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Taxonomy of Improvement Operators and the Problem of Minimal Change

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Abstract

Improvement operators is a class of belief change operators that is a generalization of the usual class of iterated belief revision operators. The idea is to relax the success property, so the new information is not necessarily believed after the improvement, but to ensure that its plausibility has increased in the epistemic state. In this paper we explore this large classby defining several different subclasses. In particular, as minimal change is a hallmark of belief change, we study what are the operators that produce the minimal change among several subclasses.

Introduction

Belief change is a key task for any any rational agent. Modeling the evolution of the beliefs of an agent when he receives new pieces of information is the aim of belief revision. The predominant approach for modeling belief revision was proposed by Alchourrón, Gärdenfors and Makinson and is known as the AGM belief revision framework (Alchourrón, Gärdenfors, and Makinson 1985; Gärdenfors 1988; Katsuno and Mendelzon 1991).

This approach has been extended in order to cope with iterated belief revision. The main approach for iterated belief revision was proposed by Darwiche and Pearl (Darwiche and Pearl 1997) (see also (Booth and Meyer 2006; Jin and Thielscher 2007) for more recent developments and (Rott 2006) for an overview of the different operators). One of the main steps for addressing the iteration of the revision process was to abandon logical belief bases, because of their lack of expressive power (see e.g. (Herzig, Konieczny, and Perussel 2003)), for epistemic states.

In (Konieczny and Pino Pérez 2008) a generalization of iterated belief revision operators, called improvement operators, has been proposed. The idea is to define operators on epistemic states that have a less drastic behavior than iterated belief revision operators. One of the major requirements of belief revision operators is the so-called success postulate, that imposes that the new pieces of information must be believed after the change. This is clearly required for a lot of scenarios. But there are also some cases where we would like to take the information into account in a more cautious way. So with improvement operators the plausibility of the new piece of information is increased, but it is not necessarily believed after the change. There areother works on belief revision operators that do not satisfy the success postulate, that are called non-prioritized revisions (see for instance (Hansson 1997) for an overview). But none of these works define such increase of the plausibility of the new information as done by improvement operators.

One major hallmark of belief change is the principle of minimal change. For belief revision this means that we do not want to allow *any* change in the beliefs of the agent in order to allow the addition of the new piece of information, we want to have a *minimal* change where the only changes are the ones really required to allow the addition. The aim of this minimal change requirement is to keep as much as possible of the old beliefs of the agent.

In this paper we explore the operators that perform a minimal change among several subclasses of improvement operators.

The rest of the paper is organized as follows: we give the preliminaries in the first section. The second section is devoted to the introduction of improvement operators. In the third section we define some subclasses of improvement operators. The fourth section introduce our criterion of minimality. The three next sections study different classes of soft improvement operators. The proofs are in the Appendix.

Preliminaries

We consider a propositional language $\mathcal{L}$ defined from a finite set of propositional variables $\mathcal{P}$ and the standard connectives. Let $\mathcal{L}^*$ denote the set of consistent formulae of $\mathcal{L}$.

An interpretation ω is a total function from $\mathcal{P}$ to $\{0,1\}$. The set of all interpretations is denoted $\mathcal{W}$. An interpretation ω is a model of a formula $\phi \in \mathcal{L}$ if and only if it makes it true in the usual truth functional way. $[\![\alpha]\!]$ denotes the set of models of the formula α, i.e., $[\![\alpha]\!] = \{\omega \in \mathcal{W} \mid \omega \models \alpha\}$. When $\{w_1, .., w_n\}$ is a set of models we denote by $\varphi_{w_1,..,w_n}$ a formula such that $[\![\varphi_{w_1,..,w_n}]\!] = \{w_1, .., w_n\}$.

We will use epistemic to represent the beliefs of the agent, as usual in iterated belief revision (Darwiche and Pearl 1997). An epistemic state Ψ represents the current beliefs of the agent, but also additional conditional information guiding the revision process (usually represented by a pre-order on interpretations, a set of conditionals, a sequence of formulae, etc). Let $\mathcal{E}$ denote the set of all epistemic states. A projection function $B : \mathcal{E} \longrightarrow \mathcal{L}^*$ associates to each epistemic state Ψ a consistent formula $B(\Psi)$, that represents the current beliefs of the agent in the epistemic state Ψ[1].

For simplicity purpose we will only consider in this paper consistent epistemic states and consistent new information. Thus, we consider change operators as functions $\circ$ mapping an epistemic state and a consistent formula into a new epistemic state, *i.e.* in symbols, $\circ : \mathcal{E} \times \mathcal{L}^* \longrightarrow \mathcal{E}$. The image of a pair (Ψ, α) under $\circ$ will be denoted by $\Psi \circ \alpha$.

We adopt the following notations:

- $\Psi \circ^n \alpha$ defined as: $\Psi \circ^1 \alpha = \Psi \circ \alpha$, $\quad \Psi \circ^{n+1} \alpha = (\Psi \circ^n \alpha) \circ \alpha$
- $\Psi \star \alpha = \Psi \circ^n \alpha$, where n is the first integer such that $B(\Psi \circ^n \alpha) \vdash \alpha$.

Note that $\star$ is undefined if there is no n such that $B(\Psi \circ^n \alpha) \vdash \alpha$, but for all operators $\circ$ considered in this work, the associated operator $\star$ will be total, that is for any pair Ψ, α there will exist n such that $B(\Psi \circ^n \alpha) \vdash \alpha$ (see postulate **(I1)** below).

Finally, let $\leq$ be a a total pre-order, i.e a reflexive ($x \leq x$), transitive ($(x \leq y \wedge y \leq z) \rightarrow x \leq z$) and total ($x \leq y \vee y \leq x$) relation over $\mathcal{W}$. Then the corresponding strict relation $<$ is defined as $x < y$ iff $x \leq y$ and $y \not\leq x$, and the corresponding equivalence relation $\simeq$ is defined as $x \simeq y$ iff $x \leq y$ and $y \leq x$. We denote $w \ll w'$ when $w < w'$ and there is no w'' such that $w < w'' < w'$. We also use the notation $\min(A, \leq) = \{w \in A \mid \nexists w' \in A \; w' < w\}$. The set of total pre-orders will be denoted $\mathcal{TP}$.

When a set $\mathcal{W}$ is equipped with a total pre-order $\leq$, then this set can be split in different levels, that gives the ordered sequence of its equivalence classes $\mathcal{W} = \langle S_0, \ldots S_n \rangle$. So $\forall x, y \in S_i \; x \simeq y$. We say in that case that x and y are at the same level of the pre-order. And $\forall x \in S_i \; \forall y \in S_j$ $i < j$ implies $x < y$. We say in this case that x is in a lower level than y. We extend straightforwardly these definitions to compare subsets of equivalence classes, i.e if $A \subseteq S_i$ and $B \subseteq S_j$ then we say that A is in a lower level than B if $i < j$.

Improvement operators

We recall in this section the definition of improvement operators

Definition 1 *An operator $\circ$ is said to be a weak improvement operator if it satisfies (I1) to (I6):*
(I1) *There exists n such that* $B(\Psi \circ^n \alpha) \vdash \alpha$
(I2) *If* $B(\Psi) \wedge \alpha \not\vdash \bot$, *then* $B(\Psi \star \alpha) \equiv B(\Psi) \wedge \alpha$
(I3) *If* $\alpha \nvdash \bot$, *then* $B(\Psi \circ \alpha) \nvdash \bot$
(I4) *For any positive integer n if* $\alpha_i \equiv \beta_i$ *for all* $i \leq n$ *then* $B(\Psi \circ \alpha_1 \circ \cdots \circ \alpha_n) \equiv B(\Psi \circ \beta_1 \circ \cdots \circ \beta_n)$
(I5) $B(\Psi \star \alpha) \wedge \beta \vdash B(\Psi \star (\alpha \wedge \beta))$
(I6) *If* $B(\Psi \star \alpha) \wedge \beta \not\vdash \bot$, *then* $B(\Psi \star (\alpha \wedge \beta)) \vdash B(\Psi \star \alpha) \wedge \beta$

Postulates (I2-I6) are very close to postulates (R*2-R*6) of usual belief revision operators (Alchourrón, Gärdenfors, and Makinson 1985; Katsuno and Mendelzon 1991; Darwiche and Pearl 1997). The important difference lies in postulate (I1) that is weaker that the usual success postulate (R*1). So postulates (I2-I6) hold for sequences of weak improvements (whereas for revision they require only one step).

