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On the Application of the Disjunctive Syllogism in Paraconsistent Logics Based on Four States of Information

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Abstract

We identify three classes of four-state paraconsistent logics according to their different approaches towards the disjunctive syllogism, and investigate three representatives of these approaches: Quasi-classical logic, which always accepts this principle, Belnap's logic, that rejects the disjunctive syllogism altogether, and a logic of inconsistency minimization that restricts its application to consistent fragments only. These logics are defined in a syntactic and a semantic style, which are linked by a simple transformation. It is shown that the three formalisms accommodate knowledge minimization, and that the most liberal formalism towards the disjunctive syllogism is also the strongest among the three, while the most cautious logic is the weakest one.

Introduction

Paraconsistent ('inconsistency-tolerant') logics are formalisms that reason with inconsistent information in a controlled and discriminating way. Thus, while classical logic, intuitionistic logic, and many other standard logics are 'explosive' in the sense that they admit the inference of any conclusion from an inconsistent set of premises, paraconsistent logics avoid this approach. In order to do so, at least one of the following two classically valid rules has to be rejected:

The Disjunctive Syllogism: from ψ *and* $\neg\psi \vee \phi$ *infer* ϕ*.*

The Introduction of Disjunction: from ψ *infer* $\psi \vee \phi$*.*

Formalisms that validate both of the rules above are necessarily explosive, as ψ entails $\psi \vee \phi$ and this together with $\neg\psi$ implies ϕ, thus the inconsistent set of assumptions $\{\psi, \neg\psi\}$ yields, in the presence of the two rules above, a derivation of an arbitrary ϕ.

In this paper, we consider different approaches of weakening the rules above in order to gain paraconsistency. We do so in a framework that is based on *four* possible states of information regarding the truth of an assertion ψ: in addition to the two standard states, in which ψ is either accepted or rejected, two additional states are allowed, reflecting the two extreme cases of uncertainty about ψ: either there is no sufficient information to determine its truth, or there are conflicting indications, causing a contradictory information about it.

In the sequel, we divide four-state paraconsistent logics by the way they apply the disjunctive syllogism (and so, indirectly, by their attitude to the introduction of disjunction). Each approach is demonstrated by a different logic, representing a particular attitude for tolerating inconsistency: The basic logic (known as Belnap's four-valued logic (Belnap 1977a; 1977b)), in which the propositional connectives have their standard lattice-theoretic interpretations, invalidates the disjunctive syllogism altogether. On other extreme, this principle is a cornerstone behind quasi-classical logic (Besnard and Hunter 1995), which is another formalism that is considered here. In between, we have intermediate approaches for cautious applications of the disjunctive syllogism. This class is represented by the logic of inconsistency minimization (Arieli and Avron 1996), according to which the disjunctive syllogism is allowed only with respect to consistent fragments of the premises.

Each one of the logics mentioned above is presented in a syntactic and a semantic style, which are linked by a simple transformation. We also show that knowledge minimization, used for reducing the amount of the relevant models without affecting the inferences, is supported by all the three logics. Interestingly, Belnap's logic, which is the most cautious in its attitude towards the disjunctive syllogism, is also the weakest logic among the three formalisms, and quasi-classical logic, which is the most liberal one, is the strongest formalism.

Consequence Relations Based on Four States of Information

Reasoning with four states of information is extensively studied in computer science and AI. Some examples for its application are in the context of symbolic model checking (Chechik et al. 2003), semantics of logic programs (Fitting 2002), natural language processing (Nelken and Francez 2002), and, of-course, inconsistency-tolerant systems. In the context of KR the latter has been investigated, e.g., in the framework of description logic (Ma, Lin, and Lin 2006), default logic (Yue, Ma, and Lin 2006), belief revision operators (Gabbay, Rodrigues, and Russo 2000), and by circumscriptive-like formulas (Arieli and Denecker

2003) or quantified Boolean formulas (Arieli 2007).

The basic idea behind our framework is to acknowledge four states of information for a given assertion: either the assertion is known to be true, known to be false, no information about it is available, or there are contradictory indications about its truth. Two common ways of representing those states are by incorporating a related (meta-) language for describing the reasoner's information (we call these approaches *syntactical*, as they do not involve truth functions), or by using truth values for the same purpose (such approaches are called *semantical*). In this section we define these approaches and describe an obvious link between them by a simple transformation.

A Syntactic Approach

Let $\mathcal{L}$ be a propositional language consisting of an alphabet $\mathcal{A}$ of propositional variables (atomic formulas), and the logical symbols $\neg, \wedge, \vee$. We denote the elements in $\mathcal{A}$ by p, q, r, literals (i.e., atomic formulas or their negations) by l_i ($i = 1, 2, \ldots$), clauses ($\vee$-disjunction of literals) by C_i ($i = 1, 2, \ldots$), and formulas in a conjunctive normal form (CNF; $\wedge$-conjunction of clauses) by ψ, ϕ. The set of CNF-formulas of $\mathcal{L}$ is denoted $\mathcal{N}$. Theories are sets of CNF-formulas, and are denoted by Γ, Δ.[1] The set of all atoms occurring in a formula ψ is denoted by $\mathcal{A}(\psi)$ and the set of all atoms occurring in a theory Γ is denoted by $\mathcal{A}(\Gamma)$ (that is, $\mathcal{A}(\Gamma) = \cup_{\psi \in \Gamma} \mathcal{A}(\psi)$). A *signed alphabet* $\mathcal{A}^{\pm}$ is a set that consists of symbols p^+, p^- for each atom $p \in \mathcal{A}$.

An *information state* I is an element in $\mathcal{P}(\mathcal{A}^{\pm})$, the power-set of $\mathcal{A}^{\pm}$. Intuitively, $p^+ \in I$ means that in I there is a reason for accepting p, and $p^- \in I$ means that in I there is a reason for accepting $\neg p$. Clearly, an information state may contain both of p^+ and p^-, or neither of them. The former situation reflects inconsistent data about p and the latter situation corresponds to incomplete data about it. This is formalized in the following definition, which also extends atomic information to CNF-formulas and theories:

Definition 1 Denote by $\Vert\!\!=_4$ the binary relation on $\mathcal{P}(\mathcal{A}^{\pm}) \times \mathcal{N}$, inductively defined as follows:

$I \Vert\!\!=_4 p$	if $p^+ \in I$,
$I \Vert\!\!=_4 \neg p$	if $p^- \in I$,
$I \Vert\!\!=_4 l_1 \vee \ldots \vee l_n$	if $I \Vert\!\!=_4 l_i$ for some $1 \leq i \leq n$,
$I \Vert\!\!=_4 C_1 \wedge \ldots \wedge C_n$	if $I \Vert\!\!=_4 C_i$ for every $1 \leq i \leq n$.

Given an information state I and a theory Γ, we denote $I \Vert\!\!=_4 \Gamma$ if $I \Vert\!\!=_4 \psi$ for every $\psi \in \Gamma$.

Definition 1 induces a corresponding relation on $\mathcal{P}(\mathcal{N}) \times \mathcal{N}$.

Notation 2 Given a theory Γ and a formula $\psi \in \mathcal{N}$, we denote:

$Mod_{\Vert\!=_4}(\Gamma) = \{I \in \mathcal{P}(\mathcal{A}^{\pm}) \mid I \Vert\!\!=_4 \Gamma\}$.

$\Gamma \Vdash_4 \psi$ if $Mod_{\Vert\!=_4}(\Gamma) \subseteq Mod_{\Vert\!=_4}(\psi)$.

In what follows we shall write $\Gamma, \phi \Vdash_4 \psi$ for abbreviating $\Gamma \cup \{\phi\} \Vdash_4 \psi$.

