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The Shapley Value in Voting Games: Computing *Single Large Party's* Power and Bounds for Manipulation by Merging

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Abstract

We consider two related and important problems in weighted voting games. For the first problem, we provide counterexamples to show that the previously proposed formulae for computing the Shapley values of *large* players in a restricted class of voting games referred to as *single large party*, are incorrect. Consequently, we present a simple and correct formula for computing the Shapley values of players in this type of games. Second, we bound the effects of *manipulation by merging*, i.e., dishonest behavior where *strategic* agents merge their weights to form a single bloc. A motivation for this problem can be found in decision-making, e.g., in negotiation settings. Consider a set of agents negotiating on how to allocate some budgets. If a subset of strategic agents merge their weights, they may be able to increase their share of the budgets. This obviously raises the question: *What is the amount of damage that is caused to the non-manipulating agents? Analogously, what is the extent of budgets, payoffs, or power that manipulators may gain?* We propose two non-trivial *tight bounds* for this problem using the well-known *Shapley-Shubik* index. The results we propose here fit under the models and mechanisms for establishing identities, which are crucial for trustworthy interactions.

1 Introduction

Cooperation among self-interested autonomous agents in multiagent environments is fundamental for agents to successfully achieve goals for which they lack all the required capabilities, skills, and knowledge to complete tasks alone. The level of skills and agents' resources vary, hence, the need for agents' cooperation to complete tasks that are otherwise difficult for individual agents to achieve, or for which better results can be attained working together as a group. One way of modeling such cooperation is via the use of *weighted voting games* (WVGs). WVGs are important in multiagent systems and human societies because of their usage in automated decision-making. See for example, (Chalkiadakis, Elkind, and Wooldridge 2012).

In a WVG, each agent has an associated *weight*. A subset of agents, called a *coalition*, wins in the game, if the sum of the weight of each agent in the coalition meets or exceeds a certain threshold, called the *quota*. Such coalitions are referred to as *winning coalitions*. The relative *power* of each agent in a WVG reflects its significance in the elicitation of winning coalitions. Agents' relative power in such games is measured using *power indices*. The *Shapley value* (Shapley 1953) and its variant, the *Shapley-Shubik* index (Shapley and Shubik 1954), are the prominent solution concepts for computing agents' power (or payoffs) in WVGs.

Prominent real-life situations where WVGs have found applications include the United Nations Security Council, the Electoral College of the United States, the International Monetary Fund, the Council of Ministers, and the European Community (Leech 2002; Alonso-Meijide and Bowles 2005). Other areas of application include economics, political science, neuroscience, threshold logic, and distributed systems (Taylor and Zwicker 1999). Also, (Nordmann and Pham 1999) have considered reliability and cost evaluation of weighted dynamic-threshold voting-systems. These systems are used in target detection, pattern recognition, safety monitoring, and human organization systems.

This paper investigates two closely related and important problems in WVGs: (1) computing the power of players in a restricted class of voting games referred to as *single large party*, and (2) bounding the effects of *manipulation by merging* (i.e., dishonest behavior by strategic players), where two or more agents merge their weights to form a bloc. Our analyses of these problems employ the well-known Shapley value and Shapley-Shubik index to measure the infuence of players (i.e., power) in the voting games. The remainder of this paper is organized as follows. Section 2 presents definitions and notation needed to provide necessary backgrounds. Section 3 considers the single large party voting games and proposes a correct Shapley value formula for computing the power of large players in such games. We investigate manipulation by merging in Section 4, and propose two new tight bounds (using the Shapley-Shubik index) to characterize the effects of this menace in Section 5. Section 6 discusses related work. We conclude in Section 7 and give directions for future work.

2 Definitions and Notation

Let $I = \{1, \ldots, n\}$ be a set of $n \in \mathbb{N}$ agents. Let $\{w_1, \ldots, w_n\}$ be the corresponding weights of these agents. The non-empty subsets, $S \subseteq I$, are called *coalitions*.

Definition 1. *Simple Game*

A simple game is a coalitional game, (I, v), where $v : 2^I \to \{0, 1\}$. A coalition $S \subseteq I$ *wins* if $v(S) = 1$ and *loses* if $v(S) = 0$.

Definition 2. *Weighted Voting Game*

A weighted voting game is a simple game which has a weighted form, (W, q), where $W = (w_1, \ldots, w_n) \in (\mathbb{R}^+)^n$ corresponds to the weights of agents in I, and $q \in \mathbb{R}^+$ is the quota of the game. A coalition S wins if the total weight of S, $w(S) = \sum_{i \in S} w_i \geq q$, which implies that $v(S) = 1$. A WVG G of n agents with quota q is denoted by $G = [q; w_1, \ldots, w_n]$. Note also that $\frac{1}{2}w(I) < q \leq w(I)$.