Definition 2 *A weak improvement operator is said to be an improvement operator if it satisfies (I7) to (I9)*[2]*:*
(I7) *If* $\alpha \vdash \mu$ *then* $B((\Psi \circ \mu) \star \alpha) \equiv B(\Psi \star \alpha)$
(I8) *If* $\alpha \vdash \neg\mu$ *then* $B((\Psi \circ \mu) \star \alpha) \equiv B(\Psi \star \alpha)$
(I9) *If* $B(\Psi \star \alpha) \not\vdash \neg\mu$ *then* $B((\Psi \circ \mu) \star \alpha) \vdash \mu$

These postulates correspond to the postulates for iterated revision (Darwiche and Pearl 1997; Jin and Thielscher 2007; Booth and Meyer 2006). Postulates (I7) and (I8) correspond to postulates (C1) and (C2) of (Darwiche and Pearl 1997), and postulate (I9) correspond to postulate (P) of (Jin and Thielscher 2007; Booth and Meyer 2006). As for the basic postulates, the difference lies in the fact that they hold for sequences of improvements.

Let us now recall the corresponding representation theorems (Konieczny and Pino Pérez 2008). Let us first define strong faithful assignements.

Definition 3 *A function $\Psi \mapsto \leq_\Psi$ that maps each epistemic state Ψ to a total pre-order on interpretations $\leq_\Psi$ is said to be a* strong faithful assignment *if and only if:*

1. *If* $w \models B(\Psi)$ *and* $w' \models B(\Psi)$, *then* $w \simeq_\Psi w'$
2. *If* $w \models B(\Psi)$ *and* $w' \not\models B(\Psi)$, *then* $w <_\Psi w'$
3. *For any positive integer n if* $\alpha_i \equiv \beta_i$ *for any* $i \leq n$ *then* $\leq_{\Psi \circ \alpha_1 \circ \cdots \circ \alpha_n} = \leq_{\Psi \circ \beta_1 \circ \cdots \circ \beta_n}$

So now we can state the representation theorem for weak improvement operators:

Theorem 1 *A change operator $\circ$ is a weak improvement operator if and only if there exists a strong faithful assignment that maps each epistemic state Ψ to a total pre-order on interpretations $\leq_\Psi$ such that*

$$[\![B(\Psi \star \alpha)]\!] = \min([\![\alpha]\!], \leq_\Psi) \quad (1)$$

Let us now give the representation theorem for improvement operators:

Definition 4 *Let $\circ$ be a weak improvement operator and $\Psi \mapsto \leq_\Psi$ its corresponding strong faithful assignment. The assignment will be called a gradual assignment if the properties S1, S2 and S3 are satisfied:*

(S1) *If* $w, w' \in [\![\alpha]\!]$ *then* $w \leq_\Psi w' \Leftrightarrow w \leq_{\Psi \circ \alpha} w'$
(S2) *If* $w, w' \in [\![\neg\alpha]\!]$ *then* $w \leq_\Psi w' \Leftrightarrow w \leq_{\Psi \circ \alpha} w'$

[1] As in most works on iterated revision, we have chosen this very general and abstract framework to define epistemic states (just objects Ψ and their logical belief $B(\Psi)$) because of its simplicity and its flexibility to capture many concrete representations of epistemic states. For a formal definition of epistemic states see (Benferhat et al. 2000).

[2] For coherence reasons we change the names of the classes of operators with respect to (Konieczny and Pino Pérez 2008), where "Improvement operators" was the class of operators satisfying (I1-I11).

(S3) *If* $w \in [\![\alpha]\!], w' \in [\![\neg\alpha]\!]$ *then* $w \leq_\Psi w' \Rightarrow w <_{\Psi\circ\alpha} w'$

Theorem 2 *A change operator* $\circ$ *is an improvement operator if and only if there exists a gradual assignment such that*

$$[\![B(\Psi \star \alpha)]\!] = \min([\![\alpha]\!], \leq_\Psi)$$

This theorem is a direct consequence of the theorem of (Konieczny and Pino Pérez 2008) (see Theorem 4 below) when removing (I10) and (I11) from the set of postulates and the corresponding conditions (S4) and (S5) from the assignment.

Belief Revision

In order to show that improvement operators are a generalization of iterated belief revision operators consider the following usual *success*postulate:

(R*1) $B(\Psi \circ \alpha) \vdash \alpha$

Note that this postulate is the one that makes a distinction between usual belief revision operators and non-prioritized revision operators. Note also that (R*1) is a particular case of (I1) where $n = 1$.

Proposition 1 *If* $\circ$ *is a weak improvement operator (i.e. it satisfies (I1-I6)) that satisfies (R*1), then it is a AGM/DP revision operator (i.e. it satisfies (R*1-R*6) of (Darwiche and Pearl 1997)).*

Proposition 2 *If* $\circ$ *is an improvement operator (i.e. it satisfies (I1-I9)) that satisfies (R*1), then it is an admissible revision operator (i.e. it satisfies (R*1-R*6) and (C1-C4) of (Darwiche and Pearl 1997) and property (P) of (Booth and Meyer 2006; Jin and Thielscher 2007)).*

Soft Improvement

As shown in the last section usual iterated belief revision operators are a special case of weak improvement operators. This is a well known subclass. But the family of weak improvement operators is much larger than that, and we want to explore further some of its subclasses. Belief revision operators are the weak improvement operators that produce the biggest change in the epistemic state. We investigate here the opposite of the weak improvement operators spectrum, i.e. operators of soft improvement, that produce the smallest change. We propose some subclasses of soft improvement operators by providing additional postulates and the corresponding representation theorems.

So we are interested in soft improvement operators defined below:

Definition 5 *An improvement operator is said to be a soft improvement operator if it satisfies the following postulate*

(I10) *If* $B(\Psi \star \alpha) \vdash \neg\mu$ *then* $B((\Psi \circ \mu) \star \alpha) \nvdash \mu$

This postulate says literally that a formula μ that is rejected by the agent after several (soft) improvements by α, can not be accepted after a soft improvement by μ and several improvements by α.

In fact, the only admissible change of status is that the formula μ that is rejected by the agent after several improvements by α can become undetermined after a soft improvement by μ and several improvements by α. At least another step of soft improvement by μ will be required in order to accept this formula by the agent after several improvements by α. This motivates the name "soft" improvement.

We can give a representation theorem for soft improvement operators:

Definition 6 *Let* $\circ$ *be a weak improvement operator and* $\Psi \mapsto \leq_\Psi$ *its corresponding strong faithful assignment. The assignment will be called a soft gradual assignment if it is a gradual assignment and if the following property holds:*

(S4) *If* $w \in [\![\alpha]\!], w' \in [\![\neg\alpha]\!]$ *then* $w' <_\Psi w \Rightarrow w' \leq_{\Psi\circ\alpha} w$

Theorem 3 *A change operator* $\circ$ *is a soft improvement operator if and only if there exists a soft gradual assignment such that*

$$[\![B(\Psi \star \alpha)]\!] = \min([\![\alpha]\!], \leq_\Psi)$$

Again, this theorem is a direct consequence of the theorem of (Konieczny and Pino Pérez 2008) (see Theorem 4 below) when removing (I11) from the set of postulates and the corresponding condition (S5) from the conditions of the assignment.

Another explanation of soft improvement can be given thanks to this representation theorem. Thinking the information in the epistemic states is ranked by degrees of belief (in a discrete scale), the degree of belief of μ after a soft improvement by μ can be at most the degree immediately inferior (more plausible) to μ in the original epistemic state.

There are several operators in the class of soft improvement, that have quite different behaviors. So we will try to identify some specific behaviors for soft improvement operators, and to study what are the minimal change operators in each of these classes.

There is one important difference between soft improvement operators: some of them can be defined locally, by looking only at the information of similar plausibility, while some of them are defined globally, i.e. they require to look at the whole epistemic state. We call this locality property *modularity*, and we express it in a general manner on strong faithful assignments.

Let f be a boolean function, i.e. $f : \{0,1\}^n \longrightarrow \{0,1\}$, then the expressions used as inputs of n has to be understood as boolean conditions (i.e. $x \leq y$ returns 1 if the relation is true and 0 if it is false).

Definition 7 *Let* $\circ$ *be a weak improvement operator. Let* $\Psi \mapsto \leq_\Psi$ *be the associated strong faithful assignment.*

Let $N_w = \{w' : w' \simeq_\Psi w\}$ *and* $N_{w+1} = \{w' : w \ll_\Psi w'\}$. *We say that the assignment* $\Psi \mapsto \leq_\Psi$ *is* modular *iff there exists a boolean function* $f : \{0,1\}^6 \longrightarrow \{0,1\}$ *such that :*

- *for any* w, *if* $w', w'' \in N_w \cup N_{w+1}$ *then*
  $w' \leq_{\Psi\circ\alpha} w'' = f(w' \leq_\Psi w'', w'' \leq_\Psi w', [\![\alpha]\!] \cap N_w = \emptyset, [\![\alpha]\!] \cap N_{w+1} = \emptyset, w' \in [\![\alpha]\!], w'' \in [\![\alpha]\!])$
- $\leq_{\Psi\circ\alpha}$ *is completely determined by the previous equality and transitivity of the relation.*

A weak improvement operator is modular iff its associated strong faithful assignment is modular.

