A Class of $\overset{\diamond}{f}$-Consistencies for Qualitative Constraint Networks

Jean-François Condotta and **Christophe Lecoutre**
Université Lille-Nord de France, Artois
CRIL-CNRS, UMR 8188
rue de l'université, SP 16, F62307 Lens
Email: {condotta,lecoutre}@cril.fr

Abstract

In this paper, we introduce a new class of local consistencies, called $\overset{\diamond}{f}$-consistencies, for qualitative constraint networks. Each consistency of this class is based on weak composition ($\diamond$) and a mapping f that provides a covering for each relation of the considered qualitative calculus. We study the connections existing between some properties of the introduced mappings and the relative inference strength of $\overset{\diamond}{f}$-consistencies. The consistency obtained by the usual closure under weak composition corresponds to the weakest element of the class, whereas $\overset{\diamond}{f}$-consistencies stronger than weak composition open new promising perspectives. Interestingly, the class of $\overset{\diamond}{f}$-consistencies is shown to form a complete lattice where the partial order denotes the relative strength of every two consistencies. We also propose a generic algorithm that allows us to compute the closure of qualitative constraint networks under any "well-behaved" consistency of the class. The experimentation that we have conducted on qualitative constraint networks from the Interval Algebra shows the interest of these new local consistencies, in particular for the consistency problem.

Introduction

Qualitative Spatial-Temporal Reasoning (QSTR) is an area of computer science dealing with qualitative information about configurations of spatial/temporal entities. A calculus in QSTR introduces particular elements for representing the entities and a finite set of base relations on these elements. Each base relation is an abstraction of concrete metric information about the relative position of entities. For applications in domains such as e.g. geographic information systems and natural language understanding, a qualitative description may reveal to be far more appropriate than a metric description, in particular when precise information is not necessary or simply not available. In the past twenty years, numerous QSTR formalisms have been proposed and studied; see e.g. (Vilain, Kautz, and van Beek 1990; Randell, Cui, and Cohn 1992; Renz 1999; Pujari, Kumari, and Sattar 1999; Renz and Nebel 2007).

In QSTR, Qualitative Constraint Networks (QCNs) are typically used to express information on spatial/temporal situations. A constraint represents a set of acceptable qualitative configurations between some variables (entities), and is then defined by a set of base relations. Given a QCN, the main problem is to determine whether the information contained in the QCN is consistent. In the general case, this problem is NP-hard. However, because the worst-case only arises within a limited range of situations, many studies have been led to develop efficient practical approaches to solve this problem.

One such approach is backtrack search combined with a constraint propagation mechanism based on tractable subclasses of relations and the closure of QCNs under weak composition, which is an operation denoted by $\diamond$ and related to path consistency (Mackworth 1977). More precisely, at each step of search, a constraint is split into relations belonging to a tractable class and closure under weak composition is an inference method applied to filter the search space (i.e. to reduce its size) by removing some inconsistent base relations. This effective approach, initiated by Nebel, has been adopted by most of the qualitative constraint solvers (Condotta, Saade, and Ligozat 2006; Gantner, Westphal, and Wolfl 2008), and in particular by GQR*, which is currently the fastest solver. This solver also uses constraint ordering heuristics based on dynamic weighting (Boussemart et al. 2004) and the concept of eligible constraints (Condotta, Ligozat, and Saade 2007). On the other hand, some recent approaches (Pham, Thornton, and Sattar 2006; Westphal and Wölfl 2009; Li, Huang, and Renz 2009) translate the consistency problem of QCNs into CSP (Constraint Satisfaction Problem) or SAT (propositional satisfiability) instances. Published results indicate that these approaches are promising.

Closure under weak composition is at the heart of the various approaches that directly handle qualitative constraint networks. It was the first inference method used to address the consistency problem of the temporal QCNs in the well-known Interval Algebra (Allen 1981). Weak composition is currently recognized as an operation that offers a good balance between the execution overhead and the filtering benefit. Besides, it has been shown to be a complete approach for most of the identified tractable classes. Nevertheless, for the hardest QCNs it may be worthwhile to consider operations stronger than $\diamond$, i.e. stronger forms of local consistency.

In this paper, we propose a new class of local consistencies adapted to qualitative calculi. Each of them is defined from $\diamond$ and a mapping f that associates with every relation r of a qualitative calculus a set of sub-relations of r forming a covering of r. Intuitively, a QCN is $\overset{\diamond}{f}$-consistent if and only if after substituting any sub-relation defined by f for the relation associated with a constraint of the QCN, the obtained QCN is closed under $\diamond$. We prove that $\diamond$ corresponds to the weakest consistency of the class whereas a local consistency similar to SAC, Singleton Arc Consistency (Debruyne and Bessiere 1997) introduced for CSP, is the strongest one. Other consistencies of the class are situated between these two bounds since the class forms a complete lattice. We also characterize an important subset of the class of $\overset{\diamond}{f}$-consistencies that contains consistencies under which closure of QCNs exists, and we propose a general-purpose algorithm to enforce any of them. A preliminary experimentation carried out using the Interval Algebra shows promising results.

The paper is organized as follows. After some preliminaries, we introduce the class of $\overset{\diamond}{f}$-consistencies and study the closure of QCNs under such consistencies. Next, we describe a general algorithm to enforce any $\overset{\diamond}{f}$-consistency, and report some preliminary results. Finally, we conclude and give some perspectives of this work.

Preliminaries

A qualitative calculus is defined from a finite set B of base relations on a domain D. Without any loss of generality, we will only consider binary relations. The elements of D represent temporal or spatial entities, and the elements of B represent all possible configurations between two entities. B is a set that satisfies the following properties (Ligozat and Renz 2004): B forms a partition of $\mathsf{D} \times \mathsf{D}$, B contains the identity relation Id, and B is closed under the converse operation ($^{-1}$). A (complex) relation is the union of some base relations, but it is customary to represent a relation as the set of base relations contained in it. Hence, the set 2^{B} will represent the set of relations of the qualitative calculus. The set 2^{B} is equipped with the weak composition operation, denoted by $\diamond$ and defined by: $\forall a, b \in \mathsf{B}, a \diamond b = \{c \in \mathsf{B} : \exists x, y, z \in \mathsf{D} \mid x\ a\ z \wedge z\ b\ y \wedge x\ c\ y\}$; $\forall r, s \in 2^{\mathsf{B}}, r \diamond s = \bigcup_{a \in r, b \in s} \{a \diamond b\}$. Note that $r \diamond s$ is the smallest relation of 2^{B} including the usual relational composition $r \circ s = \{(x, y) \in \mathsf{D} \times \mathsf{D} : \exists z \in \mathsf{D} \mid x\ r\ z \wedge z\ s\ y\}$. In some qualitative algebras (e.g. the Interval Algebra introduced below), $r \circ s$ and $r \diamond s$ are identical.

A well known temporal qualitative formalism is the Interval Algebra, also called Allen's calculus (Allen 1981). The domain $\mathsf{D}_{\mathsf{int}}$ of this calculus is the set $\{(x^-, x^+) \in \mathbb{Q} \times \mathbb{Q} : x^- < x^+\}$ since temporal entities are intervals of the rational line. The set $\mathsf{B}_{\mathsf{int}}$ of this calculus is the set $\{eq, p, pi, m, mi, o, oi, s, si, d, di, f, fi\}$ of thirteen binary relations representing all orderings of the four endpoints of two intervals; see Figure 1. For example, $m = \{((x^-, x^+), (y^-, y^+)) \in \mathsf{D}_{\mathsf{int}} \times \mathsf{D}_{\mathsf{int}} : x^+ = y^-\}$.