Proposition 3 $\Vdash_4$ *is a Tarskian consequence relation (Tarski 1941), i.e., it has the following properties, for every theory* Γ *and formulas* $\psi, \phi \in \mathcal{N}$.

Reflexivity: $\Gamma, \psi \Vdash_4 \psi$.
Monotonicity: if $\Gamma \Vdash_4 \psi$ *and* $\Gamma \subseteq \Gamma'$*, then* $\Gamma' \Vdash_4 \psi$.
Transitivity: if $\Gamma \Vdash_4 \phi$ *and* $\Gamma', \phi \Vdash_4 \psi$*, then* $\Gamma, \Gamma' \Vdash_4 \psi$.

Proof. Reflexivity immediately follows from Notation 2. Monotonicity follows from the fact that if $\Gamma \subseteq \Gamma'$ then $Mod_{\Vert\!=_4}(\Gamma') \subseteq Mod_{\Vert\!=_4}(\Gamma)$. For transitivity, suppose that $\Gamma \Vdash_4 \phi$ and $\Gamma', \phi \Vdash_4 \psi$, and let $I \in Mod_{\Vert\!=_4}(\Gamma \cup \Gamma')$. Since $I \in Mod_{\Vert\!=_4}(\Gamma)$ and $\Gamma \Vdash_4 \phi$, $I \in Mod_{\Vert\!=_4}(\{\phi\})$. Thus, as $I \in Mod_{\Vert\!=_4}(\Gamma')$, we have that $I \in Mod_{\Vert\!=_4}(\Gamma' \cup \{\phi\})$. Now, since $\Gamma, \phi \Vdash_4 \psi$, $I \in Mod_{\Vert\!=_4}(\{\psi\})$, and so $\Gamma, \Gamma' \Vdash_4 \psi$. □

Proposition 4 $\Vdash_4$ *is paraconsistent: there are* $\psi, \phi \in \mathcal{N}$ *such that* $\psi, \neg\psi \nVdash_4 \phi$.

Proof. Let $I = \{p^+, p^-\}$ for some $p \in \mathcal{A}$. For every $q \in \mathcal{A} \setminus \{p\}$ we have that $I \Vert\!\!=_4 p$ and $I \Vert\!\!=_4 \neg p$ but $I \not\Vert\!\!=_4 q$, thus $p, \neg p \nVdash_4 q$. □

A Semantic Counterpart

The paraconsistent consequence relation $\Vdash_4$ considered in the previous section is constructed in a syntactical manner, without using truth values. Yet, it is evident (e.g., by the intuitive meaning of information states) that it may be associated with a four-valued semantics. The corresponding four-valued algebraic structure, denoted $\mathcal{FOUR}$, was introduced by Belnap (1977a; 1977b) (see Figure 1). This structure is composed of four elements $\{t, f, \bot, \top\}$. Intuitively, t and f have their usual classical meanings, $\bot$ represents incomplete information and $\top$ represents inconsistent information. These values are arranged in two lattice structures. One, represented along the horizontal axis of Figure 1, is the standard logical partial order, $\leq_t$, which intuitively reflects differences in the 'measure of truth' that every value represents. We shall denote by $\wedge$ and by $\vee$ the meet and the join (respectively) of the lattice obtained by $\leq_t$, and by $\neg$ its order reversing involution, for which $\neg\top = \top$ and $\neg\bot = \bot$.

The other partial order of $\mathcal{FOUR}$, denoted $\leq_k$, is understood (again, intuitively) as reflecting differences in the amount of *knowledge* or *information* that each truth value exhibits. This partial order is represented along the vertical axis of Figure 1, and it will also have a great importance in what follows.

The next step in using $\mathcal{FOUR}$ for reasoning is to choose its set of *designated* elements. The obvious choice is the set $\mathcal{D} = \{t, \top\}$, since both of the values in $\mathcal{D}$ intuitively represent formulas that are 'known to be true'. The set $\mathcal{D}$ has the property that $a \wedge b \in \mathcal{D}$ iff both a and b are in $\mathcal{D}$, while $a \vee b \in \mathcal{D}$ iff at least one of a or b is in $\mathcal{D}$.

The semantical analogue of information states are *valuations*, which are functions that assign a truth value to each atomic formula. For a valuation ν we shall sometimes denote $\nu = \{p_i : x_i \mid i = 1, 2, \ldots\}$ instead of $\nu(p_i) = x_i$. The space of valuations is denoted Λ.

[1] The concentration on (sets of) CNF-formulas is justified by the fact that, just as in the classical semantics, in the four-valued semantics for $\mathcal{L}$, every well-formed formula is logically equivalent to a CNF-formula (see (Arieli and Avron 1996)).

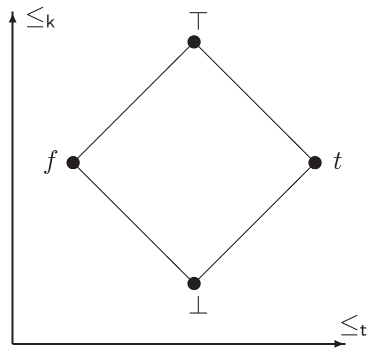

Figure 1: $\mathcal{FOUR}$

The following definition and notation should be compared with Definition 1 and Notation 2, respectively.

Definition 5 Denote by $\models_4$ the binary relation on $\Lambda \times \mathcal{N}$, inductively defined as follows:

$\nu \models_4 p$ if $\nu(p) \in \mathcal{D}$,
$\nu \models_4 \neg p$ if $\neg\nu(p) \in \mathcal{D}$,
$\nu \models_4 l_1 \vee \ldots \vee l_n$ if $\nu \models_4 l_i$ for some $1 \le i \le n$,
$\nu \models_4 C_1 \wedge \ldots \wedge C_n$ if $\nu \models_4 C_i$ for every $1 \le i \le n$.

For a valuation ν and a theory Γ, we denote $\nu \models_4 \Gamma$ if $\nu \models_4 \psi$ for every $\psi \in \Gamma$.

Notation 6 Given a theory Γ and a formula $\psi \in \mathcal{N}$, we denote:

$Mod_{\models_4}(\Gamma) = \{\nu \in \Lambda \mid \nu \models_4 \Gamma\}$.
$\Gamma \vdash_4 \psi$ if $Mod_{\models_4}(\Gamma) \subseteq Mod_{\models_4}(\psi)$.

Proposition 7 *$\vdash_4$ is a paraconsistent Tarskian consequence relation.*

Proof. A direct proof is obtained from the proofs of Propositions 3 and 4, replacing information states by valuations and $Mod_{\parallel\!\models_4}(\Gamma)$ by $Mod_{\models_4}(\Gamma)$. This proposition also follows as a corollary of Propositions 3, 4, by the correspondence between $\Vdash_4$ and $\vdash_4$, shown in Proposition 10 below. □

Relating the Two Approaches

The syntactic and the semantic approaches described above can be related by the following transformation:

Definition 8 For an information state $I \in \mathcal{P}(\mathcal{A}^{\pm})$, the valuation $\sigma(I) \in \Lambda$ is defined for every $p \in \mathcal{A}$ as follows:

$$\sigma(I)(p) = \begin{cases} t & \text{if } p^+ \in I \text{ and } p^- \notin I, \\ f & \text{if } p^+ \notin I \text{ and } p^- \in I, \\ \bot & \text{if } p^+ \notin I \text{ and } p^- \notin I, \\ \top & \text{if } p^+ \in I \text{ and } p^- \in I. \end{cases}$$

Clearly, σ is a one-to-one mapping from the space $\mathcal{P}(\mathcal{A}^{\pm})$ of the information states onto the space Λ of the four-valued valuations. Given a valuation $\nu \in \Lambda$, we shall denote by $\sigma^{-1}(\nu)$ the information state I for which $\sigma(I) = \nu$.