So this modularity property states that the plausibility relation between two interpretations in two consecutive levels of plausibility after the improvement is a function of (i.e. the only pieces of information required are): the relation between the two interpretations before the improvement (the two first inputs); the fact that there are (or not) models of the new piece of information in the two levels considered (the third and fourth inputs); and the fact that the models considered satisfy the new piece of information or not (the last two inputs).

Intuitively this property expresses the fact that for knowing the change of plausibility of interpretations it is enough to look at two consecutive levels.

Definition 8 *A soft improvement operator is* modular *if its associated soft gradual assignment is modular.*

Thus, the class of modular soft improvement operators is formed by the weak improvement operators which are soft and modular at the same time. We will see below that this class of operators have an easy and compact description.

One can state another property that identify an important difference on the behavior of soft improvement operators:

Definition 9 *A soft gradual assignment is a* systematic enhancement *if:*
(Sse) *If $w \models \neg\alpha$, $w' \models \alpha$ and $w \ll_\Psi w'$, then $w \not\ll_{\Psi\circ\alpha} w'$*

This property states that (the plausibility of) every model of the new piece of information α is systematically improved. That means that if a model of the negation of α was just a little more plausible before the improvement than a model of α, then it is no longer the case after the improvement (the model of α will be at least as plausible as the model of the negation).

So we will identify three different classes of soft improvement operators, from the most general one to the most specific one:

Soft Improvement (SI) operators that correspond to soft gradual assignments.

Modular Soft Improvement (MSI) operators that correspond to modular soft gradual assignments.

Systematic Soft Improvement (SSI) operators that correspond to systematic enhancement assignments.

In the following sections we will study the minimality of operators in these classes. But we have to define first what is our minimality criterion.

Minimality

One major objective of belief change theories is to define operators that produces minimal change in the beliefs of the agent. This is a natural requirement because beliefs are valuable, so we want to keep as much as possible the old beliefs of the agent (no unnecessary forgetting), and because we want the agent to be rational by not adding exotic beliefs (no unjustified addition).

For improvement operators, as the representation theorem states that each operator corresponds to a gradual assignment (and if we consider this representation as the canonical one), we can consider these operators as transitions between total pre-orders. In this case there is a natural measure of change: the Kemeny distance (Kemeny 1959) between the old pre-order (the pre-order associated to the old epistemic state) and the new one.

Definition 10 *The Kemeny Distance is the function $d_K : \mathcal{TP} \times \mathcal{TP} \longrightarrow \mathbb{N}$ defined as: given $\leq_1$, $\leq_2$ two total pre-orders, $d_K(\leq_1, \leq_2)$ is the cardinal of the symmetrical difference of the pre-orders,* i.e. *the number of elements in $\leq_1$ which are not in $\leq_2$ plus the number of elements in $\leq_2$ which are not in $\leq_1$. In symbols we have*

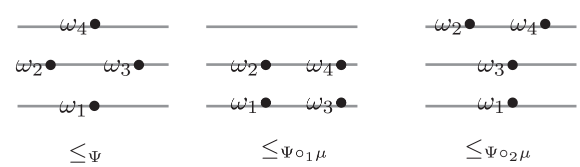
$$d_K(\leq_1, \leq_2) = |(\leq_1 \setminus \leq_2) \cup (\leq_2 \setminus \leq_1)|$$

Definition 11 *Let $\circ_1$ and $\circ_2$ be two improvement operators. We say that $\circ_1$* produces less change than *$\circ_2$ if for any epistemic state Ψ and any formula μ:*

$$d_K(\leq_\Psi, \leq_{\Psi\circ_1\mu}) \leq d_K(\leq_\Psi, \leq_{\Psi\circ_2\mu})$$

So this definition means that an operator produces less change than another one if on all possible improvements the first one produces less change (with respect to Kemeny distance) than the second one.

Example 1 *Suppose $\mathcal{W} = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Consider two improvement operators $\circ_1$ and $\circ_2$. Let Ψ be an epistemic state where $\leq_\Psi$ is its respective pre-order given by the gradual assignment (see the figure below). Let μ be a formula such that $[\![\mu]\!] = \{\omega_3, \omega_4\}$. Suppose that the pre-orders $\leq_{\Psi\circ_1\mu}$ and $\leq_{\Psi\circ_2\mu}$ are the results of the improvement of Ψ by the new information μ with respect the operators $\circ_1$ and $\circ_2$ respectively.*



$d_K(\leq_\Psi, \leq_{\Psi\circ_1\mu}) = 3$ and $d_K(\leq_\Psi, \leq_{\Psi\circ_2\mu}) = 2$. So on this example $\circ_2$ produces less change than $\circ_1$ (but to conclude that $\circ_2$ produces less change than $\circ_1$ this has to be checked for all cases).

Systematic Soft Improvement

Before giving the postulate which characterizes the behavior of operators in this class, we need some notations.

Definition 12 *Let $\circ$ be a change operator satisfying (I1). Let α, β and Ψ be two formulae and an epistemic state respectively. We say that α is below β with respect to Ψ, given $\circ$, denoted $\alpha \prec^\circ_\Psi \beta$ (or simply $\alpha \prec_\Psi \beta$ if there is no ambiguity about $\circ$) if and only if $\alpha \nvdash \bot$, $\beta \nvdash \bot$, $B(\Psi\star\alpha) \vdash B(\Psi\star(\alpha\vee\beta))$ and $B(\Psi\star\beta) \nvdash B(\Psi\star(\alpha\vee\beta))$.*

The pair (α, β) is Ψ-consecutive, denoted $\alpha \prec\!\!\prec^\circ_\Psi \beta$ (or simply $\alpha \prec\!\!\prec_\Psi \beta$ if there is no ambiguity about $\circ$) if and only if $\alpha \prec_\Psi \beta$ and there is no formula γ such that $\alpha \prec_\Psi \gamma \prec_\Psi \beta$.

So now, let us introduce an additional postulate in order to characterize operators of systematic soft improvement:

(I11) If $B(\Psi \star \alpha) \vdash \neg\mu$ and $\alpha \prec\!\!\!\prec_\Psi \alpha \wedge \mu$ then $B((\Psi \circ \mu) \star \alpha) \nvdash \neg\mu$

And we can state a corresponding representation theorem:

Theorem 4 *A change operator $\circ$ is a systematic soft improvement operator if and only if there exists a systematic enhancement such that*

$$[\![B(\Psi \star \alpha)]\!] = \min([\![\alpha]\!], \leq_\Psi)$$

This is essentially the main Theorem in (Konieczny and Pino Pérez 2008). The only difference is that condition (Sse) is used in the assignement instead of condition (S5) in (Konieczny and Pino Pérez 2008):

(S5) If $w \in [\![\alpha]\!], w' \in [\![\neg\alpha]\!]$ then $w' \ll_\Psi w \Rightarrow w \leq_{\Psi\circ\alpha} w'$

Clearly (Sse) and (S5) are equivalent in the presence of others (S1-S4) conditions.

As explained in (Konieczny and Pino Pérez 2008), there is only one operator of systematic soft improvement (once the pre-order associated to the initial epistemic state is fixed). We will call this operator one-improvement, and denote it $\odot$.

The fact that this is the only operator of this class implies straightforwardly that it is the one that produces the minimal change.

So condition (Sse) can be considered as very strong, since it defines a class of soft improvement operators that contains only one operator. But, first we consider that (Sse) is very sensible, so it is interesting to study its consequences. And secondly recall that the class of weak improvement operators is wider than soft improvement operators, and (Sse) can also prove valuable to discriminate operators in other classes. For instance we can remark that it allows to discriminate belief revision operators, since Boutilier's natural revision (Boutilier 1996) and Darwiche and Pearl $\bullet$ operator (Darwiche and Pearl 1997) do not satisfy (Sse), while Nayak's lexicographic operator (Nayak 1994; Konieczny and Pino Pérez 2000) does.

Modular Soft Improvement

Modular soft improvement are operators that can be defined locally, by looking at beliefs of similar plausibility. The following syntactical postulates try to capture this idea.

(H1) If $B(\Psi \star \alpha) \vdash \neg\mu, \alpha \prec\!\!\!\prec_\Psi \alpha \wedge \mu$ and $\neg\exists\beta(\beta \vdash \neg\mu$ and $\alpha \prec\!\!\!\prec_\Psi \beta)$, then $B((\Psi \circ \mu) \star \alpha) \nvdash \neg\mu$

This postulate means that when the revision (i.e. sequence of improvements until success) by α implies the negation of μ, if μ is just a little less plausible than its negation given α, then an improvement by μ will be enough to remove its negation from the beliefs of the agent. Note that this postulate is weaker than (I11).