Definition 3. *Critical Agent*

An agent $i \in S$ is critical to a coalition S if $w(S) \geq q$ and $w(S \backslash \{i\}) < q$.

Definition 4. *(Fatima, Wooldridge, and Jennings 2007) Single Large Party Voting Game*

A single large party voting game consists of two player types: *a player with a large weight*, and *multiple players each having the same but a small weight compared to that of the large player's weight*. A single large party voting game with quota q, having $m \in \mathbb{N}$ small players with weight w_s each, and a large player with weight w_l (where $w_l > w_s$) is denoted by $[q; w_l, \underbrace{w_s, \ldots, w_s}_{m \text{ times}}]$. It is required that $w_l < q$, otherwise, the large player can win in a game without forming coalitions with any of the small players. Similarly, $m \cdot w_s < q$, so that the small players also need the large player to win in a game.

Definition 5. *Shapley Value*

In simple games, the Shapley value is often referred to as the Shapley-Shubik power index.

Definition 6. *Shapley-Shubik Power Index*

The Shapley-Shubik index quantifies the marginal contribution of an agent to the grand coalition. Each permutation of the agents is considered. We term an agent *pivotal* in a permutation if the agents preceding it do not form a winning coalition, but by including this agent, a winning coalition is formed. We specify the computation of the index using notation of (Aziz et al. 2011). Denote by π, a permutation of the agents, so $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$, and by Π the set of all possible permutations. Denote by $S_\pi(i)$ the predecessors of agent i in π, i.e., $S_\pi(i) = \{j : \pi(j) < \pi(i)\}$. The Shapley-Shubik index, $\varphi_i(G)$, for each agent i in a WVG G:

$$\varphi_i(G) = \frac{1}{n!} \sum_{\pi \in \Pi} [v(S_\pi(i) \cup \{i\}) - v(S_\pi(i))]. \quad (1)$$

3 Power in Single Large Party Voting Games

We are concerned here with the ways in which agents that complete tasks together are compensated from their jointly derived payoffs. The Shapley value is a prominent solution concept for computing agents' payoffs in WVGs. The Shapley value is attractive, as it provides *unique* and *fair* solutions that are devoid of ambiguities to the players. However, the problem of computing the Shapley value in WVGs is known to be $\#P$-complete (Deng and Papadimitriou 1994).

In view of the intractability of this problem in the general case of WVGs, (Fatima, Wooldridge, and Jennings 2007) identify, among other things, four restricted classes of WVGs whose Shapley values can be computed in polynomial time. One such game is referred to as *a single large party*. The authors proposed two constant time formulae for computing the Shapley values of the large players in this type of games. The two formulae are for the cases when the weight of each of the small players is either 1 or greater than 1. We provide in this paper, counterexamples, to show that the two proposed formulae for the computation of the Shapley values of large players in the single large party voting games are incorrect, and then present a simple and correct formula for computing the Shapley values of large players in this class of voting games. Thus, we clarify the computation of the Shapley values of players in this class of games.

3.1 Counterexamples

We provide counterexamples to show that the previously proposed formulae for computing the Shapley values of large players in the single large party voting games are incorrect.

Example 1. *Case 1: the weight w_s of each of the small players is 1, i.e., $w_s = 1$.*

Let $[10; 8, 1, 1, 1, 1, 1, 1]$ be a single large party voting game with quota $q = 10$, having a single large player with weight $w_l = 8$, and $m = 6$ small players each with weight $w_s = 1$. According to (Fatima, Wooldridge, and Jennings 2007), the Shapley value φ_l of the large player is

$$\varphi_l = \frac{w_l}{m+1} = \frac{8}{6+1} > 1,$$

which is incorrect, as the Shapley value of an agent cannot be greater than 1. In fact, the Shapley values for all the agents in a game sum up to 1 since this solution concept is normalized.

Example 2. *Case 2: the weight w_s of each of the small players is greater than 1, i.e., $w_s > 1$.*

Let $[18; 14, 2, 2, 2, 2, 2]$ be a single large party voting game with quota $q = 18$, having a single large player with weight $w_l = 14$, and $m = 5$ small players each with weight $w_s = 2$. Again, according to (Fatima, Wooldridge, and Jennings 2007), the Shapley value φ_l of the large player is

$$\varphi_l = \frac{\lceil w_l/w_s \rceil}{m+1} = \frac{\lceil 14/2 \rceil}{5+1} = \frac{7}{6} > 1,$$

which is also incorrect.

