A Characterization of Optimality Criteria for Decision Making under Complete Ignorance

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Abstract

In this paper we present a model for decision making under complete ignorance. By complete ignorance it is meant that all that is known is the set of possible consequences associated to each action. Especially there is no set of states that can be enumerated in order to compare actions. We give two natural axioms for rational decision making under complete ignorance. We show that the optimality criteria satisfying these two axioms are the ones which consider only the extremal consequences of actions. We compare our axioms with related ones from the literature.

Introduction

Decision making under uncertainty is a fundamental issue in Artificial Intelligence. The problem is basically as follows: given a set of consequences that are strictly ordered by a decision maker and given a set of possible actions which have uncertain consequences, what is the best action to perform? A model for decision making under uncertainty is a formal framework in which the notions at work in the problem (including actions and consequences) are defined and the best actions are characterized thanks to a so-called optimality criterion.

Many models have been proposed so far for decision making under uncertainty. One can classify them according to the type of uncertainty or, more exactly, by the quantity of information required:

Probabilistic Uncertainy In the corresponding models, an action a is defined as a mapping matching a state of the world to a consequence (see e.g. (de Finetti 1937), (Ramsey 1931)). The decision maker is guided by a probability distribution p over the states of the world to make her choice. This distribution can be either objective (statistically obtained) or subjective (derived from the agent's beliefs). The decision maker then typically takes advantage of expected utility theory (von Neumann and Morgenstern 1947; Savage 1954) to evaluate her actions. The best actions are those with the highest expected utility.

Plausibility Orderings In the corresponding models, an action is still considered as a mapping associating each state of the world to a consequence, but no probability distribution over the states is available. Based on the basic assumptions underlying Savage axiomatic framework (Savage 1954), optimality criteria for such models take advantage of a preorder over the states of the world expressing their (relative) plausibility. An example of such a model is (Boutilier 1994) where decisions are made based on the most plausible states, neglecting the others. These approaches to decision making can lead to different settings depending on whether commensurability is assumed or not, i.e., whether the plausibility and utility scales can be compared. See for instance (Dubois et al. 2002; Fargier and Sabbadin 2003).

Strict Uncertainty Some works investigated the concept of strict uncertainty (sometimes also referred to as *complete ignorance*, but we keep this name for the next class of models). In the corresponding models, an action is still a mapping associating each state of the world to a consequence. But there is no plausibility ordering: only a set of states of the world is available. One of the main models pertaining to this family was proposed by Arrow and Hurwicz (Arrow and Hurwicz 1977). See also (Maskin 1979).

Complete Ignorance In all the previously cited works, one invariably considers the same definition of an action, i.e., a mapping from the states of the world to the set of consequences. Without loss of generality, each action is supposed to be deterministic and the fact that its actual consequence cannot be predicted is modeled by the fact that the true state of the world where the action is performed is only imperfectly known. However, there are scenarios for which even the notion of states in the model for decision making does not make sense, because the decision maker cannot envision them.[1] For such scenarios where each action is associated directly to a set of possible consequences, i.e., the consequences that can be reached when the action is performed, the previous families of models for decision making are not adequate. This calls for another family of models for decision making under so-

[1]The arbitrariness of identifying all contingencies of a decision by a set of states of nature has been discussed for a while in a number of papers, see e.g. (Arrow and Hurwicz 1977; Pattanaik and Peleg 1984).

called "complete ignorance". See e.g. (Nitzan and Pattanaik 1984; Gravel, Marchant, and Sen 2008).

An important point we want to stress is that these four different families of models can be ranked based on the quantity of information they require:

Probabilistic Uncertainty Set of states + probability

Plausibility Order Set of states + plausibility ordering

Strict Uncertainty Set of states

Complete Ignorance No set of states

Clearly a valuable model for decision making under uncertainty is primarily a model which *is suited to the available information*. While models based on expected utility theory are very interesting (especially because the best actions they characterize typically satisfy natural requirements), they are of no use when no probability distribution over the states of the world is available. This motivates the need for the three other families of models.