A Qualitative Constraint Network (QCN) is a pair composed of a set of variables and a set of constraints. Each variable represents a spatial/temporal entity of the system

Relation	Symbol	Inverse	Meaning
precedes	p	pi	
meets	m	mi	
overlaps	o	oi	
starts	s	si	
during	d	di	
finishes	f	fi	
equals	eq	eq	

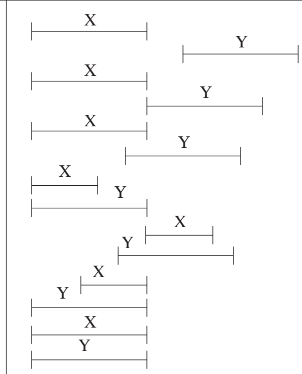


Figure 1: Base relations of the Interval Algebra.

that is modelled. Each constraint represents a set of acceptable qualitative configurations between two variables and is defined by a relation. Formally, a QCN is defined as follows:

Definition 1 *A QCN is a pair $\mathcal{N} = (V, C)$ where:*

- *$V = \{v_1, \ldots, v_n\}$ is a finite set of n variables;*
- *C is a mapping that associates a relation $C(v_i, v_j) \in 2^{\mathsf{B}}$, also denoted by C_{ij} or $\mathcal{N}[i, j]$, with each pair (v_i, v_j) of $V \times V$. C is such that $C_{ii} \subseteq \{\mathsf{Id}\}$ and $C_{ij} = C_{ji}^{-1}$.*

A *partial solution* of $\mathcal{N}$ on $V' \subseteq V$ is a mapping σ defined from V' to D such that for every pair (v_i, v_j) of variables in V', $(\sigma(v_i), \sigma(v_j))$ satisfies C_{ij}, i.e. there exists a base relation $b \in C_{ij}$ such that $(\sigma(v_i), \sigma(v_j)) \in b$. A *solution* of $\mathcal{N}$ is a partial solution of $\mathcal{N}$ on V. $\mathcal{N}$ is *consistent* iff it admits a solution. Two QCNs are *equivalent* iff they admit the same set of solutions. A *sub*QCN $\mathcal{N}'$ of $\mathcal{N}$, denoted by $\mathcal{N}' \subseteq \mathcal{N}$, is a QCN (V, C') such that $C'_{ij} \subseteq C_{ij}$, for every pair (v_i, v_j) of variables. An *atomic* QCN is a QCN such that each constraint is defined by a base relation. A *scenario* $\mathcal{S}$ of $\mathcal{N}$ is an atomic consistent subQCN of $\mathcal{N}$. A base relation b for C_{ij} is *inconsistent* iff there does not exist any scenario $\mathcal{S}$ of $\mathcal{N}$ such that $\mathcal{S}[i, j] = \{b\}$.

A QCN $\mathcal{N} = (V, C)$ is said to be *$\diamond$-consistent* or *closed under weak composition* if and only if $C_{ij} \subseteq C_{ik} \diamond C_{kj}$ $\forall v_i, v_j, v_k \in V$. The *closure* under weak composition of $\mathcal{N}$, denoted by $\diamond(\mathcal{N})$, is the greatest (w.r.t. $\subseteq$) $\diamond$-consistent subQCN of $\mathcal{N}$; $\diamond(\mathcal{N})$ is equivalent to $\mathcal{N}$. This (sub)QCN can be obtained by iterating the triangulation operation:

$$C_{ij} \leftarrow C_{ij} \cap (C_{ik} \diamond C_{kj}), \forall v_i, v_j, v_k \in V$$

until a fixed point is reached. This method can be implemented by an algorithm running in $O(n^3)$ time. Weak composition admits the following properties:

- $\diamond(\mathcal{N}) \subseteq \mathcal{N}$ ($\diamond$ is contracting),
- $\diamond(\diamond(\mathcal{N})) = \diamond(\mathcal{N})$ ($\diamond$ is idempotent),
- $\mathcal{N} \subseteq \mathcal{N}' \Rightarrow \diamond(\mathcal{N}) \subseteq \diamond(\mathcal{N}')$ ($\diamond$ is monotonic).

$\mathcal{N}_{[i,j]/r}$, with $v_i, v_j \in V$ and $r \in 2^{\mathsf{B}}$, is the QCN (V, C') defined by $C'_{ij} = r$, $C'_{ji} = r^{-1}$ and $C'_{kl} = C_{kl}$ $\forall (v_k, v_l) \in$

$V \times V \setminus \{(v_i, v_j), (v_j, v_i)\}$. The union of two QCNs $\mathcal{N} = (V, C)$ and $\mathcal{N}' = (V, C')$ is the QCN $\mathcal{N} \cup \mathcal{N}' = (V, C'')$ such that $\forall (v_i, v_j) \in V$, $C''_{ij} = C_{ij} \cup C'_{ij}$.

The Class of $\overset{\diamond}{f}$-consistencies

In this section, we introduce (for qualitative constraint networks) a general class of local consistencies, called $\overset{\diamond}{f}$-consistencies, where f is a mapping that associates a set of relations of 2^{B} with each relation of 2^{B}. Such a mapping aims at cutting each constraint into pieces from which some inferences can be made. Intuitively, a QCN is said to be $\overset{\diamond}{f}$-consistent iff for any constraint C_{ij} of the QCN, after substituting any element r' of $f(r)$ for the relation r associated with C_{ij} and computing the closure under weak composition, the relation r' associated with C_{ij} is let unchanged. Before proposing a formal definition of $\overset{\diamond}{f}$-consistencies, we introduce a set $\mathcal{F}$ that exactly contains the mappings f considered hereafter. More precisely, $\mathcal{F}$ is the set of mappings f defined from 2^{B} to $2^{2^{\mathsf{B}}}$ associating a set of relations $f(r) \in 2^{2^{\mathsf{B}}}$ with each relation $r \in 2^{\mathsf{B}}$ such that:

$\bigcup f(r) = r$, and $\emptyset \notin f(r)$ if $r \neq \emptyset$.

Note that $f(r)$ is a covering of r and $f(\{b\}) = \{\{b\}\}, \forall b \in \mathsf{B}$. Moreover, we have $f(\emptyset) = \{\emptyset\}$, which will be always implicitly assumed whenever we introduce a mapping later.

Definition 2 *Let f be an element of $\mathcal{F}$.*

- *A constraint C_{ij} of a QCN $\mathcal{N}$ is $\overset{\diamond}{f}$-consistent iff for every $s \in f(\mathcal{N}[i,j])$, $\diamond(\mathcal{N}_{[i,j]/s})[i,j] = s$.*
- *A QCN $\mathcal{N}$ is $\overset{\diamond}{f}$-consistent iff every constraint of $\mathcal{N}$ is $\overset{\diamond}{f}$-consistent.*

We obtain a new class (or family) of local consistencies since each mapping $f \in \mathcal{F}$ determines a new consistency denoted by $\overset{\diamond}{f}$. The class (set) of all $\overset{\diamond}{f}$-consistencies that can be built from elements of $\mathcal{F}$ is denoted by $\overset{\diamond}{\mathcal{F}}$. The following result shows the practical interest of the new class of consistencies: when a QCN is not $\overset{\diamond}{f}$-consistent, some base relations said to be $\overset{\diamond}{f}$-inconsistent can be identified and safely removed.

Proposition 1 *Let f be an element of $\mathcal{F}$, $\mathcal{N} = (V, C)$ be a QCN, (v_i, v_j) be a pair of variables of $\mathcal{N}$ and $s \in f(\mathcal{N}[i,j])$. Any base relation b in $s \setminus \diamond(\mathcal{N}_{[i,j]/s})[i,j]$ is inconsistent for C_{ij}.*

Proof. Let $\mathcal{S}$ be a scenario of $\mathcal{N}$ and b' be the base relation in $\mathcal{S}[i,j]$. Either we have $b' \notin s$ or $b' \in s$. If $b' \notin s$, necessarily $b' \neq b$. On the other hand, if $b' \in s$ then $b' \in \diamond(\mathcal{N}_{[i,j]/s})[i,j]$ because $\diamond$ preserves scenarios. By hypothesis, $b \notin \diamond(\mathcal{N}_{[i,j]/s})[i,j]$, which proves that $b' \neq b$. We conclude that $v_i\ b\ v_j$ cannot be true in any scenario. $\dashv$

The following mappings will be useful to illustrate our purpose. $\forall r \in 2^{\mathsf{B}} \setminus \{\emptyset\}$:

- f_{B} associates the set $f_{\mathsf{B}}(r) = \{\{b\} : b \in r\}$ with r.
- $f_{\neq}$ associates the set $f_{\neq}(r) = \{r \setminus \{b\} : b \in r\}$ with r iff $|r| > 1$; $f_{\neq}(r) = \{r\}$ otherwise.
- $f_{\diamond}$ associates the set $f_{\diamond}(r) = \{r\}$ with r.

For example, if $r = \{p, m, o\}$, then $f_{\mathsf{B}}(r) = \{\{p\}, \{m\}, \{o\}\}$ and $f_{\neq}(r) = \{\{p, m\}, \{p, o\}, \{m, o\}\}$. Moreover, given a partition $P = \{r_1, \ldots, r_k\}$ of B, the mapping f_P is defined as follows: for every relation $r \in 2^{\mathsf{B}}$, $f(r) = \{r \cap r_i : i \in \{1, \ldots, k\}\} \setminus \{\emptyset\}$. Note that $\overset{\diamond}{f_{\mathsf{B}}}$ and partition-based consistencies $\overset{\diamond}{f_P}$ can be respectively related to SAC (Debruyne and Bessière 1997) and Partition-k-AC (Bennaceur and Affane 2001), introduced for CSP, but $\overset{\diamond}{f_{\neq}}$ (as well as many other $\overset{\diamond}{f}$-consistencies) have no CSP counterpart.

