex region in the 2D-space. However, exclude a straight line (a convex region in a lower dimension) over the circle from that circle, it (circle minus the line) becomes a pre-convex region, and does not remain a convex region. Helly's theorem could be extended toward the pre-convex sets. Using such an extended Helly's theorem one can prove the Conjecture 2 by induction.

Proof sketch of Conjecture 2: Induction base case for four points is trivially true by the definition of 4-consistency. Induction hypothesis is that the assertion is true for (m-1) points, and hence all the (m-1) points have satisfiable placements in the space. Consider a new m-th point with respect to which we have (m-1) preconvex relations from the other (m-1) older points. By 4-consistency assumption we know that the three regions *wrt* every three old points have a non-null intersection. By extended Helly's theorem, that would imply the existence of a non-null region for the new m-th point. Hence, the placement of all old (m-1) points is extendable to a non-null region for the placement of the new m-th point, or the global consistency is implied. *End proof sketch.*

4-consistency Algorithm:

(Step 1) Initialize a queue Q with all constrained arcs (with non-tautology labels);
(Step 2) For each arc R_{ij} in Q do
(Step 3) For each pair of distinct nodes k and l other than i, j do
(Step 4) temp = R_{ik} □ update(R_{ik}, j, l)
(Step 5) if (temp≠ R_{ik}) then
(Step 6) if (temp = = null) return INCONSISTENCY; end if;
(Step 7) Push (Q, R_{ik});
(Step 8) R_{ij} = temp; end if;
(Step 9) Run Step 4 - 8 for all other four arcs R_{il}, R_{jk}, R_{jl}, and R_{kl};
end for; end for;
End algorithm.

Function update(R_{ij}, k, l) = [(R_{ik} . (R_{kl} . R_{lj})) □ ((R_{ik} . R_{kl}) . R_{lj})) □ ((R_{ik} . R_{kj})) □ ((R_{il} . R_{lj}))]

Complexity of the algorithm is $O(n^4)$ for n-nodes: Each arc ($O(n^2)$ arcs) gets at the most a fixed number of times (2□) to be back in the queue and the two loops in steps 2 and 3 run $O(n^2)$ times. Hence, the algorithm is polynomial.

By conjecture 2 the algorithm is correct and complete for the convex and the preconvex sub-classes of problems in the Star-algebra(□). So, [*Proposition 14*] C and P are a tractable sub-classes.

Theorem 3: The subclass P is maximally tractable.

Proof sketch: Define maximal-convex relations being the ones corresponding to a half space region on one side of a 'line' in a Star-ontology(□). For example, in Star-ontology(6) (Figure 2) regions {1, 0, 7}, {3, 0, 9} and {5, 0, 11} are three such lines, and a disjunctive relation {0, 1, 2, 3, 4, 5, 6, 7} is such a maximal-convex relation there. Now, add a corresponding adjacent two-dimensional region to each such maximal-convex relation (e.g., {0, [1 – 8]}, which will obviously be a non-convex relation, and call any such relation as m+. Next, loosen the definition of m+ by allowing some lower dimensional regions to be absent in it (e.g., {0, [1 – 4], [6 - 8]}), and call such a relation as p+. Consider the set P+ of all such p+ relations. Our proof of NP-hardness (Theorem 1) of the Star-algebra(□) uses such p+ relations and thus, shows that the reasoning problem in P+ is NP-hard. This fact, along with the Proposition 14, clearly shows that the subclass P is maximally tractable. *End proof sketch*.

3. Special Cases of the Star-ontology

Star-ontology(2) is a special case with five basic relations that could be semantically described as {*Equality, Front, Above/Left, Back, Below/Right*}. This ontology has some interesting applications in qualitative spatial reasoning. Studying the corresponding simple algebra is a future direction to our work.

Star-ontology(4) and the corresponding algebra has been extensively studied by Ligozat (1998). As in the case of Star-ontology(2), nine basic relations in Star-ontology(4) also have common sense representation: {*Equality*≡0, *East*≡1, *Northeast*≡2, *North*≡3, *Northwest*≡4, *West*≡5, *Southwest*≡6, *South*≡7, *Southeast*≡8} (Figure 1). The corresponding algebra is called as the Cardinal-directions algebra. Most of our work here generalizes Ligozat's studies of Cardinal-directions algebra. However, Ligozat's definition of convex relations is stricter than that used in the literature of Linear Algebra (Chvátal, 1983), whereas our definition follows the latter. [In Linear Algebra a convex region does not necessarily demand the "closure" property over the "lattice" representation, which is equivalent to the graphical representation G(□) in this article. For example, the disjunctive relation {East, Northeast, North, Northwest} is not closed as per Ligozat and needs West for the closure. Hence it is not a "convex" relation in his terminology, but the region represented by it is a convex set in R^2 space (in Linear Algebra) and so, we consider it as a convex relation in the Star-ontolgy(4).]

We have also worked on the Star-ontology(6) as a special concrete case (Figure 2). It has thirteen basic relations 0 (Equality) through 12. The corresponding composition table is being presented in Mitra (2002). There are 156 convex relations (including the *null* and the *tautology* (disjunction of all 13 basic relations) and 508 preconvex relations out of the total 2^{13} elements in the corresponding Star-algebra(6).

5. Conclusion

In this article we have proposed a new ontology called Star-ontology(□) for reasoning with arbitrary angular directions in 2D-space. We have discussed its properties and the complexity issues in reasoning with this ontology. Our work subsumes previously studied 2D-Cardinal directions ontology that is equivalent to the Star-ontology(4). Some interesting other ontologies that could be developed out of such a generalized framework (for different values of □) are also being suggested here.

We have also deployed a new methodology for studying the complexity issues that avoids using projections on the coordinate axes, which used to be the standard methodology before (Ligozat, 1996, 1998, Condotta, 2000) for such studies. Our technique (of relying on the Graphical representation G(□), rather than a lattice for the topology of the basic relations) has broader implications than being utilized here.

Reasoning with angular directions has a similarity to the problems of reasoning in cyclic-time ontology as developed by Balbiani and Osmani (2000). A future direction of our work is to study that similarity.

Acknowledgement: The author is supported by the NSF grant IIS-9733018. Hyoung R. Kim's and some unknown reviewers' comments have been very helpful.

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