Lemma 9 *It holds that $I \parallel\!\models_4 \Gamma$ iff $\sigma(I) \models_4 \Gamma$, and $\nu \models_4 \Gamma$ iff $\sigma^{-1}(\nu) \parallel\!\models_4 \Gamma$.*

Proof. By induction on the structure of a CNF-formula ψ, it is easily verified that $I \parallel\!\models_4 \psi$ iff $\sigma(I)(\psi) \in \mathcal{D}$, and so $I \parallel\!\models_4 \psi$ iff $\sigma(I) \models_4 \psi$. This implies that $I \parallel\!\models_4 \Gamma$ iff $\sigma(I) \models_4 \Gamma$. The second part is similar. □

Proposition 10 *For every theory Γ and formula $\psi \in \mathcal{N}$, $\Gamma \Vdash_4 \psi$ iff $\Gamma \vdash_4 \psi$.*

Proof. If $\Gamma \nvdash_4 \psi$ then $\nu \in Mod_{\models_4}(\Gamma) \setminus Mod_{\models_4}(\psi)$ for some $\nu \in \Lambda$. By Lemma 9, $Mod_{\parallel\!\models_4}(\Gamma) = \{\sigma^{-1}(\nu) \mid \nu \in Mod_{\models_4}(\Gamma)\}$, thus $\sigma^{-1}(\nu) \in Mod_{\parallel\!\models_4}(\Gamma) \setminus Mod_{\parallel\!\models_4}(\psi)$. It follows that $\Gamma \nVdash_4 \psi$. The proof of the converse is similar, using the fact that by Lemma 9, $Mod_{\models_4}(\Gamma) = \{\sigma(I) \mid I \in Mod_{\parallel\!\models_4}(\Gamma)\}$. □

Proposition 10 shows that the two approaches above actually coincide for CNF-formulas. This will be useful in what follows for showing correspondence among the different derivatives of these consequence relations.

Note 11 For relating $\vdash_4$ and $\Vdash_4$ with respect to arbitrary sets of formulas in $\mathcal{L}$ (not necessarily in a conjunctive normal form), some generalizations of the basic definitions are required. In the semantic approach this can be done rather directly by extending Definition 5 so that for *every* $\mathcal{L}$-formulas ψ, ϕ,

$\nu \models_4 \neg\psi$ if $\neg\nu(\psi) \in \mathcal{D}$,
$\nu \models_4 \psi \vee \phi$ if $\nu \models_4 \psi$ or $\nu \models_4 \phi$,
$\nu \models_4 \psi \wedge \phi$ if $\nu \models_4 \psi$ and $\nu \models_4 \phi$.

The consequence relation that is obtained by extending Definition 5 to this satisfiability relation coincides with Belnap's well-known four-valued logic (Belnap 1977a; 1977b) and has strong ties to 'first-degree entailments' in relevance logic (Anderson and Belnap 1975; Dunn and Restall 2002). Indeed, it holds that $\psi_1, \ldots, \psi_n \vdash_4 \phi$ iff $\psi_1 \wedge \ldots \wedge \psi_n \to \phi$ is a first-degree entailment.

Adapting the syntactic approach to general formulas is somewhat more cumbersome, as extra conditions have to be imposed on the satisfaction relation for assuring the well-behaving of the induced entailment with respect to general formulas (e.g., the double-negation rule, associating $\neg\neg\psi$ with ψ, has to be explicitly required; see, e.g., (Hunter 2000, Section 5)).

Quasi-Classical Logic

By their definition, paraconsistent logics are weaker than classical logic with respect to inconsistent sets of premises. However, the consequence relations considered in the previous section seem to be too weak, as they exclude some classically valid rules even when the premises are classically consistent. Thus, for instance, both of $\vdash_4$ and $\Vdash_4$ reject the disjunctive syllogism in *any* circumstance, as q is not $\vdash_4$-deducible (and, by Proposition 10, it is not $\Vdash_4$-deducible either) from $\{p, \neg p \vee q\}$.[2] In order to overcome this shortcoming, Besnard and Hunter (1995; 2000) introduced an alternative formalism, called *quasi-classical logic*, in which the satisfaction relation of the syntactic-based approach (Definition 1) is strengthen as follows:

[2]Consider a valuation that assigns $\top$ to p and f to q.

Definition 12 Let $\sim l$ be the complement of a literal l (i.e., $\sim l = \neg p$ if $l = p$, and $\sim l = p$ if $l = \neg p$). Denote by $\Vert\!\models_{\mathsf{QC}}$ the binary relation on $\mathcal{P}(\mathcal{A}^{\pm}) \times \mathcal{N}$, defined just as the relation $\Vert\!\models_4$ of Definition 1, with the only exception that $I \Vert\!\models_{\mathsf{QC}} l_1 \vee \ldots \vee l_n$ if the following two conditions hold:

- $I \Vert\!\models_4 l_1 \vee \ldots \vee l_n$, and
- for every $1 \le i \le n$, if $I \Vert\!\models_{\mathsf{QC}} \sim l_i$ then $I \Vert\!\models_{\mathsf{QC}} l_1 \vee \ldots l_{i-1} \vee l_{i+1} \vee \ldots \vee l_n$.[3]

Example 13 Let $I_1 = \{q^+\}$ and $I_2 = \{p^+, p^-\}$. Then $I_1 \Vert\!\models_{\mathsf{QC}} p \vee q$ and $I_1 \Vert\!\models_{\mathsf{QC}} \neg p \vee q$. On the other hand, $I_2 \Vert\!\not\models_{\mathsf{QC}} q$, and so $I_2 \Vert\!\not\models_{\mathsf{QC}} p \vee q$ and $I_2 \Vert\!\not\models_{\mathsf{QC}} \neg p \vee q$.

Note that according to Definition 12, the disjunction $\vee$ cease to have its usual lattice-based meaning. The idea behind this definition is, generally speaking, to link between a disjunct and its complement, and by this to preserve the meaning of the resolution principle. This gives rise to the following definition of entailments in the quasi-classical logic.

Notation 14 Given a theory Γ and a formula $\psi \in \mathcal{N}$, we denote:

$Mod_{\Vert\models_{\mathsf{QC}}}(\Gamma) = \{I \in \mathcal{P}(\mathcal{A}^{\pm}) \mid I \Vert\!\models_{\mathsf{QC}} \Gamma\}$.
$\Gamma \Vdash_{\mathsf{QC}} \psi$ iff $Mod_{\Vert\models_{\mathsf{QC}}}(\Gamma) \subseteq Mod_{\Vert\models_4}(\psi)$.

Example 15 For $\Gamma = \{p \vee q, \neg p \vee q\}$, $Mod_{\Vert\models_{\mathsf{QC}}}(\Gamma) = \{\{p^+, p^-, q^+\}, \{p^+, q^+\}, \{p^-, q^+\}, \{q^+\}\}$, thus $\Gamma \Vdash_{\mathsf{QC}} q$. Note that $\Gamma \not\Vdash_4 q$, since $\{p^+, p^-\}$ is in $Mod_{\Vert\models_4}(\Gamma)$ but not in $Mod_{\Vert\models_4}(q)$.