A simple observation justifies the incorrectness of these formulae. The formulae do not take into consideration the quotas of the games in their computation of the Shapley values of players. Thus, irrespective of the values of the quotas of the single large party voting games, the Shapley values of the large players, as computed in the examples, remain the same in both cases considered.

3.2 Correct Shapley Value Formula

Since there are $m + 1$ players in a single large party voting game (see Definition 4), a large player with weight w_l can join a coalition as the i-th member, where $1 \leq i \leq m + 1$. However, we are interested in the i's, such that the *marginal contribution* of the large player is 1 when it joins a coalition as the ith member. The marginal contribution of an agent $a \notin S$ to a coalition S is 1 if $w(S \cup \{a\}) \geq q$ and $w(S) < q$. In other words, $v(S \cup \{a\}) = 1$ and $v(S) = 0$. Consider the extreme cases first; clearly, when the large player is the first agent in any coalition, it has a marginal contribution of 0, since the agent alone cannot win in the game. Whereas, when it joins as the last agent, i.e., as the $(m + 1)$-st agent, the marginal contribution is 1, since all the small agents together cannot win without the large player. Thus, for each coalition size, we need to know the number of small players that should already be in the coalition such that when the large player joins, the agent has a marginal contribution of 1. This number is given as $\lceil \frac{q-w_l}{w_s} \rceil$ for the smallest-sized coalition that the large player can join. Thus, the large player has a marginal contribution of 1 when it joins a coalition as the i-th member where $\lceil \frac{q-w_l}{w_s} \rceil + 1 \leq i \leq m + 1$.

Now, the Shapley value φ_l of the large player can then be computed as

$$\varphi_l = \frac{m + 1 - \lceil \frac{q-w_l}{w_s} \rceil}{m + 1} \tag{2}$$

since there are a total of $m + 1$ possibilities for the player to join a coalition. This formula holds for both cases of the weights of the small agents, i.e., $w_s \geq 1$. Using (2), the Shapley values of the large players in Examples 1 and 2 are $\varphi_l = 0.7143$ and $\varphi_l = 0.6667$, respectively. The Shapley values φ_s of each of the small players can then be computed as $\varphi_s = \frac{1-\varphi_l}{m}$ (Fatima, Wooldridge, and Jennings 2007).

We have shown a simple and correct formula for computing the Shapley values of large players in a restricted class of weighted voting games referred to as a single large party. We have thus successfully set the record straight on the computation of the Shapley values of players in this class of WVGs.

4 Manipulation by Merging in WVGs

4.1 Overview

Even though WVGs are useful in modeling cooperation among players for making joint decisions, they are not immune from the vulnerability of manipulation (i.e., dishonest behavior) by some players called *manipulators*, or referred to as being *strategic*, that may be present in the games. With the possibility of manipulation, it becomes difficult to establish or maintain trust, and it becomes difficult to assure fairness in such games. This problem of insincere and manipulative behaviors among agents in WVGs has received attention of many researchers in recent years. See the works of (Bachrach and Elkind 2008; Aziz et al. 2011; Lasisi and Allan 2012; 2013; 2014).

Manipulation by merging in WVGs involves voluntary coordinated action of strategic agents who come together to form a *bloc* by merging their weights into a single weight (Machover and Felsenthal 2002; Aziz et al. 2011). Strategic agents merge their weights in anticipation of gaining more power over the outcomes of games. In a beneficial merge, merged agents are compensated commensurate with their share of the power gained by the bloc. Agents in the bloc are assumed to be working cooperatively and have transferable utility. Thus, proceeds from merging can easily be distributed among the manipulators without bickering.

This manipulation can be easy and cheap to achieve in open anonymous environments, such as the Internet. Common settings that may be vulnerable to such attack are online elections, rating systems, electronic negotiation, and auctions. We motivate this problem further by considering a real-world example from social choice domain. Consider a parliament consisting of five political parties, A, B, C, D, and E, which have 20, 30, 40, 50, and, 50 representatives (i.e., weights), respectively. This parliament is to vote on a \$100 million spending bill and how much of this amount should be controlled by each party. Further, the bill requires a minimum of 51 votes (i.e., quota) to pass. Assuming that all members of a political party votes in the same direction on a bill, the Shapley-Shubik index allocates the amount of the spending bill to be controlled by each party as follows: $A =$ \$3.33m, $B =$ \$20m, $C =$ \$20m, $D =$ \$28.33m, and $E =$ \$28.33m. Now, suppose political parties A, B, and D, merge their weights to form a bloc with weight 100. The new allocation of the amount by Shapley-Shubik index to the manipulators' bloc is \$66.67m, which is more than \$51.67m, the sum of the initial allocation to each party in the bloc.