(H2) If $B(\Psi \star \alpha) \vdash \neg\mu$, $\alpha \prec\!\!\!\prec_\Psi \alpha \wedge \mu$ and $\exists\beta(\beta \vdash \neg\mu$ and $\alpha \prec\!\!\!\prec_\Psi \beta)$, then $B((\Psi \circ \mu) \star \alpha) \vdash \neg\mu$

This postulate is very close from (H1), and deals with the case where the revision (i.e. sequence of improvements until success) by α implies the negation of μ, but μ and $\neg\mu$ are both a little less plausible than $\neg\mu$, then an improvement by μ will not be enough to remove its negation from the beliefs of the agent.

Definition 13 *A soft improvement operator which satisfies (H1) and (H2) is called a half improvement operator.*

We can also define these operators semantically:

Definition 14 *Let $\circ$ be a soft improvement operator and $\Psi \mapsto \leq_\Psi$ its corresponding soft gradual assignment. The assignment will be called a half gradual assignment if the following properties (SH1) and (SH2) are satisfied:*

(SH1) *If $\omega \in [\![\mu]\!], \omega' \in [\![\neg\mu]\!], \omega' \ll_\Psi \omega$ and $\nexists\omega'' \in [\![\neg\mu]\!]$ such that $\omega'' \simeq_\Psi \omega$, then, $\omega \leq_{\Psi\circ\mu} \omega'$.*

(SH2) *If $\omega \in [\![\mu]\!], \omega' \in [\![\neg\mu]\!], \omega' \ll_\Psi \omega$ and $\exists\omega'' \in [\![\neg\mu]\!]$ such that $\omega'' \simeq_\Psi \omega$ then, $\omega' <_{\Psi\circ\mu} \omega$.*

Note that both (SH1) and (SH2) use only information on the new formula, the old relation $\leq_\Psi$ between the two interpretations, and the interpretations that was at the same level of ω. This means that the half-gradual assignement is a modular assignement.

We can now state the representation theorem:

Theorem 5 *A change operator $\circ$ is a half improvement operator if and only if there exists a half gradual assignment such that*

$$[\![B(\Psi \star \alpha)]\!] = \min([\![\alpha]\!], \leq_\Psi)$$

In fact, just as for one-improvement we can prove that:

Proposition 3 *Once the pre-order associated to the first epistemic state is fixed, there is a unique half-improvement operator. Let us denote $\oslash$ this operator.*

Half-improvement is not the only operator in the class of modular soft improvement. Actually, in the class of modular soft improvement operators there are only two operators: the half and the one operator:

Theorem 6 *The class of modular soft improvement operators is exactly the set $\{\oslash, \odot\}$.*

The proof follows easily of the the observation that the two columns in Table 1 are the only possibilities explaining the behavior of a modular soft improvement operator and they are exactly the $\odot$ and $\oslash$ operators respectively.

We can show that half-improvement produces less changes than one-improvement:

Proposition 4 *Let Ψ be an epistemic state (a total pre-order). Then for all formula μ,*

$$d_K(\leq_\Psi, \leq_{\Psi\oslash\mu}) \leq d_K(\leq_\Psi, \leq_{\Psi\odot\mu})$$

That is, the operator $\oslash$ produces less changes than the operator $\odot$.

Actually, with Proposition 4 and Theorem 6 we have the following result:

Corollary 1 *Half-improvement operator is the minimal operator in the class of modular soft improvement operators.*

	$\odot$	$\oslash$	$\oplus$
$w \in [\![\alpha]\!]$ $w' \in [\![\alpha]\!]$	$w \leq_\Psi w' \Leftrightarrow w \leq_{\Psi\odot\alpha} w'$	$w \leq_\Psi w' \Leftrightarrow w \leq_{\Psi\oslash\alpha} w'$	$w \leq_\Psi w' \Leftrightarrow w \leq_{\Psi\oplus\alpha} w'$
$w \in [\![\neg\alpha]\!]$ $w' \in [\![\neg\alpha]\!]$	$w \leq_\Psi w' \Leftrightarrow w \leq_{\Psi\odot\alpha} w'$	$w \leq_\Psi w' \Leftrightarrow w \leq_{\Psi\oslash\alpha} w'$	$w \leq_\Psi w' \Leftrightarrow w \leq_{\Psi\oplus\alpha} w'$
$w \in [\![\alpha]\!]$ $w' \in [\![\neg\alpha]\!]$	$w <_\Psi w' \Leftrightarrow w <_{\Psi\odot\alpha} w'$ $w \simeq_\Psi w' \Rightarrow w <_{\Psi\odot\alpha} w'$ $w' \lessdot_\Psi w \ \Rightarrow w' <_{\Psi\odot\alpha} w$ $w' \ll_\Psi w \ \Rightarrow w' \simeq_{\Psi\odot\alpha} w$	$w <_\Psi w' \Leftrightarrow w <_{\Psi\oslash\alpha} w'$ $w \simeq_\Psi w' \Rightarrow w <_{\Psi\oslash\alpha} w'$ $w' \lessdot_\Psi w \ \Rightarrow w' <_{\Psi\oslash\alpha} w$ $w' \ll_\Psi w \ \Rightarrow \begin{cases} w' \simeq_{\Psi\oslash\alpha} w & \text{if } \ddot{w} \\ w' \ll_{\Psi\oslash\alpha} w & \text{if } \neg(\ddot{w}) \end{cases}$	$w <_\Psi w' \Leftrightarrow w <_{\Psi\oplus\alpha} w'$ $w \simeq_\Psi w' \Rightarrow w <_{\Psi\oplus\alpha} w'$ $w' \lessdot_\Psi w \ \Rightarrow w' <_{\Psi\oplus\alpha} w$ $w' \ll_\Psi w \ \Rightarrow \begin{cases} w' \simeq_{\Psi\oplus\alpha} w & \text{if } \ddot{\alpha} \\ w' \ll_{\Psi\oplus\alpha} w & \text{if } \neg(\ddot{\alpha}) \end{cases}$

Table 1: From $\leq_\Psi$ to $\leq_{\Psi\circ\alpha}$. We note $w \lessdot w'$ when $w < w'$ and $w \not\ll w'$. $\ddot{w}$ is true when $\nexists w'' \in [\![\neg\alpha]\!]\ w'' \simeq_\Psi w$. $\ddot{\alpha}$ means that α is separated in Ψ.

Looking for the Best Soft Improvement

Now we move to the general class of soft improvement operators. The fact of not satisfying modularity allows to define much more different operators. This allows also to define an interesting soft improvement operator producing the smallest change under certain conditions.

In order to simplify the presentation of the postulates we introduce the following definition:

Definition 15 *μ is separated in Ψ iff $\forall\beta(B(\Psi \star \beta) \vdash \mu$ or $B(\Psi \star \beta) \vdash \neg\mu)$.*

This definition of separation of a formula in an epistemic state means that any revision (and improvement) of this epistemic state will always give epistemic states that are informed about this formula (i.e. the formula or its negation can be inferred).

Definition 16 *A soft improvement operator which satisfies the following two postulates is called a best improvement operator*

(B1) *If μ is separated in Ψ, $B(\Psi \star \alpha) \vdash \neg\mu$ and $\alpha \not\ll_\Psi \alpha \wedge \mu$, then $B((\Psi \circ \mu) \star \alpha) \nvdash \neg\mu$*

(B2) *If μ is not separated in Ψ and $B(\Psi \star \alpha) \vdash \neg\mu$, then $B((\Psi \circ \mu) \star \alpha) \vdash \neg\mu$*

Postulate (B1) is close to postulates (H1) and (I11), but it holds only when the formula is separated in the epistemic state.

Postulate (B2) states that, when the formula is not separated in the epistemic state (which is the general case), the change is the same one than with (H2).

We can give a semantical counterpart to these postulates.

Definition 17 *μ is s-separated in $\leq_\Psi$ iff $\nexists\omega_1 \in [\![\mu]\!], \omega_2 \in [\![\neg\mu]\!]$ s.t. $\omega_1 \simeq_\Psi \omega_2$*

Definition 18 *Let $\circ$ be a soft improvement operator and $\Psi \mapsto \leq_\Psi$ its corresponding soft gradual assignment. The assignment will be called a best gradual assignment if the following properties*

(SB1) *If μ is s-separated in $\leq_\Psi$, $\omega \in [\![\mu]\!]$, $\omega' \in [\![\neg\mu]\!]$ and $\omega' \ll_\Psi \omega$ then $\omega \leq_{\Psi\circ\mu} \omega'$.*

(SB2) *If μ is not s-separated in $\leq_\Psi$, $\omega \in [\![\mu]\!], \omega' \in [\![\neg\mu]\!]$ and $\omega' <_\Psi \omega$ then $\omega' <_{\Psi\circ\mu} \omega$.*

The two notions of separation are clearly related:

Lemma 1 *Let $\circ$ be a weak improvement operator. Then, μ is separated in Ψ iff μ is s-separated in $\leq_\Psi$.*

Let us give now the corresponding representation theorem.