In this paper, we focus on the less informed case and consider the problem of decision making under complete ignorance. Our contribution is mainly twofold. We point out two natural axioms that the optimality criteria for decision making under complete ignorance should satisfy, and show that the optimality criteria satisfying the two axioms are exactly the ones which take only into account the extremal consequences of actions. We also compare the two axioms we provide with other axioms characterizing previous models for decision making under complete ignorance.

The main results of this paper are close to results shown by Arrrow and Hurwicz in (1977) for the strict uncertainty case. What we prove in this paper is that their results can be extended to the poorer informational case of complete ignorance. This requires different axioms (and proofs).

Let us briefly explain on a simple example why optimality critera for the strict uncertainty case cannot be exploited in the complete ignorance case. Consider four consequences A, B, C, D ordered such that $A > B > C > D$, and two actions a and b. In a strict uncertainty framework, states of the world are also available as part of the input. Assume that two states s_1 and s_2 are known as possible and that:

$R(a, s_1) = A \quad R(a, s_2) = C$
$R(b, s_1) = B \quad R(b, s_2) = D$

where R maps each action and state to a consequence (the consequence obtained by performing the action in the state). So, in the strict uncertainty case, it is possible to compare actions a and b by comparing their consequence state by state. Here one can easily check that action a Pareto-dominates action b, i.e., in each state of the world the consequence of a is strictly better than the consequence of b. This is a sufficient reason to prefer action a to action b. In the complete ignorance case, where there is no notion of state of the world, such a statewise comparison is impossible. What is available is just that $R(a) = \{A, C\}$ and $R(b) = \{B, D\}$, meaning intuitively that there exist four states of the world s'_1, s'_2, s'_3, s'_4 such that $R(a, s'_1) = A$, $R(a, s'_2) = C$, $R(b, s'_3) = B$, $R(b, s'_4) = D$. Clearly enough, the reason to prefer a to b has disappeared.

The rest of the paper is as follows. We start by some formal preliminaries about decision making under complete ignorance. Then we present the two axioms we consider for optimality criteria and a theorem characterizing the family of admissible criteria. We also give a second characterization refining the first one with strict preferences. Then we provide some examples of admissible criteria. Finally we compare our axioms with those used in the related literature and we discuss further related works.

Formal Preliminaries

Let $\mathcal{A}$ be a non empty set of actions and $\mathcal{C}$ a non empty set of consequences. Let R be a mapping from $\mathcal{A}$ to $2^{\mathcal{C}} \setminus \{\emptyset\}$ which associate to every action $a \in \mathcal{A}$ a non empty subset $R(a)$ of $\mathcal{C}$ describing the possible consequences of a. We suppose $\mathcal{C}$ ordered by a complete order, i.e., a reflexive, antisymmetric, transitive and complete relation $\geq$ expressing the decision maker preferences over the consequences $\mathcal{C}$. $>$ denotes the strict part of $\geq$ i.e., $\forall c, c' \in \mathcal{C}$, $c > c'$ iff $c \geq c'$ and $c' \ngeq c$.

A *decision frame* is given by $(\mathcal{A}, \mathcal{C}, R, \geq)$. A *decision problem* is a subset A of $\mathcal{A}$. A *decision criterion* in the frame $(\mathcal{A}, \mathcal{C}, R, \geq)$ is a choice function associating to each subset A of $\mathcal{A}$ a subset $\hat{A}$ of A (formally, it is a mapping from $2^{\mathcal{A}}$ to $2^{\mathcal{A}}$). Giving a decision criterion in the frame $(\mathcal{A}, \mathcal{C}, \mathcal{R}, \geq)$ allows to solve any decision problem: the optimal actions to realize when facing the decision problem A are those in $\hat{A}$.

Complete ignorance is expressed by the fact that given an action a, the set of its possible consequences $R(a)$ is the only available information. No probability distribution on the image of any action by R is available, which can be interpreted as "all the probability distributions on consequences are possible". Similarly, no plausibility ordering over the states of the world is available.