Previous work (Aziz et al. 2011) has shown that the problem of finding beneficial merge is NP-hard for the Shapley-Shubik index. This complexity result seems sufficient to discourage would-be strategic agents from merging. We argue in the contrary that, NP-hardness result is a worst case measure, and only shows that at least one instance of the problem requires such complexity. Thus, the real life instances of WVGs that we care about may be easy to manipulate (Lasisi and Allan 2013). Furthermore, Machover and Felsenthal (2002) characterize situations when it is advantageous or disadvantageous for agents to merge, and show that using the Shapley-Shubik index, merging can be advantageous or disadvantageous. Also, Lasisi and Allan (2011) consider evaluation of the extent of susceptibility of three power indices, namely, Shapley-Shubik, Banzhaf, and Deegan-Packel, to merging. Their results show that the Shapley-Shubik index is the most susceptible to merging.

However, none of these works provide bound on the extent of power that manipulators may gain in the case that the merging is advantageous. In contrast to these works, we propose new bounds on the extent of power that strategic agents may gain with respect to merging in WVGs using the Shapley-Shubik index to compute agents' power.

4.2 Formal Problem Definition

Let $G = [q; w_1, \ldots, w_n]$ be a WVG of n agents. Let $k \in \mathbb{N}$, $2 \leq k < n$. Consider a manipulators' coalition S of k agents which is a k-subset of the n-set I. We assume that S contains k distinct elements chosen from I. Suppose the manipulators in S merge their weights to form a bloc denoted by $\&S$, i.e., agents $i \in S$ have been assimilated into the bloc $\&S$, then, we have a new set of agents in the game after merging. Thus, the initial game G of n agents has been altered by the manipulators to give a new WVG G' of $n-k+1$ agents consisting of the bloc and other agents not in the bloc, i.e., $I\backslash S$.

Denote by $(\varphi_1(G), \ldots, \varphi_n(G)) \in [0,1]^n$ the Shapley-Shubik power of agents in a WVG G of n agents. Thus, for the manipulating agents $i \in S$ with power $\varphi_i(G)$ in game G, the sum of the power of the k manipulators in S is $\sum_{i\in S} \varphi_i(G)$, while that of the bloc formed by the manipulators in game G' is $\varphi_{\&S}(G')$. The ratio $\tau = \frac{\varphi_{\&S}(G')}{\sum_{i\in S}\varphi_i(G)}$ compares the power of the bloc formed by merging in the altered game G' to the sum of the original power of the agents in the merged bloc. τ gives a factor of the power gained or lost when strategic agents $i \in S$ alter game G to give G'. We say that φ is susceptible to manipulation if there exists a game G' such that $\tau > 1$; the merging is termed *advantageous*. If $\tau < 1$, then the merging is *disadvantageous*, while the merging is *neutral* when $\tau = 1$.

4.3 Examples of Manipulation by Merging

The strategic agents in each games are all shown in bold.

Example 3. *Advantageous Merge*

Let $G = [28; \mathbf{8}, 8, 8, 6, \mathbf{5}, 5, \mathbf{4}, \mathbf{2}, \mathbf{2}, \mathbf{2}]$ be a WVG, i.e., a game with quota, $q = 28$, and ten agents, $1, 2, \ldots, 10$. The power of the strategic agents are, $\varphi_1(G) = 0.1671$,$\varphi_5(G) = 0.0988$, $\varphi_7(G) = 0.0667$, $\varphi_8(G) = \varphi_9(G) = \varphi_{10}(G) = 0.0385$. Their cumulative power is 0.4481. Suppose the manipulators form a bloc $\&S$ and alter G by merging their weights into a single weight as follows; $G' = [28; \mathbf{23}, 8, 8, 6, 5]$. The power of this bloc is $\varphi_{\&S}(G') = \varphi_1(G') = 0.8000 > 0.4481$. The factor by which the bloc gains is $\frac{0.8000}{0.4481} = 1.79$

Example 4. *Disadvantageous Merge*

Let $G = [56; 10, 9, 9, \mathbf{9}, 8, \mathbf{7}, 6, \mathbf{6}, \mathbf{2}, \mathbf{1}]$ be a WVG of ten agents. The power of the strategic agents are, $\varphi_4(G) = 0.1214$, $\varphi_6(G) = \varphi_8(G) = 0.1175$, $\varphi_9(G) = 0.0222$, and $\varphi_{10}(G) = 0.0083$. Their cumulative power is 0.3869. Suppose the manipulators form a bloc $\&S$ and alter G as follows; $G' = [56; \mathbf{25}, 10, 9, 9, 8, 6]$. The power of this bloc is $\varphi_{\&S}(G') = \varphi_1(G') = 0.3333 < 0.3869$. The factor by which the bloc loses is $\frac{0.3333}{0.3869} = 0.86$