Theorem 7 *A change operator $\circ$ is a best improvement operator if and only if there exists a best gradual assignment such that*

$$[\![B(\Psi \star \alpha)]\!] = \min([\![\alpha]\!], \leq_\Psi)$$

And, as for one-improvement and half-improvement, we can show that there is only one best-improvement operator:

Proposition 5 *Once the pre-order associated to the first epistemic state is fixed, there is a unique best-improvement operator. We denote this operator by $\oplus$.*

These uniqueness results are important since few change operators are axiomatically defined (usual characterizations in belief revision define families of operators).

Let us now turn to the minimality issue.

Proposition 6 *Let Ψ be an epistemic state (a total pre-order). Then for all formula μ,*

$$d_K(\leq_\Psi, \leq_{\Psi\oplus\mu}) \leq d_K(\leq_\Psi, \leq_{\Psi\oslash\mu})$$

That is, the operator $\oplus$ produces less changes than the operator $\oslash$.

As a corollary from the previous propositions we have the following result:

Proposition 7 *Among the operators $\odot$, $\oslash$ and $\oplus$ the operator $\oplus$ is the operator that produces minimal change.*

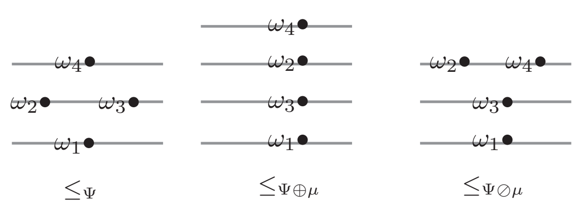
It is easy to figure out soft-improvement operators that produces less change than best-improvement. In fact the soft-improvement operator that produces the smallest change is the one that increases the plausibility of only one level of models of the new formula (this level is not randomly chosen, but is the one which produces the less change for the Kemeny distance). It seems to us that defining this minimal change operator is not of great interest, since this operator will not have a clear meaning from a logical point of view. It is of no use to look at absolute minimization if it costs too many logical properties (recall that Boutilier's natural revision operator (Boutilier 1996), although achieving the minimal change for a belief revision operator, make it at a price of bad logical properties (Darwiche and Pearl 1997)).

So, amongst improvement operators that do not add arbitrary choices in the choice of the models of the new information to be improved, best-improvement is the one that produces the smallest change:

Proposition 8 *Best-improvement operator is the soft improvement operator satisfying (B1) that produces the smallest change.*

Example 1 shows that in some cases $\oslash$ produces strictly less changes than $\odot$ ($\circ_2$ was $\oslash$ and $\circ_1$ was $\odot$). The following example shows that in some cases $\oplus$ produces strictly less changes than $\oslash$.

Example 2 *Suppose $\mathcal{W} = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Consider the improvement operators $\oplus$ and $\oslash$. Let Ψ be an epistemic state where $\leq_\Psi$ is his respective pre-order given by the gradual assignment (see the figure below). Let be μ a formula such that $[\![\mu]\!] = \{\omega_3, \omega_4\}$. The pre-orders $\leq_{\Psi\odot\mu}$ and $\leq_{\Psi\oslash\mu}$ are the results of the improvement of Ψ by the new information μ with respect the operators $\oplus$ and $\oslash$ respectively.*



It is not hard to see that $d_K(\leq_\Psi, \leq_{\Psi\oplus\mu}) = 1$ and that $d_K(\leq_\Psi, \leq_{\Psi\oslash\mu}) = 2$.

Table 1 summarizes the behavior of operators $\odot$, $\oslash$ and $\oplus$.

Example

We provide some examples of improvements in this section, in order to illustrate the behavior (and the differences) of one-improvement, half-improvement and best-improvement.

Figure 1 shows how three epistemic states, whose associated pre-orders are $\leq_{\Psi_1}, \leq_{\Psi_2}, \leq_{\Psi_3}$, are changed through the three soft-improvement operators studied in the previous sections (one-improvement $\odot$, half-improvement $\oslash$, and best-improvement $\oplus$).

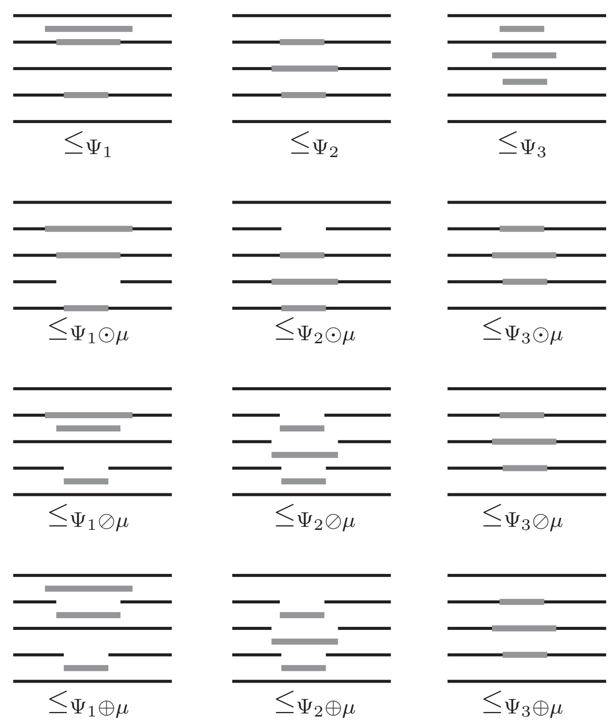


Figure 1: Examples of Soft Improvements

Interpretations are not represented on the figures, we just represent the "levels" where the interpretation are located. Gray lines represent the new formula μ, so models of this formula are located on these gray levels. Black lines represent the levels with models of $\neg\mu$.

In the $\leq_{\Psi_3}$ case, the three operators lead to the same result. This is the case where μ is separated in Ψ_3.

The $\leq_{\Psi_2}$ case shows a situation where every model of the new formula μ is equivalent to a model of its negation. In this case half-improvement and best-improvement give the same result, that produces less change than the result obtained with one-improvement. So the change is smaller. Remark that to obtain the same result as one-improvement it will just require another iteration, i.e. $\leq_{\Psi_2\odot\mu} = \leq_{\Psi_2\oslash\mu\oslash\mu} = \leq_{\Psi_2\oplus\mu\oplus\mu}$.

The $\leq_{\Psi_1}$ case is the most interesting. It is a more usual case, and it shows the difference of behaviors of the three operators. It clearly shows that in the general case best-improvement produces less change than half-improvement, that produces less change than one-improvement. To explain intuitively the change obtained by the three operators: one-improvement increase the plausibility of each model of the new formula by moving it to the first (w.r.t its current position) lower level of models of its negation. Half-improvement increase the plausibility of each model, but only by a "half-level" (i.e. if the model of the new formula was equivalent to a model of its negation then now it is strictly more plausible, but not as plausible as the lower models of the negation. And if the model of the new formula was not equivalent to a model of its negation, then now it is moved to the first lower level of models of the negation).

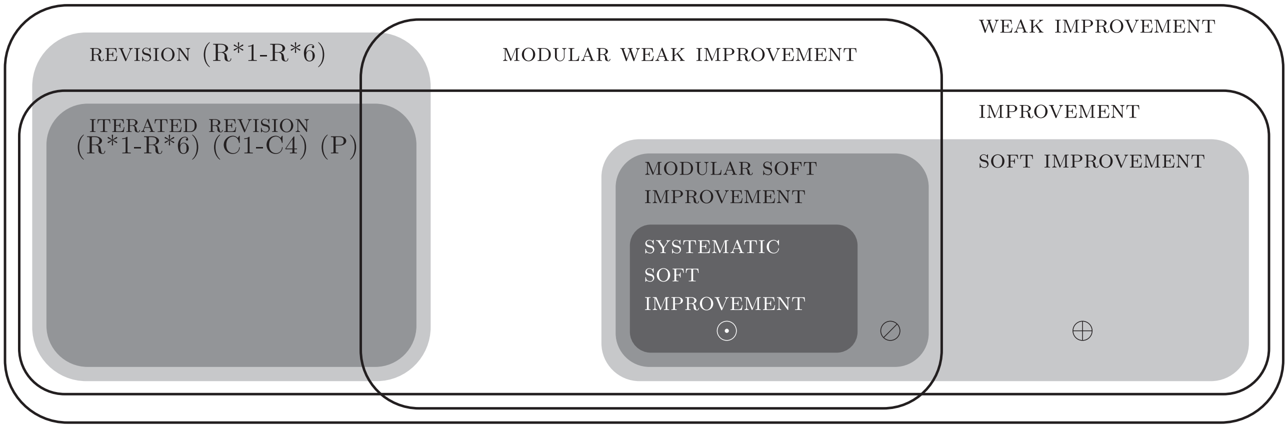


Figure 2: A map of weak improvement operators

For Best-improvement, as there are some models of the new formula that was equivalent to model of its negation, then only these models have their plausibility improved (from a "half-level").