Given a decision criterion in $(\mathcal{A}, \mathcal{C}, R, \geq)$, and a decision problem $A \subseteq \mathcal{A}$:

- If $a_1, a_2 \in A$ and $a_1 \in \hat{A}$, a_1 is said to be preferred or indifferent to a_2 with respect to A; this is noted $a_1 \trianglerighteq a_2[A]$.
- If $a_1, a_2 \in A$, $a_1 \in \hat{A}$ and $a_2 \in A \setminus \hat{A}$, a_1 is said to be preferred to a_2 with respect to A; this is noted $a_1 \triangleright a_2[A]$.
- If $a_1, a_2 \in A$, and either $a_1, a_2 \in \hat{A}$ or $a_1, a_2 \in A \setminus \hat{A}$, a_1 and a_2 are said to be equivalent with respect to A, noted $a_1 \diamond a_2[A]$.

For decision problems containing just the two actions a_1, a_2 and denoted by $[a_1, a_2]$, we simplify the notations as follows: $a_1 \diamond a_2$ if $a_1 \diamond a_2[a_1, a_2]$,
$a_1 \trianglerighteq a_2$ if $a_1 \trianglerighteq a_2[a_1, a_2]$,
$a_1 \triangleright a_2$ if $a_1 \triangleright a_2[a_1, a_2]$.

Finally, $minR(a)$ denotes the minimum of $R(a)$ and $maxR(a)$ the maximum of $R(a)$ with respect to the preference relation on consequences, i.e., $minR(a) = c$ iff $\nexists c' \in R(a)$ s.t. $c > c'$, and $maxR(a) = c$ iff $\nexists c' \in R(a)$ s.t. $c' > c$.

Decision Frames and Optimality Criteria

We make the following assumptions on decision frames:

Hypothesis (A) $\forall A \subseteq \mathcal{A}$, *if* $A \neq \emptyset$ *then* $\hat{A} \neq \emptyset$.

Hypothesis (A) states that for every decision problem we can find a non empty optimal subset (i.e., a decision is available).

Hypothesis (B) *$\forall a \in \mathcal{A}$, $minR(a)$ and $maxR(a)$ exist.*

Hypothesis (B) states that every action has a worst consequence and a best consequence. This assumption prevents us from treating cases with infinite chains of comparisons of consequences. Note that this hypothesis is straightforwardly satisfied in the finite case.

Hypothesis (C) *$\forall C \subseteq \mathcal{C}$, if $C \neq \emptyset$, $\exists a \in \mathcal{A}$ s.t. $R(a) = C$.*

Hypothesis (C) expresses some kind of density property of the set of consequences. It states that for every non empty subset of consequences we can envision an action whose possible consequences are exactly the ones in the set. This justifies the following notation: for every non empty subset $\{c_1, \ldots, c_n\}$ of $\mathcal{C}$, $[c_1, \ldots, c_n]$ denotes an action a in $\mathcal{A}$ s.t. $R(a) = \{c_1, \ldots, c_n\}$.

We now present the desirable properties the optimality criteria should satisfy and then deduce the corresponding form of these criteria.

Axiom ($\boldsymbol{\alpha}$) *If $A_1 \subseteq A_2$ and $A_1 \cap \hat{A}_2$ is not empty, then $\hat{A}_1 = A_1 \cap \hat{A}_2$.*

($\boldsymbol{\alpha}$) states that if some actions are deemed optimal in a given set of actions and if subsequently the range of actions available is contracted but these optimal actions are still available, then the optimal actions for the larger problem are still optimal (and no new optimal actions are added). This is a typical property of choice functions used for example in social choice theory (where it is known as Chernoff condition), belief revision, belief merging, etc.

Axiom ($\boldsymbol{\beta}$) *Let A be a decision problem, $a, a' \in A$ such that $\forall c \in R(a), \forall c' \in R(a'), c \geqslant c'$. Consider any $c'' \in \mathcal{C}$ and $a_1, a'_1 \in A$ s.t. $R(a_1) = R(a) \cup \{c''\}$ and $R(a'_1) = R(a') \cup \{c''\}$. Then $a_1 \trianglerighteq a'_1$.*

The rationale for ($\boldsymbol{\beta}$) is as follows. If two actions are such that they share a common consequence and if all the other consequences of the first one are preferred or indifferent to all other consequences of the second one then the first action is preferred or indifferent to the second one.