Example 5. *Neutral Merge*

Let $G = [3; 2, \mathbf{1}, \mathbf{1}, 1]$ be a WVG of four agents. The power of the strategic agents are, $\varphi_2(G) = \varphi_3(G) = 0.1666667$. Their cumulative power is 0.3333334. Suppose the manipulators form a bloc $\&S$ and alter G as follows; $G' = [3; 2, \mathbf{2}, 1]$. The power of this bloc is $\varphi_{\&S}(G') = \varphi_2(G') = 0.3333333$. Rounding the cumulative power of the manipulators (in G) and that of the bloc (in G') to 0.3333 shows that the strategic agents neither gain nor lose power.

We have shown that strategic agents may gain power, lose power, or their power may remain the same when they engage in manipulation by merging using the Shapley-Shubik index. Next, we provide tight upper and lower bounds to characterize the effect of this menace in WVGs.

5 New Bounds on Manipulation by Merging

Theorem 1. *(Upper Bound) Let $G = [q; w_1, \ldots, w_n]$ be a WVG of n agents. If two manipulators, m_1 and m_2, merge their weights to form a bloc, $\&S$, in an altered game G', then, the Shapley-Shubik power, $\varphi_{\&S}(G')$, of the bloc in the new game, $\varphi_{\&S}(G') \le \frac{n}{2}(\varphi_{m_1}(G) + \varphi_{m_2}(G))$. Moreover, this bound is asymptotically tight.*

Proof. Let $S \subset I$ be a coalition of two distinct manipulators, m_1 and m_2, from the original game G that would like to merge into a bloc $\&S$ in an altered game G'. Let Π_G be the set of all permutations of the n agents in game G. Also, let Π_{G-2} be the set of all permutations of the remaining $n-2$ non-manipulating agents in G, i.e., not including m_1 and m_2. Again, for any permutation $\pi \in \Pi_{G-2}$, let $r \in \mathbb{N}$ be the possible positions in π for insertion of m_1 or m_2 within the non-manipulating agents. Thus, $1 \le r \le n-1$.

We first bound the number of permutations in game G, for which the manipulators m_1 and m_2 are pivotal. Consider any $\pi \in \Pi_{G-2}$. Suppose we insert m_1 and m_2 arbitrarily into π to have a resulting permutation $\pi^* \in \Pi_G$ of n agents. Let Π^*_G be the set of all permutations π^* such that one of m_1 or m_2 is pivotal for π^*. Finally, let $Q(\pi^*, r, \pi)$ be the set of all permutations π^* in which at least one of m_1 or m_2 appears on the r-th position of $\pi \in \Pi_{G-2}$ and is pivotal for π^*. For example[1], consider a WVG of six agents with quota $q = 15$. Suppose $\pi = 8\ 6\ 4\ 2$, and consider an arbitrary insertion of two manipulators, **3** and **5**, into π. Let the resulting permutation $\pi^* = 8\ \mathbf{5}\ \mathbf{3}\ 6\ 4\ 2$. The manipulators are both on the 2-nd position of π (i.e., $r = 2$). Also, the manipulator with weight 3 is pivotal for π^*.

Note that $\Pi^*_G \subseteq \Pi_G$, and every permutation in Π^*_G appears in one of the sets $Q(\pi^*, r, \pi)$ for some π and r. Thus, the Shapley-Shubik power of the manipulators in game G is:

$$\varphi_{m_1}(G) + \varphi_{m_2}(G) = \frac{|\Pi^*_G|}{n!} \le \frac{1}{n!}\sum_{\pi,r}|Q(\pi^*, r, \pi)| \tag{3}$$

Now, we bound the number of permutations in the altered game G' for which the bloc $\&S$ is pivotal. Let $\pi \in \Pi_{G-2}$. Consider a permutation $f(\pi, r)$ of agents in game G' obtained from π by inserting the bloc $\&S$ at the r-th position of π. Note that if $Q(\pi^*, r, \pi)$ is not empty, then $\&S$ is pivotal for the permutation $f(\pi, r)$. Also, all the permutations π^* in the set $Q(\pi^*, r, \pi)$ derived from a permutation $\pi \in \Pi_{G-2}$ in which at least one of the manipulators appears at the r-th position of π and is pivotal for π^* corressponds to a single permutation $f(\pi, r)$ in game G'. Furthermore, it is not difficult to see that, $|Q(\pi^*, r, \pi)| \ge 2$, for all π and r when $Q(\pi^*, r, \pi) \ne \emptyset$. This is because if one of the two manipulators, say, m_1, is pivotal at position r, we can also insert the other, i.e., m_2, immediately before or after m_1 at the same position, and there are only 2 ways for them to appear together at r. Finally, we compute the Shapley-Shubik power of the bloc $\&S$ in the altered game G' by counting all the non-empty sets $Q(\pi^*, r, \pi)$ for all π and r. Hence,