We will consider later the following (representative) partitions of $\mathsf{B}_{\mathsf{int}}$:

- $P_1 = \{\{p, m, o, fi, s, d\}, \{pi, mi, oi, f, si, di, eq\}\}$
- $P_2 = \{\{p, m, o\}, \{fi, s, d\}, \{pi, mi, oi\}, \{f, si, di, eq\}\}$
- $P_3 = \{\{p\}, \{m, o\}, \{fi\}, \{s, d\}, \{pi\}, \{mi, oi\}, \{f, eq\}, \{si, di\}\}$

In order to compare the inference capability of different consistencies, we need to introduce a preorder. Let ϕ and ψ be two consistencies in $\overset{\diamond}{\mathcal{F}}$, ϕ *is stronger than* ψ, denoted by $\phi \trianglerighteq \psi$, iff whenever ϕ holds on a QCN $\mathcal{N}$ (i.e. $\mathcal{N}$ is ϕ-consistent), ψ also holds on $\mathcal{N}$; ϕ is *strictly stronger than* ψ, denoted by $\phi \vartriangleright \psi$, iff ϕ is stronger than ψ and there exists at least one QCN $\mathcal{N}$ such that ψ holds on $\mathcal{N}$ but not ϕ. Finally, ϕ and ψ are *equivalent*, denoted by $\phi \approx \psi$, iff both $\phi \trianglerighteq \psi$ and $\psi \trianglerighteq \phi$.

First, we can show that a QCN $\mathcal{N}$ is $\overset{\diamond}{f_{\diamond}}$-consistent if, and only if, $\mathcal{N}$ is closed under weak composition.

Proposition 2 *The consistency $\overset{\diamond}{f_{\diamond}}$ is equivalent to $\diamond$.*

Proof. $\mathcal{N}$ is $\diamond$-consistent $\Leftrightarrow \diamond(\mathcal{N}) = \mathcal{N} \Leftrightarrow$ for every pair (v_i, v_j) of variables of $\mathcal{N}$, $\diamond(\mathcal{N})[i,j] = \mathcal{N}[i,j] \Leftrightarrow$ for every pair (v_i, v_j) of variables of $\mathcal{N}$ and for every $s \in f_{\diamond}(\mathcal{N}[i,j])$, $\diamond(\mathcal{N}_{[i,j]/s})[i,j] = s$ (because $f_{\diamond}(r) = \{r\}$ for each relation $r \in 2^{\mathsf{B}}$) $\Leftrightarrow \mathcal{N}$ is $\overset{\diamond}{f_{\diamond}}$-consistent $\dashv$

The finer the covering of relations by an element f of $\mathcal{F}$ is, the stronger the consistency $\overset{\diamond}{f}$ is. In particular, to relate $\overset{\diamond}{f}$-consistencies, we have the following result:

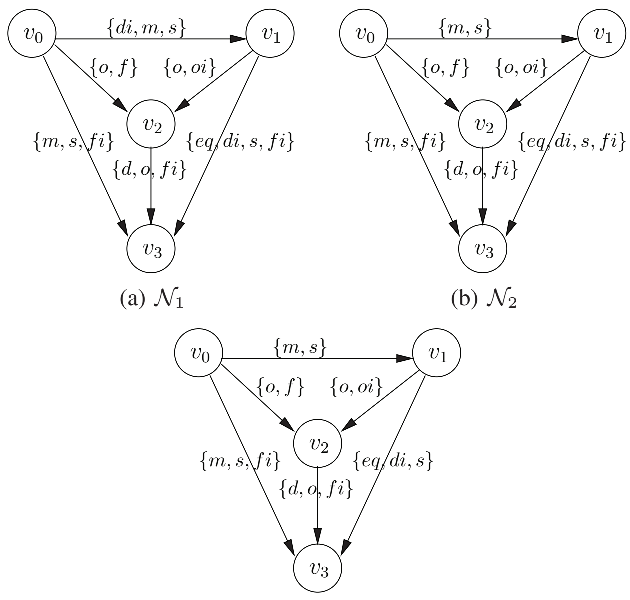
Proposition 3 *Let f, f' be two elements of $\mathcal{F}$. If for every $r \in 2^{\mathsf{B}}$ and for every $s' \in f'(r)$, there exists a set of relations $S \subseteq f(r)$ such that $s' = \bigcup S$, then $\overset{\diamond}{f} \trianglerighteq \overset{\diamond}{f'}$.*

Proof. We suppose that we have a QCN $\mathcal{N}$ that is $\overset{\diamond}{f}$-consistent. Let v_i, v_j be two variables of $\mathcal{N}$, $r = \mathcal{N}[i,j]$ and s' be an element of $f'(r)$. By hypothesis, there exists a set of relations $S \subseteq f(r)$ such that $s' = \bigcup S$. For every relation $s \in S$ we have $s \subseteq s'$, and because $\mathcal{N}_{[i,j]/s} \subseteq \mathcal{N}_{[i,j]/s'}$ and $\diamond$ is monotonic, we have $\diamond(\mathcal{N}_{[i,j]/s})[i,j] \subseteq \diamond(\mathcal{N}_{[i,j]/s'})[i,j]$. We can deduce that $\bigcup\{\diamond(\mathcal{N}_{[i,j]/s})[i,j] : s \in S\} \subseteq \diamond(\mathcal{N}_{[i,j]/s'})[i,j]$. Since $\mathcal{N}$ is $\overset{\diamond}{f}$-consistent (by hypothesis), for every relation $s \in S$, we have $\diamond(\mathcal{N}_{[i,j]/s})[i,j] = s$. Hence, $\bigcup S \subseteq \diamond(\mathcal{N}_{[i,j]/s'})[i,j]$, and as $s' = \bigcup S$, we obtain $s' \subseteq \diamond(\mathcal{N}_{[i,j]/s'})[i,j]$. On the other hand, we also know that $\diamond(\mathcal{N}_{[i,j]/s'})[i,j] \subseteq s'$ because

$\diamond$ is contracting. We can conclude that $s' = \diamond(\mathcal{N}_{[i,j]/s'})[i,j]$ and consequently that $\mathcal{N}$ is $\underset{f'}{\diamond}$-consistent. $\dashv$

For example, for the Interval Algebra, we have $\underset{f_\mathsf{B}}{\diamond} \trianglerighteq \underset{f_{P_3}}{\diamond} \trianglerighteq \underset{f_{P_2}}{\diamond} \trianglerighteq \underset{f_{P_1}}{\diamond} \trianglerighteq \underset{f_\diamond}{\diamond}$. The following corollary stipulates that $\underset{f_\mathsf{B}}{\diamond}$ is the strongest consistency (of $\underset{\mathcal{F}}{\diamond}$) and $\underset{f_\diamond}{\diamond}$ is the weakest one.

Corollary 1 *For every element $f \in \mathcal{F}$, $\underset{f_\mathsf{B}}{\diamond} \trianglerighteq \underset{f}{\diamond} \trianglerighteq \underset{f_\diamond}{\diamond}$.*



(a) $\mathcal{N}_1$ (b) $\mathcal{N}_2$ (c) $\mathcal{N}_3$

Figure 2: $\mathcal{N}_3 \subset \mathcal{N}_2 \subset \mathcal{N}_1$.

From this result, we can deduce in particular that $\underset{f_\mathsf{B}}{\diamond} \trianglerighteq \underset{f_{\neq}}{\diamond} \trianglerighteq \underset{f_\diamond}{\diamond}$. Now, let us consider the three QCNs of the Interval Algebra depicted in Figure 2. On each of these graphs, a variable is represented by a node, and a constraint by an arc labelled with the associated relation; note that, for simplicity, there is no arc going from v_i to v_j when either there is already an arc going from v_j to v_i or $i = j$. We can check that:

- $\mathcal{N}_1$ is $\underset{f_\diamond}{\diamond}$-consistent but not $\underset{f_{\neq}}{\diamond}$-consistent because $di \notin \diamond(\mathcal{N}_{1[0,1]/\{di,m\}})[0,1]$,
- $\mathcal{N}_2$ is $\underset{f_{\neq}}{\diamond}$-consistent but not $\underset{f_\mathsf{B}}{\diamond}$-consistent because $\diamond(\mathcal{N}_{2[1,3]/\{fi\}})[1,3] = \emptyset$,
- $\mathcal{N}_3$ is $\underset{f_\mathsf{B}}{\diamond}$-consistent.