Conclusion and Related Work

In this paper we have started the investigation of soft improvement operators. Soft improvement operators are a subclass of weak improvement operators, just as belief revision operators are. See Figure 2 for a map of weak improvement subclasses. We defined two subclasses of soft improvement operators: modular soft improvement operators and systematic soft improvement operators. For each of this class we provide a prototypal operator, that we characterize logically and for which we provide a representation theorem. We also study these operators with respect to minimal change, when this minimality is computed using the Kemeny distance between the pre-orders obtained through the assignments.

Ideas close to the ones behind the definition of the one-improvement operator already appeared in some works such as (Cantwell 1997; van Ditmarsch 2005; Laverny and Lang 2005), but there was no logical characterization in all these works. As far as we know there is no work mentioning ideas close to half-improvement or best-improvement.

Acknowledgements

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Appendix

Proof of Theorem 5:

We need to state some auxiliary results before the main proof. First of all, remark that the following Proposition is a consequence of Theorem 1 (The proof is in (Konieczny and Pino Pérez 2008)):

Proposition 9 *Let $\circ$ be a weak improvement operator. Then*

$$B(\Psi \star (\alpha \vee \beta)) = \begin{cases} B(\Psi \star \alpha) \text{ or} \\ B(\Psi \star \beta) \text{ or} \\ B(\Psi \star \alpha) \vee B(\Psi \star \beta) \end{cases}$$

With the help of the previous Proposition is easy to obtain the following two Corollaries giving characterizations of relations $\prec_\Psi$ and $\prec\!\!\prec_\Psi$.

Corollary 2 *Let $\circ$ be a weak improvement operator. Then $\alpha \prec_\Psi \beta$ if and only if there exist w, w' such that $w \in [\![B(\Psi \star \alpha)]\!]$, $w' \in [\![B(\Psi \star \beta)]\!]$, $w <_\Psi w'$.*

Corollary 3 *Let $\circ$ be a weak improvement operator. Then $\alpha \prec\!\!\prec_\Psi \beta$ if and only if there exist w, w' such that $w \in [\![B(\Psi \star \alpha)]\!]$, $w' \in [\![B(\Psi \star \beta)]\!]$, $w <_\Psi w'$ and there is no w'' such that $w <_\Psi w'' <_\Psi w'$.*

Let us now turn to the proof of the theorem.

(**only if**) Let $\circ$ be an half-improvement operator, *i.e.* a soft improvement operator that satisfies (H1) and (H2). We want to prove that $\circ$ and his respective soft gradual assignment satisfy the semantic conditions (SH1) and (SH2).

(SH1): Let $\omega \in [\![\mu]\!]$, $\omega' \in [\![\neg\mu]\!]$ such that $\omega' \ll_\Psi \omega$. And suppose there is no $\omega'' \in [\![\neg\mu]\!]$ such that $\omega'' \simeq_\Psi \omega$. Consider $\alpha = \varphi_{\omega,\omega'}$ and $\mu = \varphi_\omega$. Since $\omega' <_\Psi \omega$, $\min([\![\alpha]\!], \leq_\Psi) = \{\omega'\}$. Thus, by Theorem 1 $B(\Psi \star \alpha) \vdash \neg\mu$. Since $\alpha \wedge \mu \nvdash \bot$, and $\{\omega\} = \min([\![\alpha \wedge \mu]\!], \leq_\Psi)$ by Corollary 3, $\alpha \prec\!\!\prec_\Psi \alpha \wedge \mu$.

Towards a contradiction, suppose that there exists β such that $\beta \vdash \neg\mu$ and $\alpha \prec\!\!\prec_\Psi \beta$. By Corollary 3, $\exists\omega''' \in [\![B(\Psi \star \alpha)]\!]$ and $\exists\omega'' \in [\![B(\Psi \circ \beta)]\!]$ such that $\omega''' \ll_\Psi \omega''$. As $\omega''' \simeq_\Psi \omega'$, necessarily $\omega' \ll_\Psi \omega''$. As, by hypothesis, $\omega' \ll_\Psi \omega$, necessarily $\omega \simeq_\Psi \omega''$, but $\omega'' \models \neg\mu$ a contradiction. Thus, there is no a formula β such that $\beta \vdash \neg\mu$ and $\alpha \prec\!\!\prec_\Psi \beta$. Then, by (H1), $B((\Psi \circ \mu) \star \alpha) \nvdash \neg\mu$. Thus, by Theorem 1, $\min([\![\alpha]\!], \leq_{\Psi\circ\mu}) \not\subseteq [\![\neg\mu]\!]$, *i.e.* $\min([\![\alpha]\!], \leq_{\Psi\circ\mu}) \cap [\![\mu]\!] \neq \emptyset$. Then, $\min([\![\alpha]\!], \leq_{\Psi\circ\mu}) \cap [\![\mu]\!] = \{\omega\}$ and therefore, $\omega \leq_{\Psi\circ\mu} \omega'$.

(SH2): Let $\omega \in [\![\mu]\!]$, $\omega' \in [\![\neg\mu]\!]$, be such that $\omega' \ll_\Psi \omega$ and suppose that there is $\omega'' \in [\![\neg\mu]\!]$ such that $\omega \simeq_\Psi \omega''$. Consider $\alpha = \varphi_{\omega,\omega'}$ and $\mu = \varphi_\omega$. Then, by Corollary 3, we have $\alpha \prec\!\!\prec_\Psi \alpha \wedge \mu$. Furthermore, $\{\omega'\} = [\![B(\Psi \star \alpha)]\!]$ and $\alpha \wedge \mu \nvdash \bot$. Consider the formula $\beta = \varphi_{\omega''}$, and note that $\beta \vdash \neg\mu$ and $\omega' \ll_\Psi \omega''$. Since $\omega'' \in [\![B(\Psi \star \beta)]\!]$, by

Corollary 3, $\alpha \not\ll_\Psi \beta$. Then, by postulate (H2), $B((\Psi \circ \mu) \star \alpha) \vdash \neg\mu$. Thus, $\min([\![\alpha]\!], <_{\Psi\circ\mu}) \subseteq [\![\neg\mu]\!]$. Then, $\min([\![\alpha]\!], <_{\Psi\circ\mu}) = \{\omega'\}$ and therefore $\omega' <_{\Psi\circ\mu} \omega$.

(**if**) We know, by Theorem 3, that $\circ$ is a soft improvement operator. It remains to prove that $\circ$ satisfies (H1) and (H2).

(H1): Suppose that $B(\Psi \star \alpha) \vdash \neg\mu$, $\alpha \not\ll_\Psi \alpha \wedge \mu$ and there is no β such that $\beta \vdash \neg\mu$ and $\alpha \not\ll_\Psi \beta$. We want to show that $B((\Psi \circ \mu) \star \alpha) \not\vdash \neg\mu$.

Since $\alpha \not\ll_\Psi \alpha \wedge \mu$, by Corollary 3, there are ω and ω' such that $\omega \in [\![B(\Psi \star (\alpha \wedge \mu))]\!]$, $\omega' \in [\![B(\Psi \star \alpha)]\!]$ and $\omega' \ll_\Psi \omega$. This, together with the hypothesis, entails $\omega \in [\![\mu]\!]$ and $\omega' \in [\![\neg\mu]\!]$. Towards a contradiction, suppose that there is $\omega'' \in [\![\neg\mu]\!]$ such that $\omega'' \simeq_\Psi \omega$. Consider the formula $\varphi_{\omega''}$. Then $\varphi_{\omega''} \vdash \neg\mu$ and $\omega'' \in [\![B(\Psi \star \varphi_{\omega''})]\!]$. Since $\omega' \ll_\Psi \omega$ and $\omega' \ll_\Psi \omega''$, by Corollary 3, we obtain $\alpha \not\ll_\Psi \varphi_{\omega''}$, a contradiction. Thus, $\nexists \omega'' \in [\neg\mu]$ such that $\omega'' \simeq_\Psi \omega$. Then, from (SH1) follows $\omega \leq_{\Psi\circ\mu} \omega'$. Since $\star$ satisfies the property (I10), by Theorem 3, $\star$ also satisfies (S4). Therefore, $\omega' \leq_{\Psi\circ\mu} \omega$. Thus, $\omega' \simeq_{\Psi\circ\mu} \omega$.