Proposition 16 *$\Vdash_{\mathsf{QC}}$ is paraconsistent.*

Proof. It holds that $\{p^+, p^-\} \in Mod_{\Vert\models_{\mathsf{QC}}}(\{p, \neg p\})$ but $\{p^+, p^-\} \notin Mod_{\Vert\models_4}(\{q\})$, therefore $p, \neg p \not\Vdash_{\mathsf{QC}} q$. □

Note that, as demonstrated in Example 15, unlike $\Vdash_4$, the entailment $\Vdash_{\mathsf{QC}}$ admits the principle of resolution, so for arbitrary clauses C_1, C_2, C_3, C_4 we have:

$$\big\{C_1 \vee l \vee C_2 \,,\ C_3 \vee \sim l \vee C_4\big\} \ \Vdash_{\mathsf{QC}}\ C_1 \vee C_2 \vee C_3 \vee C_4.$$

The clause $C_1 \vee C_2 \vee C_3 \vee C_4$ is called a *resolvent* of $C_1 \vee l \vee C_2$ and $C_3 \vee \sim l \vee C_4$. By the monotonicity of quasi-classical logic we have, then, the next result.

Proposition 17 *Any resolvent of two clauses in a theory Γ is $\Vdash_{\mathsf{QC}}$-deducible from Γ.*

The 'price' of this is that $\Vdash_{\mathsf{QC}}$ is not a consequence relation, since (although it is reflexive and monotonic) it is not transitive.[4] Thus, proofs cannot be composed for more complicated proofs, and so inference in quasi-classical logic is not an iterative process, but rather a one-step procedure. We refer to (Hunter 2000) for a more detailed discussion on the properties of $\Vdash_{\mathsf{QC}}$ and a corresponding sound and complete proof theory.

By the next lemma, shown in (Hunter 2000), we can give a semantic counterpart to $\Vdash_{\mathsf{QC}}$.

[3] In (Besnard and Hunter 1995; Hunter 2000) the relations $\Vert\!\models_{\mathsf{QC}}$ and $\Vert\!\models_4$ are called, respectively, strong satisfiability and weak satisfiability.

[4] Indeed, $p \Vdash_{\mathsf{QC}} p \vee q$ and $\neg p, p \vee q \Vdash_{\mathsf{QC}} q$, but $p, \neg p \not\Vdash_{\mathsf{QC}} q$.

Lemma 18 *$I \Vert\!\models_{\mathsf{QC}} l_1 \vee \ldots \vee l_n$ iff*

- *there is $1 \le i \le n$, such that $l_i^+ \in I$ and $l_i^- \notin I$, or*
- *for all $1 \le i \le n$, $l_i^+ \in I$ and $l_i^- \in I$.*

Definition 19 Denote by $\models_{\mathsf{QC}}$ the binary relation on $\Lambda \times \mathcal{N}$, inductively defined as follows:

$\nu \models_{\mathsf{QC}} p$	if $\nu(p) \in \mathcal{D}$
$\nu \models_{\mathsf{QC}} \neg p$	if $\neg\nu(p) \in \mathcal{D}$
$\nu \models_{\mathsf{QC}} l_1 \vee \ldots \vee l_n$	if $\nu(l_i) = t$ for some $1 \le i \le n$ or $\nu(l_i) = \top$ for all $1 \le i \le n$
$\nu \models_{\mathsf{QC}} C_1 \wedge \ldots \wedge C_n$	if $\nu \models_{\mathsf{QC}} C_i$ for all $1 \le i \le n$

Given a valuation ν and a theory Γ, we denote $\nu \models_{\mathsf{QC}} \Gamma$ if $\nu \models_{\mathsf{QC}} \psi$ for every $\psi \in \Gamma$.

Notation 20 Given a theory Γ and a formula $\psi \in \mathcal{N}$, we denote:

$Mod_{\models_{\mathsf{QC}}}(\Gamma) = \{\nu \in \Lambda \mid \nu \models_{\mathsf{QC}} \Gamma\}$.
$\Gamma \vdash_{\mathsf{QC}} \psi$ iff $Mod_{\models_{\mathsf{QC}}}(\Gamma) \subseteq Mod_{\models_4}(\psi)$.

Proposition 21 *For every theory Γ and a formula $\psi \in \mathcal{N}$, $\Gamma \Vdash_{\mathsf{QC}} \psi$ iff $\Gamma \vdash_{\mathsf{QC}} \psi$.*

Proof. For a literal l we have that $I \Vert\!\models_{\mathsf{QC}} l$ iff $\sigma(I)(l) \in \mathcal{D}$, thus $I \Vert\!\models_{\mathsf{QC}} l$ iff $\sigma(I) \models_{\mathsf{QC}} l$. Also, by Lemma 18, $I \Vert\!\models_{\mathsf{QC}} l_1 \vee \ldots \vee l_n$ iff $\sigma(I) = t$ for some $1 \le i \le n$, or $\sigma(I) = \top$ for every $1 \le i \le n$, iff $\sigma(I) \models_{\mathsf{QC}} l_1 \vee \ldots \vee l_n$. Thus, by induction on the structure of a CNF-formula ψ, we have that $I \Vert\!\models_{\mathsf{QC}} \psi$ iff $\sigma(I) \models_{\mathsf{QC}} \psi$. It follows that $Mod_{\Vert\models_{\mathsf{QC}}}(\Gamma) = \{\sigma^{-1}(\nu) \mid \nu \in Mod_{\models_{\mathsf{QC}}}(\Gamma)\}$ and $Mod_{\models_{\mathsf{QC}}}(\Gamma) = \{\sigma(I) \mid I \in Mod_{\Vert\models_{\mathsf{QC}}}(\Gamma)\}$. This implies (as in the proof of Proposition 10) that $\Vdash_{\mathsf{QC}}$ and $\vdash_{\mathsf{QC}}$ coincide. □

Logic of Minimal Inconsistency

As noted before, Belnap's logic excludes some classically valid rules even for classically consistent theories, and so it may be considered as too weak. Quasi classical logic, on the other hand, may be considered as too strong. This is shown in the following example.

Example 22 Consider the theory $\Gamma = \{p, \neg p, p \vee q\}$. Here, $Mod_{\models_{\mathsf{QC}}}(\Gamma) = \{\{p : \top, q : t\}, \{p : \top, q : \top\}\}$, thus $\Gamma \vdash_{\mathsf{QC}} q$. This may look counter-intuitive, as there is no justification to infer q from $p \vee q$ (the only assertion in Γ that mentions q) as long as one already 'knows' p.

The formalism that is considered in this section overcomes this problem by applying classically valid rules (in particular, the disjunctive syllogism) only to classically valid fragments of the theory (see Corollary 29 and Note 30 below). The idea here is to minimize the inconsistent states of information of the reasoner, following the recognition that although inconsistency is unavoidable, it should be minimized as much as possible.

Definition 23 For $\nu, \mu \in \Lambda$ we denote by $\nu \le_\top \mu$ that for every atom $p \in \mathcal{A}$, $\mu(p) = \top$ whenever $\nu(p) = \top$. We denote by $\nu <_\top \mu$ that $\nu \le_\top \mu$ and there is an atomic formula $p \in \mathcal{A}$, such that $\mu(p) = \top$ while $\nu(p) \neq \top$.

Notation 24 Given a theory Γ and a formula $\psi \in \mathcal{N}$, we denote:

$$Mod_{\models_{\mathsf{MI}}}(\Gamma) = \{\nu \in Mod_{\models_4}(\Gamma) \mid \text{if } \mu <_\top \nu \text{ then } \mu \notin Mod_{\models_4}(\Gamma)\}.$$

$\Gamma \vdash_{\mathsf{MI}} \psi$ iff $Mod_{\models_{\mathsf{MI}}}(\Gamma) \subseteq Mod_{\models_4}(\psi)$.

The logic $\vdash_{\mathsf{MI}}$ was introduced in (Arieli and Avron 1996; 1998) as a generalization to the four-valued case of Priest's three-valued logic LPm of inconsistency minimization (Priest 1991).