From Corollary 1 and QCNs $\mathcal{N}_1$ and $\mathcal{N}_2$, we deduce that (for the Interval Algebra) $\underset{f_\mathsf{B}}{\diamond} \triangleright \underset{f_{\neq}}{\diamond} \triangleright \underset{f_\diamond}{\diamond}$ (note the strict order).

The equivalence classes of $\approx$ form a partition of $\underset{\mathcal{F}}{\diamond}$; the set of all equivalence classes is denoted by $\underset{\mathcal{F}}{\diamond}|^{\approx}$. Note that $\underset{\mathcal{F}}{\diamond}|^{\approx}$ is a finite set since the set B of base relations is considered to be finite. The relation $\trianglerighteq^{\approx}$ defined on $\underset{\mathcal{F}}{\diamond}|^{\approx}$ by $\forall[\phi], [\psi] \in \underset{\mathcal{F}}{\diamond}|^{\approx}$, $[\phi] \trianglerighteq^{\approx} [\psi]$ iff $\phi \trianglerighteq \psi$ where ϕ and ψ are any representatives (elements) in $[\phi]$ and $[\psi]$, is a partial order. We have the following result:

Proposition 4 *($\underset{\mathcal{F}}{\diamond}|^{\approx}, \trianglerighteq^{\approx}$) is a complete lattice with $[\underset{f_\mathsf{B}}{\diamond}]$ as greatest element and $[\underset{f_\diamond}{\diamond}]$ as least element.*

Proof.
(Existence of binary joins) Let $\underset{f_1}{\diamond}$ and $\underset{f_2}{\diamond}$ be two elements of $\underset{\mathcal{F}}{\diamond}$, and let us define f as $\forall r \in 2^\mathsf{B}$, $f(r) = f_1(r) \cup f_2(r)$. First, we can observe that $f \in \mathcal{F}$ by construction. From Proposition 3, we deduce that $\underset{f}{\diamond} \trianglerighteq \underset{f_1}{\diamond}$ and $\underset{f}{\diamond} \trianglerighteq \underset{f_2}{\diamond}$. Now, let us consider a mapping $f' \in \mathcal{F}$ such that $\underset{f'}{\diamond} \trianglerighteq \underset{f_1}{\diamond}$ and $\underset{f'}{\diamond} \trianglerighteq \underset{f_2}{\diamond}$. By definition, any $\underset{f'}{\diamond}$-consistent QCN $\mathcal{N}$ is $\underset{f_1}{\diamond}$-consistent and $\underset{f_2}{\diamond}$-consistent. Hence, for every pair (v_i, v_j) of variables of $\mathcal{N}$, $s \in f_1(\mathcal{N}[i,j]) \Rightarrow \diamond(\mathcal{N}_{[i,j]/s})[i,j] = s$ and $s \in f_2(\mathcal{N}[i,j]) \Rightarrow \diamond(\mathcal{N}_{[i,j]/s})[i,j] = s$. So, for every $s \in f_1(\mathcal{N}[i,j]) \cup f_2(\mathcal{N}[i,j])$, $\diamond(\mathcal{N}_{[i,j]/s})[i,j] = s$. We deduce that $\mathcal{N}$ is $\underset{f}{\diamond}$-consistent, and $\underset{f'}{\diamond} \trianglerighteq \underset{f}{\diamond}$. Then, $[\underset{f}{\diamond}]$ is the least upper bound of $[\underset{f_1}{\diamond}]$ and $[\underset{f_2}{\diamond}]$.
(Existence of binary meets) Let $\underset{f_1}{\diamond}$ and $\underset{f_2}{\diamond}$ be two elements of $\underset{\mathcal{F}}{\diamond}$, and let us define the set E as $E = \{f' \in \mathcal{F} : \underset{f_1}{\diamond} \trianglerighteq \underset{f'}{\diamond} \wedge \underset{f_2}{\diamond} \trianglerighteq \underset{f'}{\diamond}\}$. Note that $E \neq \emptyset$ since $f_\diamond \in E$. Next, let us define f as $\forall r \in 2^\mathsf{B}$, $f(r) = \bigcup\{f'(r) : f' \in E\}$. From this definition and Proposition 3, we deduce that $\underset{f}{\diamond} \trianglerighteq \underset{f'}{\diamond}$ for every $f' \in E$. We now prove by contradiction that $\underset{f_1}{\diamond} \trianglerighteq \underset{f}{\diamond}$ and $\underset{f_2}{\diamond} \trianglerighteq \underset{f}{\diamond}$. Let us suppose that $\underset{f_1}{\diamond} \trianglerighteq \underset{f}{\diamond}$ does not hold. This means that there exists a $\underset{f_1}{\diamond}$-consistent QCN $\mathcal{N}$ that is not $\underset{f}{\diamond}$-consistent. Hence, there exist two variables v_i, v_j of $\mathcal{N}$ such that $\diamond(\mathcal{N}_{[i,j]/s})[i,j] \neq s$ with $s \in f(\mathcal{N}[i,j])$. From construction of f, we know that there exists a mapping $f' \in E$ such that $s \in f'(\mathcal{N}[i,j])$. Hence, $\mathcal{N}$ is not $\underset{f'}{\diamond}$-consistent. On the other hand, as $f' \in E$ we have $\underset{f_1}{\diamond} \trianglerighteq \underset{f'}{\diamond}$, and $\mathcal{N}$ $\underset{f'}{\diamond}$-consistent since $\mathcal{N}$ is $\underset{f_1}{\diamond}$-consistent. This is a contradiction (because $\mathcal{N}$ is both $\underset{f'}{\diamond}$-consistent and not $\underset{f'}{\diamond}$-consistent) so $\underset{f_1}{\diamond} \trianglerighteq \underset{f}{\diamond}$ does hold. Similarly, we can show that $\underset{f_2}{\diamond} \trianglerighteq \underset{f}{\diamond}$. Then, $[\underset{f}{\diamond}]$ is the greatest lower bound of $[\underset{f_1}{\diamond}]$ and $[\underset{f_2}{\diamond}]$. $\dashv$

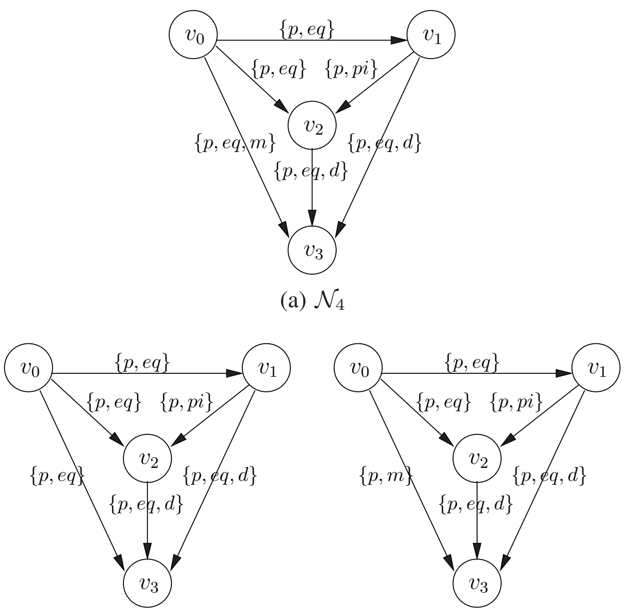
To conclude this section, let us prove the following result for atomic QCNs.