[1]The numbers in the permutations are the weights of the agents.

$$
\begin{aligned}
\varphi_{\&S}(G') &\leq \frac{1}{(n-1)!} \sum_{\pi,r:Q(\pi^*,r,\pi)\neq\emptyset} 1 \\
&= \frac{1}{(n-1)!} \sum_{\pi,r} |Q(\pi^*,r,\pi)| \cdot \frac{1}{|Q(\pi^*,r,\pi)|} \\
&= \frac{1}{(n-1)!} \sum_{\pi,r} |Q(\pi^*,r,\pi)| \cdot \frac{1}{2} \\
&= \frac{1}{(n-1)!} \cdot \frac{n!}{2} \cdot \frac{1}{n!} \sum_{\pi,r} |Q(\pi^*,r,\pi)| \\
&= \frac{n}{2}(\varphi_{m_1}(G) + \varphi_{m_2}(G)).
\end{aligned}
$$

We prove that this bound is asymptotically tight. To do so, we need only show that there exists at least one game where manipulators achieve the proposed bound. Consider a WVG $G = [2n-3; 2, \ldots, 2, 1, 1]$ of n agents, and having two manipulators, say, m_1 and m_2, each with weight 1 in the game. m_1 is pivotal for any permutation of agents in G if and only if it appears at the $(n-1)$-th position and immediately followed by m_2 at the last position. Observe that the sum of the weights of all the agents before the $(n-1)$-th position is $2n-4$, which is less than the desired quota of the game. Thus, there are $(n-2)!$ ways to arrange the non-manipulating agents in G such that m_1 is pivotal at the $(n-1)$-th position. By the same argument, m_2 is pivotal for the same number of permutations. Hence, $\varphi_{m_1}(G) + \varphi_{m_2}(G) = \frac{2(n-2)!}{n!}$. Suppose now that m_1 and m_2 merge their weights to form a bloc, $\&S$, of weight 2, resulting in the game, $G' = [2n-3; 2, \ldots, 2]$ of $n-1$ agents. Clearly, this game is *unanimity*[2], and requires all agents in G' to form a winning coalition. Thus, $\varphi_i(G') = \frac{1}{n-1}$ for all agents i in game G'. Finally, $\varphi_{\&S}(G') = \frac{1}{n-1} = \frac{1}{n-1} \cdot \frac{n!}{2(n-2)!} \cdot \frac{2(n-2)!}{n!} = \frac{n}{2}(\varphi_{m_1}(G) + \varphi_{m_2}(G))$. □

Theorem 2. *(Lower Bound). Let $G = [q; w_1, \ldots, w_n]$ be a WVG of n agents. If two manipulators, m_1 and m_2, merge their weights to form a bloc, $\&S$, in an altered game G', then, the Shapley-Shubik power, $\varphi_{\&S}(G')$, of the bloc in the new game, $\varphi_{\&S}(G') \geq \frac{n}{2(n-1)}(\varphi_{m_1}(G) + \varphi_{m_2}(G))$. Moreover, this bound is asymptotically tight.*

Proof. Let $S \subset I$ be a coalition of two distinct manipulators, m_1 and m_2, from the original game G that would like to merge into a bloc $\&S$ in an altered game G'. Let $\Pi_{G'}$ be the set of all permutations of the $n-1$ agents (including the merge bloc) in game G'. Also, let $\Pi^*_{G'}$ be the set of all permutations in game G' such that the bloc $\&S$ is pivotal for the permutations. Note that $\Pi^*_{G'} \subseteq \Pi_{G'}$. Finally, let Π^*_G, $Q(\pi^*, r, \pi)$, and $f(\pi, r)$ be as defined in Theorem 1.