Proposition 25 $\vdash_{\mathsf{MI}}$ *is paraconsistent.*

Proof. Indeed, $\{p:\top, q:f\} \in Mod_{\models_{\mathsf{MI}}}(\{p,\neg p\})$ but $\{p:\top, q:f\} \notin Mod_{\models_4}(\{q\})$, and so $p,\neg p \not\vdash_{\mathsf{MI}} q$. □

Example 26 Consider again the theory $\Gamma = \{p, \neg p, p \vee q\}$ of Example 22. As intuitively expected (and unlike quasi-classical logic), q is *not* a $\vdash_{\mathsf{MI}}$-conclusion of Γ, as $Mod_{\models_{\mathsf{MI}}}(\Gamma) = \{\{p:\top, q:t\}, \{p:\top, q:f\}, \{p:\top, q:\bot\}\}$. On the other hand, $r, \neg r, \neg p, p \vee q \vdash_{\mathsf{MI}} q$. The latter is a particular case of the next proposition that shows that every non-tautological clause that can be inferred from a consistent fragment of a theory, can also be inferred by $\vdash_{\mathsf{MI}}$ from the whole theory, provided that the clause is independent of the inconsistent part of the theory.

Definition 27 We say that Γ' and Γ'' are *independent*, if $\mathcal{A}(\Gamma') \cap \mathcal{A}(\Gamma'') = \emptyset$. Two independent theories Γ' and Γ'' are a *partition* of Γ, if $\Gamma = \Gamma' \cup \Gamma''$.

Proposition 28 *Denote by* $\vdash$ *the standard entailment of classical logic. Let* Γ' *and* Γ'' *be a partition of* Γ. *Suppose that* Γ' *is classically consistent, and that* ψ *is a non-tautological clause*[5] *that is independent of* Γ''. *If* $\Gamma' \vdash \psi$ *then* $\Gamma \vdash_{\mathsf{MI}} \psi$.[6]

Proof. By (Arieli and Avron 1998, Proposition 64), if Γ' is classically consistent and ψ is as in the proposition, then $\Gamma' \vdash \psi$ iff $\Gamma' \vdash_{\mathsf{MI}} \psi$. It is enough to show, then, that under the conditions of the proposition, if $\Gamma' \vdash_{\mathsf{MI}} \psi$ then $\Gamma \vdash_{\mathsf{MI}} \psi$. Indeed, suppose otherwise that $\Gamma \not\vdash_{\mathsf{MI}} \psi$. Then there is a valuation $\nu \in Mod_{\models_{\mathsf{MI}}}(\Gamma) \setminus Mod_{\models_4}(\psi)$. Note that this in particular implies that $\nu \in Mod_{\models_4}(\Gamma')$ and $\nu \in Mod_{\models_4}(\Gamma'')$. Now, consider the following valuation:

$$\mu(p) = \begin{cases} \nu(p) & \text{if } p \in \mathcal{A}(\Gamma') \cup \mathcal{A}(\psi), \\ t & \text{otherwise.} \end{cases}$$

As $\mu(\phi) = \nu(\phi)$ for every formula $\phi \in \Gamma' \cup \{\psi\}$, we have that $\mu \in Mod_{\models_4}(\Gamma')$ and $\mu \notin Mod_{\models_4}(\psi)$. Now, since $\Gamma' \vdash_{\mathsf{MI}} \psi$, $\mu \notin Mod_{\models_{\mathsf{MI}}}(\Gamma')$, and so there is a valuation $\mu' \in Mod_{\models_4}(\Gamma')$ such that $\mu' <_\top \mu$. Now, consider the following valuation:

$$\nu'(p) = \begin{cases} \mu'(p) & \text{if } p \in \mathcal{A}(\Gamma') \cup \mathcal{A}(\psi), \\ \nu(p) & \text{otherwise.} \end{cases}$$

Clearly, $\nu' <_\top \nu$, and since ν' is the same as μ' on $\mathcal{A}(\Gamma')$, ν' is also a model of Γ'. Moreover, using the facts that both Γ' and ψ are independent of Γ'', it follows that $\nu' \in Mod_{\models_4}(\Gamma'')$. Thus, $\nu' \in Mod_{\models_4}(\Gamma)$ and $\nu' <_\top \nu$, which implies that $\nu \notin Mod_{\models_{\mathsf{MI}}}(\Gamma)$, but this is a contradiction to the choice of ν. □

Corollary 29 *Suppose that* Γ' *and* Γ'' *are a partition of* Γ *and that* Γ' *is classically consistent. Then any non-tautological resolvent of two clauses in* Γ' *is* $\vdash_{\mathsf{MI}}$*-deducible from* Γ.

Proof. If ψ is a resolvent of two clauses in Γ' then $\Gamma' \vdash \psi$. Also, $\mathcal{A}(\psi) \subseteq \mathcal{A}(\Gamma')$, thus ψ is independent of Γ''. By Proposition 28, then, $\Gamma \vdash_{\mathsf{MI}} \psi$. □

Note 30 As Example 26 shows, Corollary 29 does not hold in general (i.e., for resolvents of clauses in any theory), so $\vdash_{\mathsf{MI}}$ admits a restricted form of resolution (and so of the disjunctive syllogism), adhering consistency (cf. Proposition 17).

The syntactical counterpart $\Vdash_{\mathsf{MI}}$ of $\vdash_{\mathsf{MI}}$ is defined as follows:

Definition 31 For $I, J \in \mathcal{P}(\mathcal{A}^\pm)$ we denote by $I \leq_\top J$ that $\{p \in \mathcal{A} \mid p^+, p^- \in I\} \subseteq \{p \in \mathcal{A} \mid p^+, p^- \in J\}\}$. If the above containment is proper, we write $I <_\top J$.

Notation 32 Given a theory Γ and a formula $\psi \in \mathcal{N}$, we denote:

$$Mod_{\Vdash_{\mathsf{MI}}}(\Gamma) = \{I \in Mod_{\Vdash_4}(\Gamma) \mid \text{if } J <_\top I \text{ then } J \notin Mod_{\Vdash_4}(\Gamma)\}.$$

$\Gamma \Vdash_{\mathsf{MI}} \psi$ iff $Mod_{\Vdash_{\mathsf{MI}}}(\Gamma) \subseteq Mod_{\models_4}(\psi)$.

Again, it is easy to see that for information states $I, J \in \mathcal{P}(\mathcal{A}^\pm)$ it holds that $I <_\top J$ (in the sense of Definition 31) iff $\sigma(I) <_\top \sigma(J)$ (in the sense of Definition 23). Similarly, for valuations $\nu, \mu \in \Lambda$, $\nu <_\top \mu$ iff $\sigma^{-1}(\nu) <_\top \sigma^{-1}(\mu)$. Thus, we have that $Mod_{\Vdash_{\mathsf{MI}}}(\Gamma) = \{\sigma^{-1}(\nu) \mid \nu \in Mod_{\models_{\mathsf{MI}}}(\Gamma)\}$ and that $Mod_{\models_{\mathsf{MI}}}(\Gamma) = \{\sigma(I) \mid I \in Mod_{\Vdash_{\mathsf{MI}}}(\Gamma)\}$, which implies that $\Vdash_{\mathsf{MI}}$ and $\vdash_{\mathsf{MI}}$ coincide.

Knowledge Minimization

Since $Mod_{\models_{\mathsf{MI}}}(\Gamma) \subseteq Mod_{\models_4}(\Gamma)$ and $Mod_{\models_{\mathsf{QC}}}(\Gamma) \subseteq Mod_{\models_4}(\Gamma)$, the set of models needed for drawing conclusions by $\vdash_{\mathsf{MI}}$ and by $\vdash_{\mathsf{QC}}$ is never bigger than those needed for drawing conclusions by $\vdash_4$. In this section we present a natural approach for reducing the number of models even further, without changing the logic. The idea is to consider, in each logic, the $\subseteq$-minimal information states of the premises, or – equivalently – to take advantage of the knowledge order $\leq_k$ of $\mathcal{FOUR}$ and to consider the $\leq_k$-minimal valuations among the models of the premises. The intuition behind this refinement is that one should not assume anything that is not really known.