Proposition 5 *Let f be an element of $\mathcal{F}$, and $\mathcal{N}$ be an atomic* QCN*. If $\mathcal{N}$ is consistent then $\mathcal{N}$ is $\underset{f}{\diamond}$-consistent.*

Proof. For any element f of $\mathcal{F}$ and any base relation b, we know that $f(\{b\}) = \{\{b\}\}$. It means that all consistencies in $\underset{\mathcal{F}}{\diamond}$ are equivalent when restricted to atomic QCNs. As $\underset{f_\diamond}{\diamond}$ is equivalent to $\diamond$ (see Proposition 2) and as it is known that an atomic consistent QCN is necessarily closed under weak composition (i.e. $\diamond$-consistent), we deduce that an atomic consistent QCN is necessarily $\underset{f}{\diamond}$-consistent, whatever f is. $\dashv$

Closure of QCNs under $\underset{f}{\diamond}$-consistencies

A consistency ϕ is *well-behaved* iff for any QCN $\mathcal{N}$, there exists a (unique) largest ϕ-consistent QCN $\mathcal{N}'$ smaller than or equal to $\mathcal{N}$ (w.r.t. $\subseteq$). $\mathcal{N}'$ is called the ϕ-closure of $\mathcal{N}$, and denoted by $\phi(\mathcal{N})$. In this section, we are concerned with the closure of QCNs under $\underset{f}{\diamond}$-consistencies. Are $\underset{f}{\diamond}$-consistencies well-behaved? In other words, given a QCN $\mathcal{N}$ and a consistency $\overset{\diamond}{f}$ in $\overset{\diamond}{\mathcal{F}}$, does the $\overset{\diamond}{f}$-closure of $\mathcal{N}$ exist? We first show that this is not always the case with an example taken from the Interval Algebra. We consider $f \in \mathcal{F}$ such that $f(\{p, eq, m\}) = \{\{p, eq, m\}, \{eq\}\}$ and $f(r) = \{r\}$ for every relation $r \in 2^{\mathsf{B}_{\mathsf{int}}} \setminus \{\{p, eq, m\}\}$. Figure 3 shows three distinct QCNs. The first QCN $\mathcal{N}_4$ is not $\overset{\diamond}{f}$-consistent because $\diamond(\mathcal{N}_{4[0,3]/\{eq\}})[0, 3] = \emptyset$. Now let us turn to the two distinct QCNs $\mathcal{N}_5$ and $\mathcal{N}_6$: both QCNs are $\overset{\diamond}{f}$-consistent and (strictly) smaller than $\mathcal{N}_4$. Observing that there does not exist any $\overset{\diamond}{f}$-consistent QCN strictly greater than $\mathcal{N}_5$ and $\mathcal{N}_6$ and smaller than $\mathcal{N}_4$, we have just proved that $\overset{\diamond}{f}$ is not well-behaved.



(a) $\mathcal{N}_4$

(b) $\mathcal{N}_5$ (c) $\mathcal{N}_6$

Figure 3: $\mathcal{N}_4 = \mathcal{N}_5 \cup \mathcal{N}_6$.

Nevertheless, there exist some mappings f for which the consistencies $\overset{\diamond}{f}$ are guaranteed to be well-behaved. This is the case for the elements of the set $\mathcal{F}^*$ introduced below. Roughly speaking, for every relation r, $f(r)$ cannot be finer than the set of $f(r')$ with r' contained in r.

Definition 3 *$\mathcal{F}^*$ is the set of mappings f in $\mathcal{F}$ such that for every $r, r' \in 2^{\mathsf{B}}$ with $r' \subset r$ and for every $s \in f(r)$, we have $s \cap r' \neq \emptyset \Rightarrow \exists S \subseteq f(r')$ such that $s \cap r' = \bigcup S$.*

For example, all mappings mentioned in our previous illustrations belong to $\mathcal{F}^*$, except the last one that has been introduced above to prove that some $\overset{\diamond}{f}$-consistencies are not well-behaved. We first show the following result.

Proposition 6 *Let f be a mapping of $\mathcal{F}^*$. If $\mathcal{N}_1$ and $\mathcal{N}_2$ are two $\overset{\diamond}{f}$-consistent* QCN*s defined on the same set of variables, then $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$ is a $\overset{\diamond}{f}$-consistent* QCN.

Proof. Let v_i, v_j be two variables of $\mathcal{N}$ (and consequently of $\mathcal{N}_1$ and $\mathcal{N}_2$), $r = \mathcal{N}[i, j]$ and $s \in f(r)$. We have to show that $\diamond(\mathcal{N}_{[i,j]/s})[i, j] = s$. Let $r_1 = \mathcal{N}_1[i, j]$, $r_2 = \mathcal{N}_2[i, j]$ and let s_1 and s_2 be the two relations defined as $s_1 = s \cap r_1$ and $s_2 = s \cap r_2$. As $s \in f(r)$, we have $s \subseteq r$ and as $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$, we have $r = r_1 \cup r_2$. We can deduce that $s = s_1 \cup s_2$, and also that $\mathcal{N}_{1[i,j]/s_1} \subseteq \mathcal{N}_{[i,j]/s}$ and $\mathcal{N}_{2[i,j]/s_2} \subseteq \mathcal{N}_{[i,j]/s}$. Because $\diamond$ is monotonic, we have $\diamond(\mathcal{N}_{1[i,j]/s_1}) \subseteq \diamond(\mathcal{N}_{[i,j]/s})$ and $\diamond(\mathcal{N}_{2[i,j]/s_2}) \subseteq \diamond(\mathcal{N}_{[i,j]/s})$.

On the other hand, as $f \in \mathcal{F}^*$ there exist $S_1 \subseteq f(r_1)$ and $S_2 \subseteq f(r_2)$ such that $\bigcup S_1 = s_1$ and $\bigcup S_2 = s_2$ (if we assume that $s_1 \neq \emptyset$ and $s_2 \neq \emptyset$). From $\mathcal{N}_1$ and $\mathcal{N}_2$ being $\overset{\diamond}{f}$-consistent, we deduce that $\diamond(\mathcal{N}_{1[i,j]/s'_1})[i, j] = s'_1$, $\forall s'_1 \in S_1$ and $\diamond(\mathcal{N}_{2[i,j]/s'_2})[i, j] = s'_2$, $\forall s'_2 \in S_2$. Moreover, because $\diamond$ is monotonic, we have $\diamond(\mathcal{N}_{1[i,j]/s'_1})[i, j] \subseteq \diamond(\mathcal{N}_{1[i,j]/s_1})[i, j]$, $\forall s'_1 \in S_1$ and $\diamond(\mathcal{N}_{2[i,j]/s'_2})[i, j] \subseteq \diamond(\mathcal{N}_{2[i,j]/s_2})[i, j]$, $\forall s'_2 \in S_2$. From this, we obtain $s'_1 \subseteq \diamond(\mathcal{N}_{1[i,j]/s_1})[i, j]$, $\forall s'_1 \in S_1$ and $s'_2 \subseteq \diamond(\mathcal{N}_{2[i,j]/s_2})[i, j]$, $\forall s'_2 \in S_2$. Consequently, $s_1 \subseteq \diamond(\mathcal{N}_{1[i,j]/s_1})[i, j]$ and $s_2 \subseteq \diamond(\mathcal{N}_{2[i,j]/s_2})[i, j]$. As $\diamond$ is contracting, we also have $\diamond(\mathcal{N}_{1[i,j]/s_1})[i, j] \subseteq s_1$ and $\diamond(\mathcal{N}_{2[i,j]/s_2})[i, j] \subseteq s_2$. Finally, $\diamond(\mathcal{N}_{1[i,j]/s_1})[i, j] = s_1$ and $\diamond(\mathcal{N}_{2[i,j]/s_2})[i, j] = s_2$.

From what precedes, we obtain $s_1 \subseteq \diamond(\mathcal{N}_{[i,j]/s})[i, j]$ and $s_2 \subseteq \diamond(\mathcal{N}_{[i,j]/s})[i, j]$. So, $s = s_1 \cup s_2 \subseteq \diamond(\mathcal{N}_{[i,j]/s})[i, j]$. The same result can be obtained when $s_1 = \emptyset$ or $s_2 = \emptyset$. Moreover, we also know that $\diamond(\mathcal{N}_{[i,j]/s})[i, j] \subseteq s$ because $\diamond$ is contracting. We can conclude that $\diamond(\mathcal{N}_{[i,j]/s})[i, j] = s$, and consequently that $\mathcal{N}$ is $\overset{\diamond}{f}$-consistent. $\dashv$

From the previous result, we can show that for every QCN $\mathcal{N}$ and every f in $\mathcal{F}^*$, the QCN $\bigcup\{\mathcal{N}' : \mathcal{N}' \subseteq \mathcal{N}$ and $\mathcal{N}'$ is $\overset{\diamond}{f}$-consistent$\}$ is the largest $\overset{\diamond}{f}$-consistent subQCN of $\mathcal{N}$, i.e. the $\overset{\diamond}{f}$-closure of $\mathcal{N}$.

Corollary 2 *If $\overset{\diamond}{f}$ is a consistency in $\overset{\diamond}{\mathcal{F}^*}$, then $\overset{\diamond}{f}$ is well-behaved.*

Observing that $f_{\neq}$ and f_{B} do belong to $\mathcal{F}^*$, we can show that the QCNs from Figure 2 are such that $\overset{\diamond}{f_{\neq}}(\mathcal{N}_1) = \mathcal{N}_2$ and $\overset{\diamond}{f_{\mathsf{B}}}(\mathcal{N}_2) = \mathcal{N}_3$.

Importantly, every consistency in $\overset{\diamond}{\mathcal{F}^*}$ preserves the set of scenarios. This is not very surprising since Proposition 1 already indicates that identified base $\overset{\diamond}{f}$-inconsistent relations can be safely discarded.