In order to prove that $B((\Psi \circ \mu) \star \alpha) \not\vdash \neg\mu$, we will prove that $[\![B((\Psi \circ \mu)) \star \alpha)]\!] \cap [\![\mu]\!] \neq \emptyset$. We claim that $\omega \in \min([\![\alpha]\!], \leq_{\Psi\circ\mu})$ and since $\omega \models \mu$ we get the previous inequality. Towards a contradiction, suppose the claim false, *i.e.* there is a model $\omega_1 \in \min([\![\alpha]\!], \leq_{\Psi\circ\mu})$ such that $\omega_1 <_{\Psi\circ\mu} \omega$. We have two cases: $\omega_1 \models \mu$ or $\omega_1 \models \neg\mu$.
Case 1: $\omega_1 \models \mu$. In this case we have $\omega_1 \models \alpha \wedge \mu$. Since $\omega \in [\![B(\Psi \star (\alpha \wedge \mu))]\!]$, by Theorem 1, necessarily $\omega \in \min([\![\alpha \wedge \mu]\!], \leq_\Psi)$. In particular, $\omega \leq_\Psi \omega_1$. Thus, by (S1), $\omega \leq_{\Psi\circ\mu} \omega_1$, a contradiction.
Case 2: $\omega_1 \models \neg\mu$. In this case we have $\omega_1 \models \alpha \wedge \neg\mu$. Since $\omega' \in [\![B(\Psi \star \alpha)]\!]$, by Theorem 1, necessarily $\omega' \in \min([\![\alpha]\!], \leq_\Psi)$. In particular, $\omega' \leq_\Psi \omega_1$. But $\omega', \omega_1 \models \neg\mu$, then (S2) gives $\omega' \leq_{\Psi\circ\mu} \omega_1$. As we has seen before, $\omega \simeq_{\Psi\circ\mu} \omega'$. Thus, by transitivity, $\omega \leq_{\Psi\circ\mu} \omega_1$, a contradiction.

(H2): Suppose that $B(\Psi \star \alpha) \vdash \neg\mu$, $\alpha \not\ll_\Psi \alpha \wedge \mu$ and there exists β such that $\beta \vdash \neg\mu$ $\alpha \not\ll_\Psi \beta$. By Corollary 3, there are $\omega \in [\![B(\Psi \star (\alpha \wedge \mu))]\!]$, $\omega', \omega'' \in [\![B(\Psi \star \alpha)]\!]$ and $\omega''' \in [\![B(\Psi \star \beta)]\!]$ such that $\omega' \ll_\Psi \omega$ and $\omega'' \ll_\Psi \omega'''$. Thus, by hypothesis, we have $\omega \models \mu$ and $\omega', \omega'', \omega''' \models \neg\mu$. Since $\omega' \simeq_\Psi \omega''$, that is the case that $\omega \simeq_\Psi \omega'''$. Thus, we have a model of $\neg\mu$ that is in the same level than ω, then by (SH2) $\omega' <_{\Psi\circ\mu} \omega$. Now, if we suppose that there is a model $\omega_4 \in [\![B((\Psi \circ \mu) \star \alpha)]\!] \cap [\![\mu]\!]$, then $\omega_4 \leq_{\Psi\circ\mu} \omega$. But $\omega \leq_\Psi \omega_4$ since $\omega_4 \models \alpha \wedge \mu$. Then, by (S2) and the fact that $\omega, \omega_4 \models \mu$, $\omega \leq_{\Psi\circ\mu} \omega_4$. Thus $\omega \simeq_{\Psi\circ\mu} \omega_4$ but $\omega' <_{\Psi\circ\mu} \omega$, then $\omega' <_{\Psi\circ\mu} \omega_4$, in contradiction with the minimality of ω_4. ∎

Proof of Lemma 1:

Suppose that μ is s-separated for $\leq_\Psi$. Towards a contradiction, suppose that μ is not separated for $\leq_\Psi$. Thus, we can find a formula β such that $B(\Psi \star \beta) \not\vdash \mu$ and $B(\Psi \star \beta) \not\vdash \neg\mu$, that is, there are models $\omega_1, \omega_2 \in [\![B(\Psi \star \beta)]\!]$ such that $\omega_1 \in [\![\mu]\!]$ and $\omega_2 \in [\![\neg\mu]\!]$. But the fact that $\omega_1, \omega_2 \in [\![B(\Psi \star \beta)]\!]$ entails, by Theorem 1, $\omega_1 \simeq_\Psi \omega_2$, in contradiction with our hypothesis.

Conversely, suppose that μ is separated for $\leq_\Psi$. Towards a contradiction, suppose that μ is not s-separated for $\leq_\Psi$. Thus, we can find models ω_1, ω_2 such that $\omega_2 \models \neg\mu$ and $\omega_1 \models \mu$ and $\omega_1 \simeq_\Psi \omega_2$. Put $\beta = \varphi_{\omega_1, \omega_2}$. By the hypothesis, $\min([\![\beta]\!], \leq_\Psi) = \{\omega_1, \omega_2\}$. Thus, by Theorem 1, $B(\Psi \star \beta) \not\vdash \mu$ and $B(\Psi \star \beta) \not\vdash \neg\mu$ in contradiction with the separability of μ for $\leq_\Psi$. ∎

Proof of Theorem 7:

(**only if**) Let $\circ$ be an best improvement operator, *i.e.* a soft improvement operator that satisfies (B1) and (B2). We want to prove that $\circ$ and his respective soft gradual assignment satisfy the semantic conditions (SB1) and (SB2).

(SB1): Let $\omega \in [\![\mu]\!]$, $\omega' \in [\![\neg\mu]\!]$ such that $\omega' \ll_\Psi \omega$. Furthermore, suppose that μ is s-separated in Ψ. We want to show that $\omega \leq_\Psi \omega'$. Consider $\alpha = \varphi_{\omega,\omega'}$. Since $\omega' <_\Psi \omega$, $\min([\![\alpha]\!], \leq_\Psi) = \{\omega'\}$. Thus, by Theorem 1, $B(\Psi \star \alpha) \vdash \neg\mu$. Since $\alpha \wedge \mu \not\vdash \perp$, and $\{\omega\} = \min([\![\alpha \wedge \mu]\!], \leq_\Psi)$, by Corollary 3, $\alpha \not\ll_\Psi \alpha \wedge \mu$. Moreover, by Lemma 1, μ is separated in Ψ. Thus the hypotheses of (B1) hold and we can conclude $B((\Psi \circ \mu) \star \alpha) \not\vdash \neg\mu$. So, by Theorem 1, $\min([\![\alpha]\!], \leq_{\Psi\circ\mu}) \not\subseteq \neg\mu$, *i.e.* $\omega \leq_{\Psi\circ\mu} \omega'$.

(SB2): Assume that μ is not s-separated in Ψ. In addition, suppose that $\omega \models \mu, \omega' \models \neg\mu$ and $\omega' <_\Psi \omega$. We want to show that $\omega' <_{\Psi\circ\mu} \omega$. Consider $\alpha = \varphi_{\omega,\omega'}$. Since $\omega' <_\Psi \omega$, $\min([\![\alpha]\!], \leq_\Psi) = \{\omega'\}$. Thus, by Theorem 1, $B(\Psi \star \alpha) \vdash \neg\mu$. By Lemma 1, μ is not separated in Ψ. Then, by (B2), $B((\Psi \circ \mu) \star \alpha) \vdash \neg\mu$. Then, by Theorem 1, $\omega \notin \min([\![\alpha]\!], \leq_{\Psi\circ\mu})$, and as by(I3) $\min([\![\alpha]\!], \leq_{\Psi\circ\mu}) \neq \emptyset$, we conclude $\omega' <_{\Psi\circ\mu} \omega$.

(**if**) We know, by Theorem 3, that $\circ$ is a soft improvement operator. It remains to prove that $\circ$ satisfies (B1) and (B2).

(B1): Suppose that μ is separated in Ψ, $B(\Psi \star \alpha) \vdash \neg\mu$ and $\alpha \not\ll_\Psi \alpha \wedge \mu$. We want to show that $B((\Psi \circ \mu) \star \alpha) \not\vdash \neg\mu$. Let $\omega \in \min([\![\alpha]\!], \leq_\Psi)$. By Theorem 1, $\omega \models \neg\mu$. By the hypothesis and Corollary 3, there exists $\omega' \in [\![\alpha \wedge \mu]\!]$ such that $\omega \ll_\Psi \omega'$.Then, by (SB1), $\omega' \leq_{\Psi\circ\mu} \omega$. In fact, we have $\omega' \simeq_{\Psi\circ\mu} \omega$. Note that after soft improvement by μ the minimal elements of $[\![\alpha]\!]$ with respect to $\leq_{\Psi\circ\mu}$ are at the level of ω. Therefore, $\omega' \in \min([\![\alpha]\!], \leq_{\Psi\circ\mu})$. Then, by Theorem 1, $B((\Psi \circ \mu) \star \alpha) \not\vdash \neg\mu$.