Definition 33 For $\nu, \mu \in \Lambda$ we denote by $\nu \leq_k \mu$ that for every $p \in \mathcal{A}$, $\nu(p) \leq_k \mu(p)$. We denote by $\nu <_k \mu$ that $\nu \leq_k \mu$ and there is a $p \in \mathcal{A}$ such that $\nu(p) <_k \mu(p)$.

Notation 34 Given a set $\mathcal{S} \subseteq \mathcal{P}(\mathcal{A}^\pm)$ of information states and a set $\mathcal{V} \subseteq \Lambda$ of valuations, we denote:

[5] That is, none of the disjunctions in ψ contains an atomic formula and its negation.

[6] The aspiration to preserve classically valid inferences whenever this is reasonably possible, even in the presence of inconsistency, is sometimes called 'adaptivity' (Batens 2000); Adaptive formalisms presuppose the consistency of all the assertions 'unless and until proven otherwise'.

$Min_{\subseteq}(\mathcal{S}) = \{I \in \mathcal{S} \mid$ if $J \subset I$ then $J \notin \mathcal{S}\}$.

$Min_{\leq_k}(\mathcal{V}) = \{\nu \in \mathcal{V} \mid$ if $\mu <_k \nu$ then $\mu \notin \mathcal{V}\}$.

Using Notation 34 we can now introduce a variety of derivatives of the entailments relations considered previously, adhering knowledge-minimization.

Notation 35 Given a theory Γ and a formula $\psi \in \mathcal{N}$, for every logic $\mathsf{x} \in \{4, \mathsf{QC}, \mathsf{MI}\}$ we denote:

$\Gamma \Vdash^{\subseteq}_{\mathsf{x}} \psi$ iff $Min_{\subseteq}(Mod_{\Vdash_{\mathsf{x}}}(\Gamma)) \subseteq Mod_{\Vdash_4}(\psi)$.

$\Gamma \vdash^{\leq_k}_{\mathsf{x}} \psi$ iff $Min_{\leq_k}(Mod_{\models_{\mathsf{x}}}(\Gamma)) \subseteq Mod_{\models_4}(\psi)$.

Proposition 36 *Knowledge minimization with respect to $\subseteq$ and $\leq_k$ are dual:*

a) $\Gamma \Vdash^{\subseteq}_4 \psi$ *iff* $\Gamma \vdash^{\leq_k}_4 \psi$.

b) $\Gamma \Vdash^{\subseteq}_{\mathsf{MI}} \psi$ *iff* $\Gamma \vdash^{\leq_k}_{\mathsf{MI}} \psi$.

c) $\Gamma \Vdash^{\subseteq}_{\mathsf{QC}} \psi$ *iff* $\Gamma \vdash^{\leq_k}_{\mathsf{QC}} \psi$.

Proof. We show Part (a); The proof of the other two parts is similar. For each information states I, J,

$I \subseteq J$ iff
$\forall p \in \mathcal{A}^{\pm}$ $p^+ \in I \Rightarrow p^+ \in J$ and $p^- \in I \Rightarrow p^- \in J$, iff
$\forall p \in \mathcal{A}$ $\sigma(I)(p) \leq_k \sigma(J)(p)$, iff
$\sigma(I) \leq_k \sigma(J)$.

By Lemma 9, $Mod_{\Vdash_4}(\Gamma) = \{\sigma^{-1}(\nu) \mid \nu \in Mod_{\models_4}(\Gamma)\}$ and $Mod_{\models_4}(\Gamma) = \{\sigma(I) \mid I \in Mod_{\Vdash_4}(\Gamma)\}$, thus:

$Min_{\subseteq}(Mod_{\Vdash_4}(\Gamma)) = \{\sigma^{-1}(\nu) \mid \nu \in Min_{\leq_k}(Mod_{\models_4}(\Gamma))\}$,
$Min_{\leq_k}(Mod_{\models_4}(\Gamma)) = \{\sigma(I) \mid I \in Min_{\subseteq}(Mod_{\Vdash_4}(\Gamma))\}$.

It follows, then, that:

- If $\Gamma \nVdash^{\subseteq}_4 \psi$ then there is an $I \in Min_{\subseteq}Mod_{\Vdash_4}(\Gamma)$ such that $I \notin Mod_{\Vdash_4}(\psi)$, thus $\sigma(I) \in Min_{\leq_k}Mod_{\models_4}(\Gamma)$ and $\sigma(I) \notin Mod_{\models_4}(\Gamma)$, so $\Gamma \nvdash^{\leq_k}_4 \psi$.
- If $\Gamma \nvdash^{\leq_k}_4 \psi$ then there is a $\nu \in Min_{\leq_k}Mod_{\models_4}(\Gamma)$ and $\nu \notin Mod_{\models_4}(\psi)$. Thus, $\sigma^{-1}(\nu) \in Min_{\subseteq}Mod_{\Vdash_4}(\Gamma)$ and $\sigma^{-1}(\nu) \notin Mod_{\Vdash_4}(\psi)$, so $\Gamma \nVdash^{\subseteq}_4 \psi$. □

Proposition 37 *All the logics considered above are preserved under knowledge minimization: for every theory Γ (not necessarily finite) and a formula ψ,*

a) $\Gamma \vdash_4 \psi$ *iff* $\Gamma \vdash^{\leq_k}_4 \psi$.

b) $\Gamma \vdash_{\mathsf{MI}} \psi$ *iff* $\Gamma \vdash^{\leq_k}_{\mathsf{MI}} \psi$.

c) $\Gamma \vdash_{\mathsf{QC}} \psi$ *iff* $\Gamma \vdash^{\leq_k}_{\mathsf{QC}} \psi$.

Proof. Parts (a) and (b) follow, respectively, from Corollary 32 and Proposition 59 of (Arieli and Avron 1998). For Part (c) we adjust a similar proof in (Arieli and Avron 1998) to quasi-classical logic. First, we show the following lemma:

Lemma 38 *For every $\nu \in Mod_{\models_{\mathsf{QC}}}(\Gamma)$ there is a $\mu \in Min_{\leq_k}(Mod_{\models_{\mathsf{QC}}}(\Gamma))$ such that $\mu \leq_k \nu$.*

Proof. Indeed, for a valuation $\nu \in Mod_{\models_{\mathsf{QC}}}(\Gamma)$ consider the set $S_\nu = \{\nu_i \mid \nu_i \in Mod_{\models_{\mathsf{QC}}}(\Gamma),\ \nu_i \leq_k \nu\}$. Suppose that $C \subseteq S_\nu$ is a descending chain with respect to $\leq_k$. We shall show that C is bounded in S_ν, so by Zorn's lemma S_ν has a minimal element, which is the required valuation in $Min_{\leq_k}(Mod_{\models_{\mathsf{QC}}}(\Gamma))$. Let μ be the following valuation: for every $p \in \mathcal{A}$, $\mu(p) = \min_{\leq_k}\{\nu_i(p) \mid \nu_i \in C\}$. This μ is well defined since C is a chain of four-valued valuations. Obviously, μ bounds C. It remains to show that $\mu \in S_\nu$. Suppose that $\psi \in \Gamma$ and $\mathcal{A}(\psi) = \{p_1, \dots, p_n\}$. Then: $\mu(p_1) = \nu_{i_1}(p_1), \dots, \mu(p_n) = \nu_{i_n}(p_n)$. Since C is a chain we may assume, without a loss of generality, that $\nu_{i_1} \geq_k \dots \geq_k \nu_{i_n}$, and so μ is the same as ν_{i_n} on every atom in $\mathcal{A}(\psi)$. Since $\nu_{i_n} \in Mod_{\models_{\mathsf{QC}}}(\psi)$, so is μ. This is true for every $\psi \in \Gamma$, and so $\mu \in S_\nu$ as required.