Proposition 7 *Let f be an element of $\mathcal{F}^*$. For every* QCN *$\mathcal{N}$, $\overset{\diamond}{f}(\mathcal{N})$ is equivalent to $\mathcal{N}$.*

Proof. Suppose that there exist two variables (v_i, v_j) of $\mathcal{N}$ and a base relation $b \in \mathsf{B}$ such that $b \in \mathcal{N}[i, j]$, $b \notin \overset{\diamond}{f}(\mathcal{N})[i, j]$ and a scenario $\mathcal{S}$ of $\mathcal{N}$ with $\mathcal{S}[i, j] = \{b\}$. From Proposition 5, we know that $\mathcal{S}$ is $\overset{\diamond}{f}$-consistent, and from $\mathcal{S}$ being a scenario of $\mathcal{N}$, we know that $\mathcal{S} \subseteq \mathcal{N}$. Hence, by closure definition, we have $\mathcal{S} \subseteq \overset{\diamond}{f}(\mathcal{N})$. This leads to a contradiction since $\mathcal{S}[i, j] = \{b\}$ and $b \notin \overset{\diamond}{f}(\mathcal{N})[i, j]$. $\dashv$

Generic Algorithm

In this section, we present a basic $\overset{\diamond}{f}$-algorithm, that is to say an algorithm that allows us to compute the $\overset{\diamond}{f}$-closure $\overset{\diamond}{f}(\mathcal{N})$ of any given QCN $\mathcal{N}$. Such a closure is guaranteed to exist

Function `df-revise(`C_{ij}`)` : Boolean

in/out : C_{ij}, a constraint of the QCN $\mathcal{N}$
output: $true$ iff the revision of C_{ij} is effective

1 $r \leftarrow \emptyset$
2 **foreach** *relation* $s \in f(C_{ij})$ **do**
3     $\mathcal{N}' \leftarrow \diamond(\mathcal{N}_{[i,j]/s})$ // $\overset{\diamond}{f}$-check on s
4     $r \leftarrow r \cup (s \setminus \mathcal{N}'[i,j])$ // $\overset{\diamond}{f}$-inconsistent base relations are collected
5 **if** $r \neq \emptyset$ **then**
6     $r' \leftarrow \mathcal{N}_{[i,j]} \setminus r$
7     $\mathcal{N} \leftarrow \mathcal{N}_{[i,j]/r'}$ // C_{ij} becomes r'
8   **return** *true*
9 **else**
10   **return** *false*

Function `df-closure(`$\mathcal{N}$,f`)` : Boolean

in/out : $\mathcal{N} = (V, C)$, a QCN
in : f, an element of $\mathcal{F}^*$
output: $true$ iff $\overset{\diamond}{f}(\mathcal{N}) \neq \bot$

1 $\mathcal{N} \leftarrow \diamond(\mathcal{N})$ // $\diamond$ enforced
2 **if** $\mathcal{N} = \bot$ **then**
3   **return** *false*
4 **repeat**
5     $modified \leftarrow false$
6     $Q \leftarrow \{C_{ij} \in C \mid i < j \wedge |f(C_{ij})| > 1\}$
7   **while** $Q \neq \emptyset$ **do**
8   select and remove a constraint C_{ij} from Q
9   **if** *df-revise*(C_{ij}) **then**
10             $\mathcal{N} \leftarrow \diamond(\mathcal{N})$ // $\diamond$ maintained
11   **if** $\mathcal{N} = \bot$ **then**
12   **return** *false*
13             $modified \leftarrow true$
14 **until** $\neg modified$
15 **return** *true*

since f is assumed to belong to $\mathcal{F}^*$; see Corollary 2. We introduce a constraint-oriented propagation scheme for enforcing the consistency $\overset{\diamond}{f}$. The constraint-oriented propagation scheme is characterized by revision of constraints that are successively picked from a dedicated set Q called the queue of the propagation.

The revision of a constraint C_{ij} removes from C_{ij} some base relations that are $\overset{\diamond}{f}$-inconsistent (if any). A revision is said to be effective if it removes at least one base relation. This is the role of function df-revise. For each element s of $f(C_{ij})$, a $\overset{\diamond}{f}$-check on s is performed, that is to say, $\mathcal{N}' = \diamond(\mathcal{N}_{[i,j]/s})$ is computed (line 3), which enables the identification of $\overset{\diamond}{f}$-inconsistent base relations, those in $s \setminus \mathcal{N}'[i,j]$ (line 4). The variable r collects $\overset{\diamond}{f}$-inconsistent base relations from C_{ij}, and if r is not empty, C_{ij} is updated (line 7) and $true$ is returned.

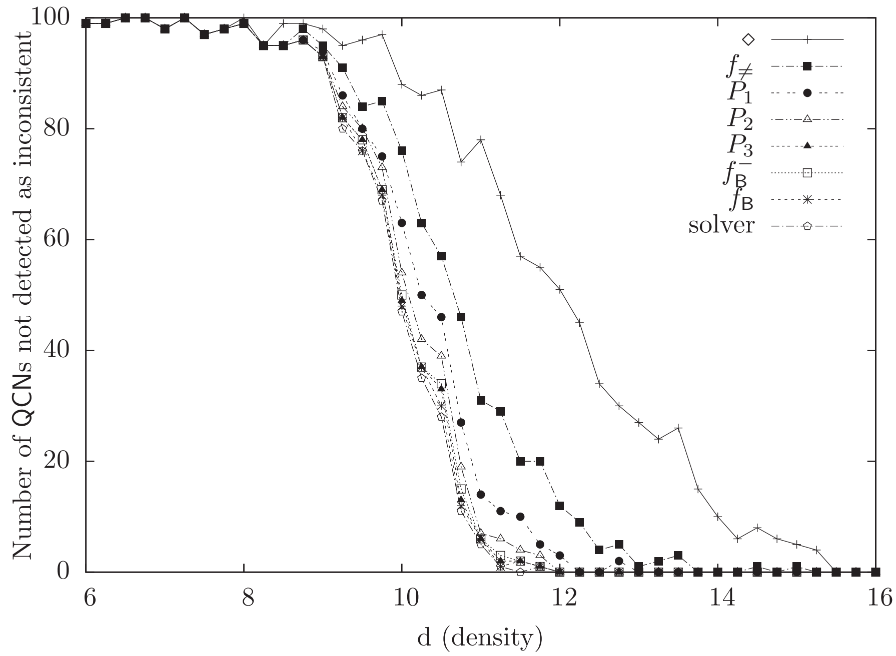
The main function, called df-closure, performs one or several turns (passes) of the main loop. At each pass, all constraints are revised in turn: constraints are iteratively selected from Q (line 8) and df-revise is called to perform revisions (line 9). When an inference is performed (i.e. a revision is effective), the Boolean variable $modified$ is set to $true$, which determines that a next pass is necessary. The algorithm stops when no inference is performed during a pass, or when an inconsistency is detected (lines 3 and 12). Note that when a QCN $\mathcal{N}$ is trivially inconsistent because there exists an empty constraint in $\mathcal{N}$, we note $\mathcal{N} = \bot$. Initially (line 1), and after each effective revision (line 10), $\diamond$ (closure under weak composition) is applied on $\mathcal{N}$. This is sound because we know that for any $f \in \mathcal{F}^*$, $\overset{\diamond}{f}(\mathcal{N}) \subseteq \overset{\diamond}{f_\diamond}(\mathcal{N}) = \diamond(\mathcal{N})$. When initializing Q (line 6), a constraint C_{ji} with $i < j$ is ignored because it can be deducted from C_{ij} by means of the inverse operation. Also, a constraint C_{ij} such that $|f(C_{ij})| = 1$ is ignored because it is necessarily $\overset{\diamond}{f}$-consistent (recall that closure under weak composition is maintained during search).

We can prove that the algorithm df-closure is correct, i.e. enforce $\overset{\diamond}{f}$. Indeed, the algorithm is sound because every base relation removed in df-revise is $\overset{\diamond}{f}$-inconsistent. On the other hand, the algorithm is complete because, as soon as an inference is performed, a new pass is run (and all constraints are revised). However, it is important to note that the function df-revise removes at least one $\overset{\diamond}{f}$-inconsistent base relation from a given $\overset{\diamond}{f}$-inconsistent constraint. Consequently, this guarantess completeness although the function df-revise does not systematically render the given constraint $\overset{\diamond}{f}$-consistent.