The following lemma gives an interesting consequence of axiom ($\boldsymbol{\beta}$). It states that, when considering an action and its corresponding set of consequences, it is enough to look at the extremal ones, i.e., the pair composed of the worst consequence and the best consequence (their existence is ensured by Hypothesis (B)).

Lemma 1 *Assume an optimality criterion satisfying ($\boldsymbol{\beta}$). Let $a \in \mathcal{A}$. $a \diamond [minR(a), maxR(a)]$.*

As a direct consequence of Lemma 1, we easily get the following corollary:

Corollary 1 *Let a, a' be two actions in a decision problem A. If $R(a) = R(a')$ then $a \diamond a'$.*

This corollary shows that the only relevant information for comparing two actions are their sets of consequences. Thus, actions with the same set of consequences are undistinguishable. This implies that actions can be identified with sets of consequences.

Finally, the following lemma mainly states that the comparison between two actions depends only of these two actions (and their consequences), and not of any other available action.

Lemma 2 *Assume an optimality criterion satisfying ($\boldsymbol{\alpha}$). Then we have:*

1. *if $a_1 \diamond a_2$, then $a_1 \diamond a_2[A]$ for all A s.t. $a_1, a_2 \in A$.*
2. *if $a_1 \trianglerighteq a_2[A]$ for some A, then $a_1 \trianglerighteq a_2$.*
3. *if $a_1 \triangleright a_2[A]$ for some A, then $a_1 \triangleright a_2$.*

Lemma 1 allows us to prove also the equivalence between ($\boldsymbol{\beta}$) and a second axiom that may seem stronger at a first glance:

Axiom ($\boldsymbol{\beta}$*) *Let A be a decision problem, $a, a' \in A$ such that $\forall c \in R(a), \forall c' \in R(a'), c \geqslant c'$. Consider any $C \subseteq \mathcal{C}$ and $a_1, a'_1 \in A$ s.t. $R(a_1) = R(a) \cup C$ and $R(a'_1) = R(a') \cup C$. Then $a_1 \trianglerighteq a'_1$.*

Proposition 1 *If an optimality criterion satisfies ($\boldsymbol{\alpha}$), then it satisfies ($\boldsymbol{\beta}$) if and only if it satisfies ($\boldsymbol{\beta}$*).*

This proposition enlightens another interpretation of ($\boldsymbol{\beta}$). It states that when comparing two actions a and a' by an optimality criterion satisfying ($\boldsymbol{\beta}$), if by erasing their common consequences, one obtains that all remaining consequences of a are preferred or indifferent to all those of a', then a should be preferred or indifferent to a'.

Characterization of Optimality Criteria

In this section we prove the main theorem of the paper, which characterizes the family of optimality criteria meeting the requirements of ($\boldsymbol{\alpha}$) and ($\boldsymbol{\beta}$).

Theorem 1 *Under Hypotheses (A), (B) and (C), an optimality criterion satisfies ($\boldsymbol{\alpha}$) and ($\boldsymbol{\beta}$) if and only if there exists a complete preorder $\succcurlyeq$ on the set of consequence pairs (m, M) with $M \geq m$ satisfying:*

(1) If $m_1 \geq m_2$ and $M_1 \geq M_2$ then $(m_1, M_1) \succcurlyeq (m_2, M_2)$, such that

(2) for every non empty subset A of $\mathcal{A}$,

$$\hat{A} = \{a | (minR(a), maxR(a)) \succcurlyeq (minR(a'), maxR(a')), \forall a' \in A\}.$$

This theorem shows that any criterion satisfying ($\boldsymbol{\alpha}$) and ($\boldsymbol{\beta}$) selects in a decision problem A the non dominated elements with respect to a preorder on actions that is linked to a preorder $\succcurlyeq$ on pairs of consequences by the property that an action a is preferred or indifferent to another action a' if and only if $(minR(a), maxR(a)) \succcurlyeq (minR(a'), maxR(a'))$.

Clearly enough, the theorem characterizes not a single criterion but a family of possible criteria. This is due to the condition on extremal consequences of the theorem that is only an implication (and not an equivalence) enabling further distinctions to be made.

The strength of this theorem is that it uses only relatively simple requirements for characterizing a family of criteria which depend only on the extremal consequences with respect to the agent's preference relation on consequences.