Now, back to the proof of Part (c) in Proposition 37: One direction is obvious: if $\Gamma \vdash_{\mathsf{QC}} \psi$ then $Mod_{\models_{\mathsf{QC}}}(\Gamma) \subseteq Mod_{\models_4}(\psi)$. But $Min_{\leq_k}(Mod_{\models_{\mathsf{QC}}}(\Gamma)) \subseteq Mod_{\models_{\mathsf{QC}}}(\Gamma)$, hence $Min_{\leq_k}(Mod_{\models_{\mathsf{QC}}}(\Gamma)) \subseteq Mod_{\models_4}(\psi)$, and so $\Gamma \vdash^{\leq_k}_{\mathsf{QC}} \psi$. For the converse, let $\nu \in Mod_{\models_{\mathsf{QC}}}(\Gamma)$. By Lemma 38 there is a valuation $\mu \in Min_{\leq_k}(Mod_{\models_{\mathsf{QC}}}(\Gamma))$ such that $\mu \leq_k \nu$. As $\Gamma \vdash^{\leq_k}_{\mathsf{QC}} \psi$, $\mu \in Mod_{\models_4}(\psi)$, that is, $\mu(\psi) \in \mathcal{D} = \{t, \top\}$. Since all the operators that correspond to the connectives of ψ are monotone with respect to $\leq_k$, $\nu(\psi) \geq_k \mu(\psi)$. But $\mathcal{D}$ is upwards-closed with respect to $\leq_k$, therefore $\nu(\psi) \in \mathcal{D}$ as well, i.e., $\nu \in Mod_{\models_4}(\psi)$. It follows that $Mod_{\models_{\mathsf{QC}}}(\Gamma) \subseteq Mod_{\models_4}(\psi)$, and so $\Gamma \vdash_{\mathsf{QC}} \psi$. □

Example 39 Let $\Gamma = \{p_i \vee p_{i+1}, p_i \vee \neg p_{i+1} \mid i \geq 1\}$. This theory has an infinite number of QC-models, as $\{\nu_1, \nu_2, \dots\} \subset Mod_{\models_{\mathsf{QC}}}(\Gamma)$, where $\forall i, j \geq 1$ $\nu_i(p_j) = \top$ if $j \geq i$ and $\nu_i(p_j) = t$ if $j < i$. On the other hand, there is only one $\leq_k$-minimal QC-model of Γ, denote it μ, in which $\mu(p_j) = t$ for every $p_j \in \mathcal{A}$. By Proposition 37, then, it suffices to consider only μ for the QC-inferences of Γ, that is, $\Gamma \vdash_{\mathsf{QC}} \psi$ iff $\mu \models_4 \psi$.

Knowledge minimization is therefore admitted by all the formalisms that we have considered:

Corollary 40 *For a theory Γ and a formula $\psi \in \mathcal{N}$,*

a) $\Gamma \vdash_4 \psi$ *iff* $\Gamma \Vdash_4 \psi$ *iff* $\Gamma \Vdash^{\subseteq}_4 \psi$ *iff* $\Gamma \vdash^{\leq_k}_4 \psi$.

b) $\Gamma \vdash_{\mathsf{MI}} \psi$ *iff* $\Gamma \Vdash_{\mathsf{MI}} \psi$ *iff* $\Gamma \Vdash^{\subseteq}_{\mathsf{MI}} \psi$ *iff* $\Gamma \vdash^{\leq_k}_{\mathsf{MI}} \psi$.

c) $\Gamma \vdash_{\mathsf{QC}} \psi$ *iff* $\Gamma \Vdash_{\mathsf{QC}} \psi$ *iff* $\Gamma \Vdash^{\subseteq}_{\mathsf{QC}} \psi$ *iff* $\Gamma \vdash^{\leq_k}_{\mathsf{QC}} \psi$.

Proof. Item (a) follows from Propositions 10, 36(a), and 37(a); Item (b) follows from the note at the end of the previous section and from Propositions 36(b) and 37(b); Item (c) follows from Propositions 21, 36(c), 37(c). □

Comparing the Logics

As noted before, an important difference among the three formalisms considered above is their attitude towards the disjunctive syllogism. The following example illustrates this.

Example 41 Let $\Gamma = \{p, \neg p, q, \neg p \vee r, \neg q \vee s\}$. Belnap's consequence relation excludes any application of the disjunctive syllogism, thus $\Gamma \nvdash_4 r$ and $\Gamma \nvdash_4 s$. On the other extreme, quasi-classical logic supports every application of resolution in Γ, therefore $\Gamma \vdash_{\mathsf{QC}} r$ and $\Gamma \vdash_{\mathsf{QC}} s$. The logic of inconsistency minimization is situated in an intermediate level: as Γ can be partitioned to two independent fragments $\Gamma' = \{q, \neg q \vee s\}$ and $\Gamma' = \{p, \neg p, \neg p \vee r\}$, the former of which is classically consistent, then by Proposition 28 the disjunctive syllogism is supported only with respect to its consistent fragment, hence $\Gamma \vdash_{\mathsf{MI}} s$ while $\Gamma \not\vdash_{\mathsf{MI}} r$.

Next, we investigate the relative strength of the three formalisms.

Proposition 42 *For every theory Γ and $\psi \in \mathcal{N}$,*

a) if $\Gamma \vdash_4 \psi$ then $\Gamma \vdash_{\mathsf{MI}} \psi$,

b) if $\Gamma \vdash_{\mathsf{MI}} \psi$ then $\Gamma \vdash_{\mathsf{QC}} \psi$.

Note 43 The two items of Proposition 42 imply that $\Gamma \vdash_{\mathsf{QC}} \psi$ whenever $\Gamma \vdash_4 \psi$, and so $\vdash_4 \subseteq \vdash_{\mathsf{MI}} \subseteq \vdash_{\mathsf{QC}}$. Note that by Example 41 these containments are, in fact, strict (see also Examples 22, 26). Moreover, unlike $\vdash_4$ and $\vdash_{\mathsf{QC}}$ that are monotonic (see, respectively, (Arieli and Avron 1996, Proposition 3.10) and (Hunter 2000, Proposition 4.32)), $\vdash_{\mathsf{MI}}$ is nonmonotonic. This is demonstrated in Example 26: for $\Gamma = \{\neg p, p \vee q\}$ we have that $\Gamma \vdash_{\mathsf{MI}} q$ but $\Gamma \cup \{p\} \not\vdash_{\mathsf{MI}} q$.

Proof Outline of Proposition 42. Part (a) simply follows from the fact that $Mod_{\models_{\mathsf{MI}}}(\Gamma) \subseteq Mod_{\models_4}(\Gamma)$. For Part (b), suppose first that Γ consists only of clauses. We consider two operators that construct a theory from a theory and a valuation:

- $T_1(\Gamma, \nu)$ is the theory that is obtained from Γ by removing all the *clauses* in which there is a literal l such that $\nu(l) = \top$.