We first bound $|Q(\pi^*, r, \pi)|$ for any π and $1 \leq r \leq (n-1)$. There are ${}^2P_2 = 2$ permutations[3] in $Q(\pi^*, r, \pi)$ such that m_1 and m_2 appear at the r-th position of π. There are ${}^2P_1 \cdot {}^{(n-2)}P_1 = 2(n-2)$ permutations in $Q(\pi^*, r, \pi)$ such that one of the two manipulators appears at the r-th position of π and the other is elsewhere in π. In all, $|Q(\pi^*, r, \pi)| \leq 2 + 2(n-2) \leq 2(n-1)$. Again, as in Theorem 1, all the permutations π^* in the set $Q(\pi^*, r, \pi)$ derived from a permutation $\pi \in \Pi_{G-2}$ in which at least one of the manipulators appears at the r-th position of π and is pivotal for π^* corresponds to a single permutation $f(\pi, r)$ in game G'. Thus, it is clear that $|\Pi^*_{G'}| \geq \frac{1}{|Q(\pi^*,r,\pi)|} \cdot |\Pi^*_G|$. Hence,

$$
\begin{aligned}
\frac{|\Pi^*_{G'}|}{(n-1)!} &\geq \frac{1}{|Q(\pi^*,r,\pi)|} \cdot \frac{1}{(n-1)!} \cdot |\Pi^*_G| \\
\varphi_{\&S}(G') &= \frac{1}{2(n-1)} \cdot \frac{n!}{(n-1)!} \cdot \frac{|\Pi^*_G|}{n!} \\
&= \frac{n}{2(n-1)} \cdot \frac{|\Pi^*_G|}{n!} \\
&= \frac{n}{2(n-1)}(\varphi_{m_1}(G) + \varphi_{m_2}(G)).
\end{aligned}
$$

We prove that this bound is asymptotically tight. Consider a WVG $G = [n; 1, 1, \ldots, 1]$ of n agents, and having two manipulators, say, m_1 and m_2, in the game. Clearly, this game is unanimity, and requires all agents in game G to form a winning coalition. Thus, $\varphi_i(G) = \frac{1}{n}$ for all agents i in game G, and in particular, $\varphi_{m_1}(G) + \varphi_{m_2}(G) = \frac{2}{n}$. Suppose the manipulators merge their weights to form a bloc, $\&S$, of weight 2, resulting in the game, $G' = [n; 2, 1, \ldots, 1]$ of $n-1$ agents. The game remains unanimity, and $\varphi_{\&S}(G') = \frac{1}{n-1} = \frac{1}{n-1} \cdot \frac{n}{2} \cdot \frac{2}{n} = \frac{n}{2(n-1)}(\varphi_{m_1}(G) + \varphi_{m_2}(G))$. □

6 Related Work

Many international economic organizations have employed WVGs in decision making. See Section 1 for some examples. While applying WVGs, it is found that the distribution of power of players is not directly related to their weights. Thus, there is the need to fairly assign power to players in WVGs. Power indices have been developed to measure the strength of each player in a WVG. The *Shapley value* (Shapley 1953) and its variant, the *Shapley-Shubik* index (Shapley and Shubik 1954), are the prominent solution concepts for computing agents' power in WVGs.

Since the general problem of computing the Shapley value in WVGs is known to be #P-complete, researchers have concentrated on such issues as *pseudo-polynomial algorithms* (Matsui and Matsui 2000), efficient *approximation algorithms* (Bachrach et al. 2010), or even considering special cases of the general problem that can be solved efficiently as done by the work of Fatima, Wooldridge, and Jennings (2007), that we extend in the first part of this paper.

On the other hand, WVGs are vulnerable to various forms of dishonest behaviors, referred to as *manipulations*, by strategic players that may be present in the games. Prominent among these forms of dishonest behaviors are manipulations by *splitting* and *merging* (Bachrach and Elkind 2008; Aziz et al. 2011; Lasisi and Allan 2012; 2013; 2014). Unlike in merging, however, where two or more strategic agents merge their weights to form a single bloc, manipulation by

[2] A WVG is unanimity if there is a single winning coalition in the game, and every agent in the game is critical to the coalition.

[3] $P(n,r) = \frac{n!}{(n-r)!}$.

splitting involves a strategic agent splitting its weight among two or more false agents in anticipation of gaining more power. It is important to note that, even though these two forms of manipulation have received attention of many researchers for the cases when the number, k, of strategic agents involved in the manipulation is either $k = 2$ or $k > 2$, none of these works has considered the bounds on the extent of power that strategic agents may gain when they merge their weights in non-unanimity WVGs that we consider in this paper. Table 1 provides a summary of the state of the arts on the bounds for the two forms of manipulation in WVGs.