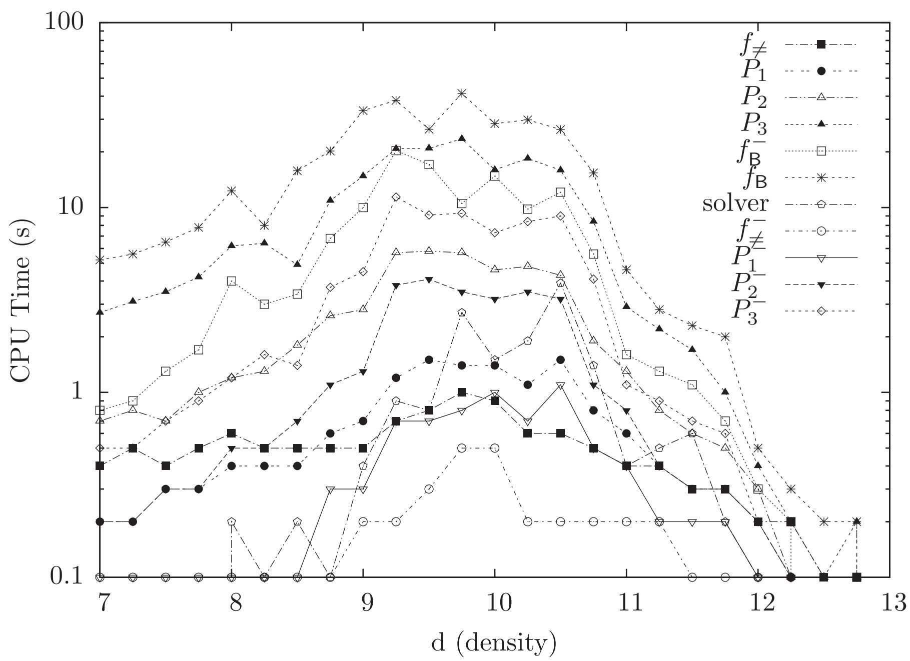
The worst-case time complexity of the function df-revise is $O(\mathsf{s}\lambda)$ where s is the greatest size (cardinality) of sets in $\{f(r) : r \in 2^B\}$ and λ the worst-case time complexity of enforcing $\diamond$, i.e. $O(n^3)$ for binary relations. Indeed, at most s $\overset{\diamond}{f}$-checks are performed. At each pass of the function df-closure, the number of calls to df-revise is $O(n^2)$, so the worst-case time complexity of one pass of df-closure is $O(\mathsf{s}\lambda n^2)$. Although the number of passes is bounded by $O(|\mathsf{B}|n^2)$ (only one base relation removed at each pass), we think that it is a small number in practice (this will be confirmed in our experimentations). Besides, we believe that the basic algoritm presented here can be refined so as to make it incremental (similarly to what is done for SAC (Bessiere et al. 2010)).

Experiments

In our experimentation, we have focused on qualitative constraint networks from the Interval Algebra, randomly generated following Model A (Nebel 1996). This model involves the generation of QCNs according to three parameters: n the number of variables, d the density and s the average number of base relations in each constraint. First, a graph with n nodes is built; the average degree of nodes is d. Then, every edge of the graph is labelled with a relation of s base relations (on average), randomly generated according to a binomial distribution. Finally, the graph is transformed into

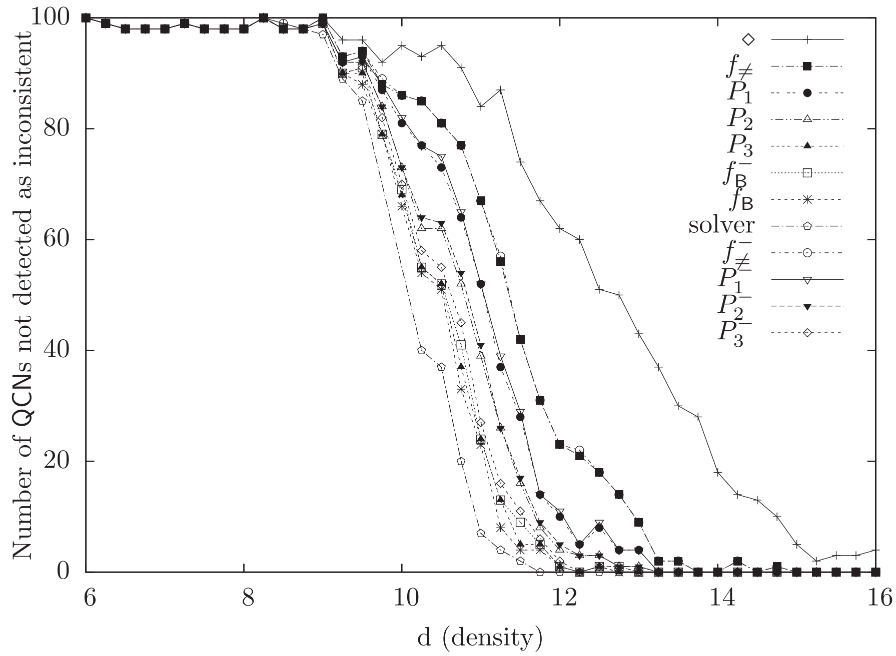


(a) Filtering Strength for $A(75, d, 6.5)$

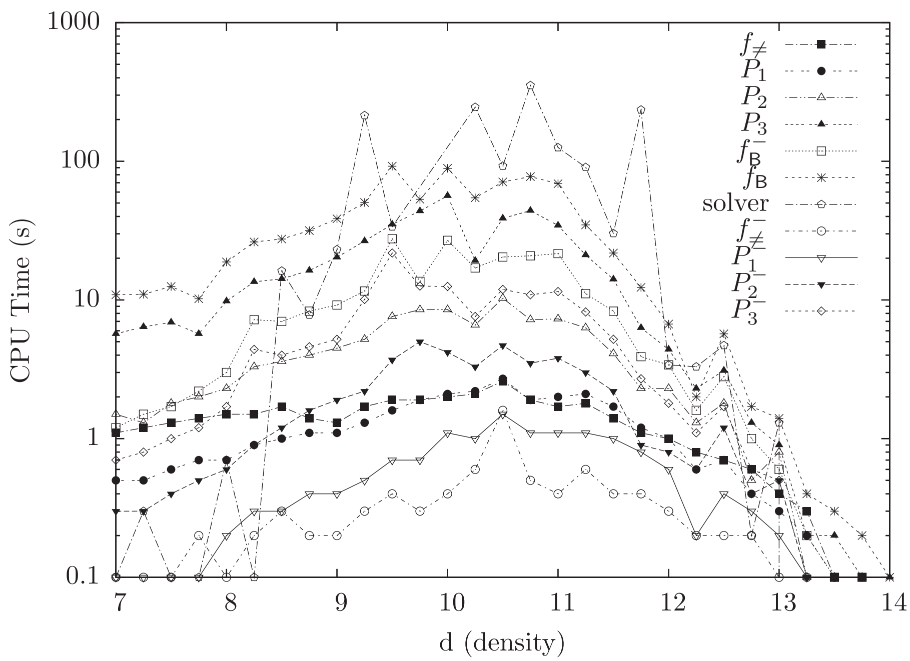


(b) CPU Time for $A(75, d, 6.5)$

Figure 4: Experimental results for series $A(75, d, 6.5)$.



(a) Filtering Strength for $A(100, d, 6.5)$



(b) CPU Time for $A(100, d, 6.5)$

Figure 5: Experimental results for series $A(100, d, 6.5)$.

a QCN: each node becomes a variable, and each edge becomes a constraint whose associated relation is given by the label of the edge. When there is no edge between two nodes in the graph, a universal constraint (i.e. a constraint whose associated relation is the total relation) is also introduced in the QCN. The set (or series) of QCNs that can be generated from n, d and s is denoted by $A(n, d, s)$. The experimental results presented in this section concern QCN instances from series $A(75, d, 6.5)$ and $A(100, d, 6.5)$ for d varying from 2 to 24 with a step of 0.25. For these series, the hardest instances are located in a region where the density ranges from 8 to 11: this is where a phase transition occurs between an under-constrained region where all instances are (almost surely) consistent and an over-constrained region where all networks are inconsistent. For each series, we generated 100 instances.

The main objective of our experimental study is to compare both the filtering strength and the time efficiency of some $\overset{\diamond}{f}$-algorithms (those based on consistencies introduced in previous sections). The first criterion used for our comparisons is the number of QCNs detected as inconsistent, within the phase transition. This informs us about the relative filtering strength[1] of different consistencies. Note that the exact number of inconsistent QCNs will be computed using a complete solving method: this represents the ideal filtering capability for a consistency. This method, called $solver$ afterwards, is the solver proposed in (Nebel 1996). Basically, it performs search by successively reducing each constraint relation to a tractable one (using a splitting of the initial relations) and maintaining the QCN closed under weak composition. In our context, we used the tractable sets of the Ord-Horn relations as split elements, and sought the best control parameters of $solver$ to solve our instances. The second criterion used for our comparisons is the CPU time (given in seconds) taken by $\overset{\diamond}{f}$-algorithms (and $solver$).