(B2): Suppose that μ is not separated in Ψ and $B(\Psi \star \alpha) \vdash \neg\mu$. We want to show that $B((\Psi \circ \mu) \star \alpha) \vdash \neg\mu$. Towards a contradiction suppose there exists $\omega' \in [\![B((\Psi \circ \mu) \star \alpha)]\!] \cap [\![\mu]\!]$. By the hypothesis and Theorem 1, $\min([\![\alpha]\!], \leq_\Psi) \subseteq [\![\neg\mu]\!]$. Thus $\omega' \notin \min([\![\alpha]\!], \leq_\Psi)$. Let ω be a model in $\min([\![\alpha]\!], \leq_\Psi)$. Then, $\omega <_\Psi \omega'$. Since, by Lemma 1, μ is not separated in $\leq_\Psi$, by (SB2) we have $\omega <_{\Psi\circ\mu} \omega'$m in contradiction with the minimality of $\omega' \in [\![\alpha]\!]$ with respect to $\leq_{\Psi\circ\mu}$. ∎

Proof of Proposition 4:

The idea for proving Propositions 4 and 6, comes from the analysis of the number of elements in the symmetrical difference between the old epistemic state and the new epistemic state: each unit in $d_K(\leq_\Psi, \leq_{\Psi\circ\mu})$ has its origin in two operations: *creation* and *destruction*. The *creation* corresponds to the fact that there exists a new couple in $\leq_{\Psi\circ\mu}$ which is not in $\leq_\Psi$; the *destruction* corresponds to the fact that there was a couple in $\leq_\Psi$ which is not in $\leq_{\Psi\circ\mu}$. More precisely, we will say that the couple (ω, ω') is created by un operator $\circ$ (in the context $\leq_\Psi$, μ) if $\omega \leq_{\Psi\circ\mu} \omega'$ and $\omega \not\leq_\Psi \omega'$. We will say that the couple (ω, ω') is destroyed by un operator $\circ$ (in the context $\leq_\Psi$, μ) if $\omega \not\leq_{\Psi\circ\mu} \omega'$ and $\omega \leq_\Psi \omega'$. The following result, the proof of which is straightforward, summarizes this discussion:

Lemma 2 $d_K(\leq_\Psi, \leq_{\Psi\circ\mu})$ *is the number of couples created plus the number of couples destroyed.*

Now we have the tools for establishing the key results which allow us to prove the Propositions 4 and 6.

Lemma 3 *Consider the context* $\leq_\Psi$ *and* μ. *Then,*
(i) Every couple created by $\oslash$ *is also created by* $\odot$.
(ii) Every couple destroyed by $\oslash$ *is also destroyed by* $\odot$.

Proof of Lemma 3:

First we prove part (i). Suppose that the couple (ω, ω') has been created by $\oslash$, that is $\omega \leq_{\Psi\oslash\mu} \omega'$ and $\omega \not\leq_\Psi \omega'$. Then, by the totality of $\leq_\Psi$, $\omega' <_\Psi \omega$. Necessarily, $\omega \in [\![\mu]\!]$, $\omega' \notin [\![\mu]\!]$ and $\omega' \ll_\Psi \omega$. Therefore, $\omega \leq_{\Psi\odot\mu} \omega'$, *i.e.* the couple (ω, ω') has also been created by $\odot$.
Now we prove part (ii). Suppose that the couple (ω, ω') has been destroyed by $\oslash$, that is $\omega \not\leq_{\Psi\oslash\mu} \omega'$ and $\omega \leq_\Psi \omega'$. By the totality of $\leq_{\Psi\oslash\mu}$, we have $\omega' <_{\Psi\oslash\mu} \omega$. Necessarily, $\omega' \in [\![\mu]\!]$, $\omega \notin [\![\mu]\!]$ and $\omega' \simeq_\Psi \omega$. Thus, by S3, $\omega' <_{\Psi\odot\mu} \omega$. By the totality of $\leq_{\Psi\odot\mu}$, $\omega \not\leq_{\Psi\odot\mu} \omega'$, *i.e.* the couple (ω, ω') has also been destroyed by $\odot$. ∎

Lemma 4 *Consider the context* $\leq_\Psi$ *and* μ. *Then,*
(i) Every couple created by $\oplus$ *is also created by* $\oslash$.
(ii) Every couple destroyed by $\oplus$ *is also destroyed by* $\oslash$.

Proof of Lemma 4:

First we prove part (i). Suppose that the couple (ω, ω') has been created by $\oplus$, that is, $\omega \leq_{\Psi\oplus\mu} \omega'$ and $\omega \not\leq_\Psi \omega'$. Then, by the totality of $\leq_\Psi$, $\omega' <_\Psi \omega$. Thus, the only possibility of creation arrives when $\omega' \ll_\Psi \omega$, $\omega \in [\![\mu]\!]$, $\omega' \notin [\![\mu]\!]$ and for any models ω_1 and ω_2 such that $\omega_1 \in [\![\mu]\!]$ and $\omega_2 \notin [\![\neg\mu]\!]$, we have $\omega_1 \not\simeq_\Psi \omega_2$ (application of SB1). In this case, it is clear that we can apply SH1 and get $\omega \leq_{\Psi\oslash\mu} \omega'$, *i.e.* the couple (ω, ω') has also been created by $\oslash$.
Now we prove part (ii). Suppose that the couple (ω, ω') has been destroyed by $\oplus$, that is $\omega \not\leq_{\Psi\oplus\mu} \omega'$ and $\omega \leq_\Psi \omega'$. By the totality of $\leq_{\Psi\oplus\mu}$, we have $\omega' <_{\Psi\oslash\mu} \omega$. Necessarily, $\omega' \in [\![\mu]\!]$, $\omega \notin [\![\mu]\!]$ and $\omega' \simeq_\Psi \omega$. Thus, by S3, $\omega' <_{\Psi\oslash\mu} \omega$. By the totality of $\leq_{\Psi\oslash\mu}$, $\omega \not\leq_{\Psi\oslash\mu} \omega'$, *i.e.* the couple (ω, ω') has also been destroyed by $\oslash$. ∎

Proposition 4 follows straightforwardly from Lemmas 2 and 3. ∎

Proof of Proposition 6:

Proposition 6 follows straightforwardly from Lemmas 2 and 4. ∎

Proof of Proposition 8:

The idea is to consider two cases: separated and non separated. In the first case all the soft improvement operators have the same behavior because of (B1).

In the case non separated, the proof proceeds by cases in the same style as Lemmas 3 and 4. The two columns of Figure 2, explain at a glance, the reasons of this minimality. ∎

References

Alchourrón, C. E.; Gärdenfors, P.; and Makinson, D. 1985. On the logic of theory change: Partial meet contraction and revision functions. *Journal of Symbolic Logic* 50:510–530.

Benferhat, S.; Konieczny, S.; Papini, O.; and Pino Pérez, R. 2000. Iterated revision by epistemic states: axioms, semantics and syntax. In *Fourteenth European Conference on Artificial Intelligence (ECAI'2000)*, 13–17.

Booth, R., and Meyer, T. 2006. Admissible and restrained revision. *Journal of Artificial Intelligence Research* 26:127–151.

Boutilier, C. 1996. Iterated revision and minimal change of conditional beliefs. *Journal of Philosophical Logic* 25(3):262–305.

Cantwell, J. 1997. On the logic of small changes in hypertheories. *Theoria* 63(1-2):54–89.

Darwiche, A., and Pearl, J. 1997. On the logic of iterated belief revision. *Artificial Intelligence* 89:1–29.

Gärdenfors, P. 1988. *Knowledge in flux*. MIT Press.

Hansson, S. O. 1997. What's new isn't always best. *Theoria* 1–13. Special issue on non-prioritized belief revision.

Herzig, A.; Konieczny, S.; and Perussel, L. 2003. On iterated revision in the agm framework. In *Seventh European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU'03)*, 477–488.

Jin, Y., and Thielscher, M. 2007. Iterated belief revision, revised. *Artificial Intelligence* 171:1–18.

Katsuno, H., and Mendelzon, A. O. 1991. Propositional knowledge base revision and minimal change. *Artificial Intelligence* 52:263–294.

Kemeny, J. G. 1959. Mathematics without numbers. *Daedalus* 88:571–591.

Konieczny, S., and Pino Pérez, R. 2000. A framework for iterated revision. *Journal of Applied Non-Classical Logics* 10(3-4):339–367.

Konieczny, S., and Pino Pérez, R. 2008. Improvement operators. In *Proceedings of the Eleventh International Conference on Principles of Knowledge Representation And Reasoning (KR'08).*, 177–187.

Laverny, N., and Lang, J. 2005. From knowledge-based programs to graded belief-based programs, part i: On-line reasoning*. *Synthese* 147(2):277–321.

Nayak, A. C. 1994. Iterated belief change based on epistemic entrenchment. *Erkenntnis* 41:353–390.

Rott, H. 2006. Shifting priorities: Simple representations for 27 iterated theory change operators. *Modality matters: twenty-five essays in honour of Krister Segerberg, Uppsala Philosophical Studies* 35:359–384.

van Ditmarsch, H. 2005. Prolegomena to dynamic logic for belief revision. *Synthese* 147(2):229–275.