Merging	Lower Bound	Upper Bound
$k = 2$	This paper	This paper
$k > 2$	?	?
Splitting	Lower Bound	Upper Bound
$k = 2$	Bachrach & Elkind '08	Bachrach & Elkind '08
$k > 2$	Lasisi & Allan '14	Lasisi & Allan '14

Table 1: Summary of bounds for manipulations in WVGs

7 Conclusions

We investigate two closely related and important problems in WVGs. In the first part of the paper, we present a simple and correct formula for computing the Shapley values of *large* players in a restricted class of WVG referred to as *single large party*. Our result here clarifies the computation of the Shapley values of players in this class of games as opposed to previously incorrect results for the game.

Furthermore, in the second part of the paper, we consider manipulation by merging in WVGs. Manipulation by merging in WVGs refers to a dishonest behavior where two or more strategic agents merge their weights to form a single bloc. We propose two new and non-trivial bounds for this problem. The two bounds we propose are also shown to be asymptotically tight, i.e., there exists at least a game where strategic agents achieve the proposed bounds when they engage in such manipulation. The manipulation we consider in this research is natural, and has practical applications, that motivate interests from both the game theory and artificial intelligence communities. The proposed results in the research fit under the models of deception and fraud, as well as models and mechanisms for establishing identities, which are crucial for maintaining trustworthy interactions.

We have considered the case when the number, k, of strategic agents in a WVG is 2. As demonstrated in Examples 3 and 4, it is also possible for the number of strategic agents to be more than 2. Thus, it will be interesting to see non-trivial upper and lower bounds for this problem for the case, $k > 2$, using the Shapley-Shubik power index. Another interesting future work is to extend these new bounds to the *Banzhaf* index, which is another prominent power index in WVGs. Finally, developing methods to reduce the effects of this problem in WVGs is an interesting research problem.

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References

Alonso-Meijide, J. M., and Bowles, C. 2005. Generating functions for coalitional power indices:An application to the IMF. *Annals of Operations Research* 137:21–44.

Aziz, H.; Bachrach, Y.; Elkind, E.; and Paterson, M. 2011. False-name manipulations in weighted voting games. *Journal of Artificial Intelligence Research* 40:57–93.

Bachrach, Y., and Elkind, E. 2008. Divide and conquer: False-name manipulations in weighted voting games. In *7th AAMAS Conference*, 975–982.

Bachrach, Y.; Markakis, E.; Procaccia, A. D.; Rosenschein, J. S.; and Saberi, A. 2010. Approximating power indices - theoretical and empirical analysis. *Autonomous Agents and Multiagent Systems* 20(2):105–122.

Chalkiadakis, G.; Elkind, E.; and Wooldridge, M. 2012. *Computational Aspects of Cooperative Game Theory.*

Deng, X., and Papadimitriou, C. H. 1994. On the complexity of cooperative solution concepts. *Mathematics of Operations Research* 19(2):257–266.

Fatima, S. S.; Wooldridge, M.; and Jennings, N. R. 2007. A randomized method for the shapely value for the voting game. In *6th AAMAS Conference*, 955–962.

Lasisi, R. O., and Allan, V. H. 2011. Annexations and merging in weighted voting games - The extent of susceptibility of power indices. In *3rd ICAART Conference*, 124–133.

Lasisi, R. O., and Allan, V. 2012. A search-based approach to annexation and merging in weighted voting games. In *4th Intl. Conf. on Agents and Artificial Intelligence*, 44–53.

Lasisi, R. O., and Allan, V. H. 2013. Experimental evaluation of the effects of manipulation by merging in weighted voting games. In *5th ICAART Conference*, 196–203.

Lasisi, R. O., and Allan, V. H. 2014. New bounds on false-name manipulation in weighted voting games. In *27th FLAIRS Conference*, 57–62.

Leech, D. 2002. Voting power in the governance of the IMF. *Annals of Operations Research* 109(1):375–397.

Machover, M., and Felsenthal, D. S. 2002. Annexation and alliances: When are blocs advantageous a priori? *Social Choice and Welfare* 19(2):295–312.

Matsui, T., and Matsui, Y. 2000. A survey of algorithms for calculating power indices of weighted majority games. *Journal of the Operations Research Society of Japan* 43(1).

Nordmann, L., and Pham, H. 1999. Weighted voting systems. *IEEE Transactions on Reliability* 48(1):42–49.

Shapley, L. S., and Shubik, M. 1954. A method for evaluating the distribution of power in a committee system. *American Political Science Review* 48:787–792.

Shapley, L. S. 1953. A value for n-person games. *In Kuhn, H. W., and Tucker, A. W. (Eds.), Contributions to the Theory of Games* II:pp. 307–317.

Taylor, A. D., and Zwicker, W. S. 1999. *Simple Games: Desirability Relations, Trading, Pseudoweightings, first edition ed.* Princeton University Press.