When enforcing $\overset{\diamond}{f}$-consistencies, we may decide to ignore universal constraints so as to limit the computation effort of the algorithms. This means that when there is a constraint between two variables v_i, v_j such that $C_{ij} = \mathsf{B}$ then no check on C_{ij} is performed by means of f. Pragmatically, for every mapping f, we can introduce a related so-called *reduced* mapping f^- defined as: $f^-(r) = f(r)$ if $r \neq \mathsf{B}$, and $f^-(r) = \{\mathsf{B}\}$ otherwise. Intuitively, we may expect to save time whereas limiting the loss of inferences due to the universal nature of these constraints.

Figures 4(a) and 5(a) show the filtering capabilities of various $\overset{\diamond}{f}$-consistencies on series $A(75, d, 6.5)$ and $A(100, d, 6.5)$, respectively. Setting the value 6.5 to s permits us to compare our experimental results with similar works; e.g. see (Westphal and Wölfl 2009). A first observation is that consistencies based on reduced mappings are quite close to unreduced ones. For $n = 75$ variables, this was so striking that we decided (for clarity reasons) to not plot the curves corresponding to reduced mappings (except for f_{B}^-). For $n = 100$ variables, a small difference is visible. This means that for a a given mapping f, the mapping f^- allows us to detect almost the same number of inconsistent QCNs. However, we conjecture that this is less and less true when n increases. A second observation is that $\overset{\diamond}{f_{\mathsf{B}}}$ (theoretically shown to be the strongest local consistency in $\overset{\diamond}{\mathcal{F}}$) is very effective as it almost detects all inconsistent QCNs (as identified by $solver$) from series $A(75, d, 6.5)$, and a lot of them from series $A(100, d, 6.5)$. Other consistencies stronger than $\diamond =\overset{\diamond}{f_\diamond}$ are, in order, P_3, P_2, P_1 and $\overset{\diamond}{f_{\neq}}$. Interestingly, there is even so a significant gap between $\overset{\diamond}{f_{\neq}}$ and $\diamond$, which motivates us to further study each of these new consistencies. Finally, note that the number of passes executed by the function df-closure is very limited (around 3.5 on average).

Figures 4(b) and 5(b) show the CPU time taken by the $\overset{\diamond}{f}$-algorithms on the same series. Note the use of a log scale on the y-axis in order to better distinguish between the behaviour of all algorithms. For $n = 75$ variables, the use of the strongest consistencies such as $\overset{\diamond}{f_{\mathsf{B}}}$, $\overset{\diamond}{f_{P_3}}$ and $\overset{\diamond}{f_{P_2}}$ involve a large overhead with respect to $solver$, but when reduced mappings are used the $\overset{\diamond}{f}$ algorithms are far faster. For $n = 100$ variables, $solver$ becomes clearly slower than all other algorithms, but recall that $solver$ performs a complete search whereas $\overset{\diamond}{f}$-algorithms are incomplete since they can only perform some inferences. However, it is fair to compare $solver$ and $\overset{\diamond}{f}$-algorithms on instances shown to be inconsistent by both approaches. This is the case for most of the instances of series $A(100, d, 6.5)$ with d around 11.75 or higher; see Figure 5(a). For such instances, algorithms such as $\overset{\diamond}{f_{P_1}}$ and $\overset{\diamond}{f_{\neq}}$ (and their reduced variants) are about two orders of magnitude faster than $solver$. Finally, $\overset{\diamond}{f_\diamond} = \diamond$ is clearly the fastest algorithm as it is usually enforced within 0.1s (we did not plot its CPU curves because this flattens the figures) but remember that it is far weaker than other introduced $\overset{\diamond}{f}$-consistencies as shown in Figures 4(a) and 5(a). Besides, $\overset{\diamond}{P_1^-}$ and $\overset{\diamond}{f_{\neq}^-}$ are also cheap to enforce.

To summarize, our (preliminary) experimentation shows how promising $\overset{\diamond}{f}$-consistencies may be, and in particular those based on reduced mappings that offer a good compromise between time overhead and filtering capability. Maintaining such consistencies during search is a perspective that we envision using a fast solver like GQR*.

Conclusion

In this paper, we have introduced the class of $\overset{\diamond}{f}$-consistencies for qualitative constraint networks. This class forms a complete lattice and contains original local consistencies (even when considering their CSP counterparts) such as $\overset{\diamond}{f_{\neq}}$, all being stronger than weak composition. Looking for the $\overset{\diamond}{f}$-consistency that is the most appropriate to solve hard instances (from different qualitative algebras) is a pragmatic perspective of this work. On the other hand, we may imagine additional new classes built from coverings where $\diamond$ is substituted by another local consistency. Studying the connections between all these consistencies and the problems of

[1] We could also assess the filtering strength of a given local consistency ϕ in terms of the number of base relations deleted when applying ϕ, but this information is closely related to our first criterion.

(global) consistency and minimality of QCNs is an exciting theoretical perspective.

References

Allen, J.-F. 1981. An interval-based representation of temporal knowledge. In *Proceedings of IJCAI'81*, 221–226.

Bennaceur, H., and Affane, M. 2001. Partition-k-AC: An efficient filtering technique combining domain partition and arc consistency. In *Proceedings of CP'01*, 560–564.

Bessiere, C.; Cardon, S.; Debruyne, R.; and Lecoutre, C. 2010. Efficient algorithms for singleton arc consistency. *Constraints* 15:to appear.

Boussemart, F.; Hemery, F.; Lecoutre, C.; and Sais, L. 2004. Boosting systematic search by weighting constraints. In *Proceedings of ECAI'04*, 146–150.

Condotta, J.-F.; Ligozat, G.; and Saade, M. 2007. Eligible and frozen constraints for solving temporal qualitative constraint networks. In *Proceedings of CP'07*, 806–814.

Condotta, J.-F.; Saade, M.; and Ligozat, G. 2006. A generic toolkit for n-ary qualitative temporal and spatial calculi. In *Proceedings of TIME'06*, 78–86.

Debruyne, R., and Bessiere, C. 1997. Some practical filtering techniques for the constraint satisfaction problem. In *Proceedings of IJCAI'97*, 412–417.

Gantner, Z.; Westphal, M.; and Wolfl, S. 2008. GQR a fast reasoner for binary qualitative constraint calculi. In *Proceedings of AAAI'08 Workshop on Spatial and Temporal Reasoning*.

Li, J.-J.; Huang, J.; and Renz, J. 2009. A divide-and-conquer approach for solving interval algebra networks. In *Proceedings of IJCAI'09*, 572–577.

Ligozat, G., and Renz, J. 2004. What is a qualitative calculus? a general framework. In *Proceedings of PRICAI*, 53–64.

Mackworth, A. 1977. Consistency in networks of relations. *Artificial Intelligence* 8(1):99–118.

Nebel, B. 1996. Solving hard qualitative temporal reasoning problems: Evaluating the efficienty of using the ORD-Horn class. In *Proceedings of ECAI'96*, 38–42.

Pham, D.-N.; Thornton, J.; and Sattar, A. 2006. Towards an efficient SAT encoding for temporal reasoning. In *Proceedings of CP'06*, 421–436.

Pujari, A.; Kumari, G.; and Sattar, A. 1999. INDU: an interval and duration network. In *Proceedings of AI'99*, 291–303.

Randell, D.; Cui, Z.; and Cohn, A. 1992. A spatial logic based on regions and connection. In *Proceeding of KR'92*, 165–176.

Renz, J., and Nebel, B. 2007. Qualitative spatial reasoning using constraint calculi. In *Handbook of Spatial Logics*. Springer. 161–215.

Renz, J. 1999. Maximal tractable fragments of the region connection calculus: A complete analysis. In *Proceedings of IJCAI'99*, 448–455.

Vilain, M.; Kautz, H.; and van Beek, P. 1990. Constraint propagation algorithms for temporal reasoning: a revised report. In *Readings in qualitative reasoning about physical systems*. Morgan Kaufmann Publishers Inc. 373–381.

Westphal, M., and Wölfl, S. 2009. Qualitative CSP, finite CSP and SAT: Comparing methods for qualitative constraint-based reasoning. In *Proceedings of IJCAI'09*, 628–633.