- $T_2(\Gamma, \nu)$ is the theory that is obtained from Γ by removing from every clause in Γ the *literals* l such that $\nu(l) = \top$.[7]

The following facts are easily verified:

Fact 1: If $\nu \in Mod_{\models_{\mathsf{MI}}}(T_1(\Gamma, \nu))$ there is $\mu \in Mod_{\models_{\mathsf{MI}}}(\Gamma)$ such that $\mu \leq_\top \nu$ and $\mu \leq_k \nu$ (either $\mu = \nu$ or μ is obtained from ν by letting $\mu(p_i) \in \{t, f, \bot\}$ for some $p_i \in \mathcal{A}(\Gamma) \setminus \mathcal{A}(T_1(\Gamma, \nu))$ such that $\nu(p_i) = \top$).

Fact 2: If $\nu \in Mod_{\models_{\mathsf{QC}}}(\Gamma)$ then $\nu \in Mod_{\models_{\mathsf{QC}}}(T_2(\Gamma, \nu))$.

Also, by the definitions of T_1 and T_2, for every clause C_1 in $T_1(\Gamma, \nu)$ there is a clause C_2 in $T_2(\Gamma, \nu)$ such that C_1 is a subformula of C_2. Thus:

Fact 3: $Mod_{\models_4}(T_2(\Gamma, \nu)) \subseteq Mod_{\models_4}(T_1(\Gamma, \nu))$.

Now, if $\Gamma \not\vdash_{\mathsf{QC}} \psi$, there is $\nu \in Mod_{\models_{\mathsf{QC}}}(\Gamma) \setminus Mod_{\models_4}(\psi)$. By Fact 2, $\nu \in Mod_{\models_{\mathsf{QC}}}(T_2(\Gamma, \nu)) \setminus Mod_{\models_4}(\psi)$, thus $\nu \in Mod_{\models_4}(T_2(\Gamma, \nu)) \setminus Mod_{\models_4}(\psi)$, and by Fact 3, $\nu \in Mod_{\models_4}(T_1(\Gamma, \nu)) \setminus Mod_{\models_4}(\psi)$. Note that this implies, in particular, that there is no $p \in \mathcal{A}(T_1(\Gamma, \nu)) \cup \mathcal{A}(\psi)$ such that $\nu(p) = \top$. Consider a valuation $\nu' \in \Lambda$ that is identical to ν on $\mathcal{A}(T_1(\Gamma, \nu)) \cup \mathcal{A}(\psi)$ and $\nu(p) \neq \top$ elsewhere. Then $\nu' \in Mod_{\models_{\mathsf{MI}}}(T_1(\Gamma, \nu)) \setminus Mod_{\models_4}(\psi)$. Now, by Fact 1, there is a valuation $\mu \in Mod_{\models_{\mathsf{MI}}}(\Gamma)$ such that $\mu \leq_k \nu'$. Since $\nu' \notin Mod_{\models_4}(\psi)$, $\mu \notin Mod_{\models_4}(\psi)$ as well. Thus $Mod_{\models_{\mathsf{MI}}}(\Gamma) \not\subseteq Mod_{\models_4}(\psi)$, and so $\Gamma \not\vdash_{\mathsf{MI}} \psi$.

The proof for sets of CNF-formulas follows from the proof above by the fact that $C_1 \wedge \ldots \wedge C_n \vdash_{\mathsf{x}} \psi$ iff $C_1, \ldots, C_n \vdash_{\mathsf{x}} \psi$ for both $\mathsf{x} = \mathsf{QC}$ and $\mathsf{x} = \mathsf{MI}$. $\square$

A graphic representation of the results in Corollary 40 and Proposition 42 is given in Figure 2.

As the next result shows (see Proposition 1–3 in (Coste-Marquis and Marquis 2005)), the formalisms considered above also differ in their computational complexity.

Proposition 44 *Deciding whether a formula $\psi \in \mathcal{N}$ follows from a theory Γ is in P for $\vdash_4$, is coNP-complete for $\vdash_{\mathsf{QC}}$, and is Π_2^p-complete for $\vdash_{\mathsf{MI}}$.*[8]

Summary

Paraconsistent reasoning requires the weakening of either the disjunctive syllogism or the law of disjunction introduction. In this paper, we have considered three different approaches for doing so in the context of four-states logics, and investigated the relative expressive power of three logics, each one represents a different approach. More specifically, we compared quasi-classical logic – whose definition is syntactical in nature and motivated by the need to preserve the resolution principle – to a logic of inconsistency minimization, that is based on purely semantical considerations, and which tolerates classically valid rules only with respect to classically consistent fragments of the premises.

Despite their different attitudes to the application of the disjunctive syllogism, all the investigated formalisms share the knowledge minimization property, which allows to reduce the number of models without affecting the conclusions. This also may be captured, in the context of four possible states of information, as a kind of consistency preserving method: As long as one keeps the redundant information as minimal as possible the tendency of getting into conflicts decreases.

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[7] Consider for instance the theory $\Gamma = \{p, \neg p, p \vee q\}$ of Examples 22 and 26, and let $\nu \in \Lambda$ such that $\nu(p) = \top$. Then $T_1(\Gamma, \nu) = \emptyset$ and $T_2(\Gamma, \nu) = \{q\}$.

[8] When Γ and ψ are an arbitrary set of formulas and a formula in $\mathcal{L}$, the decision problem for $\vdash_4$ becomes coNP-complete, and remains Π_2^p-complete for $\vdash_{\mathsf{MI}}$.

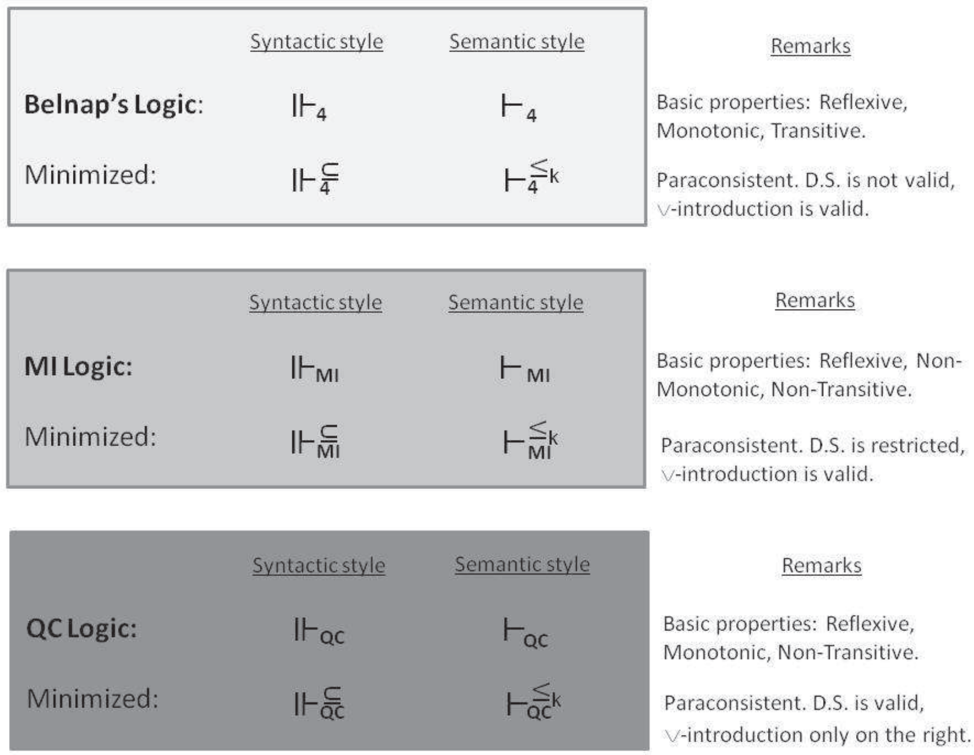

Figure 2: Classes of paraconsistent entailments in four-valued logics (Entailments within the same box are equivalent; Entailments in a higher box are weaker than those in a lower box).

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