Another Characterization Theorem

Consider now the following axiom:

Axiom ($\boldsymbol{\beta'}$) *Let A be a decision problem, $a, a' \in A$ such that $\forall c \in R(a), \forall c' \in R(a'), c > c'$. Consider any $c'' \in \mathcal{C}$ and $a_1, a'_1 \in A$ s.t. $R(a_1) = R(a) \cup \{c''\}$ and $R(a'_1) = R(a') \cup \{c''\}$. Then $a_1 \rhd a'_1$.*

This axiom is similar to ($\boldsymbol{\beta}$) with the difference that except for the common consequence, all consequences of the first action are strictly preferred to all those of the second action.

We can obtain a theorem similar to Theorem 1 based on axiom ($\boldsymbol{\beta'}$) which leads to impose a further condition on the preorder defined in Theorem 1:

Theorem 2 *Under Hypotheses (A), (B) and (C), an optimality criterion satisfies ($\boldsymbol{\alpha}$), ($\boldsymbol{\beta'}$) and ($\boldsymbol{\beta}$) if and only if there exists a complete preorder $\succcurlyeq$ on the set of consequence pairs (m, M) with $M \geqslant m$ satisfying conditions:*

(1a) If $m_1 \geq m_2$ and $M_1 \geq M_2$ then $(m_1, M_1) \succcurlyeq (m_2, M_2)$.
(1b) If $m_1 \geq m_2$ and $M_1 \geq M_2$ and ($m_1 > m_2$ or $M_1 > M_2$) then $(m_1, M_1) \succ (m_2, M_2)$.

such that

(2) for every non empty subset A of $\mathcal{A}$, $\hat{A}$ is defined by

$$\hat{A} = \{a | (minR(a), maxR(a)) \succcurlyeq (minR(a'), maxR(a')), \forall a' \in A\}.$$

Adding axiom ($\boldsymbol{\beta'}$) restricts the set of admissible criteria. Theorem 1 involves only comparisons with respect to $\geq$ over extremal consequences to characterize optimal actions. Theorem 2 introduces strict comparisons in condition (1b) and then excludes other actions from the optimal set. Thus, an action a such that there exists another action a' with $minR(a') \geq minR(a)$ and $maxR(a') \geq maxR(a)$ can be in the optimal set for a criterion admitted by Theorem 1, even if $minR(a') > minR(a)$ or $maxR(a') > maxR(a)$. This is not possible for a criterion admitted by Theorem 2.

Examples of Optimality Criteria

In this section we consider some simple decision criteria and investigate whether they meet the requirements of the theorems given in the previous sections.

Typically, we define each criterion c by specifying a binary relation over $\mathcal{A}$, that we write $\trianglerighteq_c$ (abusing notations). The associated choice function is defined by: $\forall A \subseteq \mathcal{A}$, $\hat{A}^c = \{a \in A \mid \nexists a' \in A \; a' \rhd_c a\}$.

The first criterion we focus on is a worst-case one, which is usual in strict uncertainty cases:

Definition 1 (Min) *Let $a_1, a_2 \in \mathcal{A}$.*
$a_1 \trianglerighteq_{min} a_2$ *iff* $minR(a_1) \geq minR(a_2)$.

The Min criterion is in a sense the safest one in case of complete ignorance. It is usually known as Wald criterion (Wald 1950).

Definition 2 (Max) *Let $a_1, a_2 \in \mathcal{A}$.*
$a_1 \trianglerighteq_{max} a_2$ *iff* $maxR(a_1) \geq maxR(a_2)$.

The Max criterion is the dual of the Min criterion expressing some kind of optimistic state of mind.

It is easy to check that these two criteria satisfy ($\boldsymbol{\alpha}$) and ($\boldsymbol{\beta}$).

Proposition 2 *Min and Max satisfy ($\boldsymbol{\alpha}$) and ($\boldsymbol{\beta}$).*

It is easy to see that that none of Min and Max satisfies ($\boldsymbol{\beta'}$) in general. Indeed, let us consider two actions a_1, a_2 such that $minR(a_1) \geq minR(a_2)$ and $maxR(a_1) > maxR(a_2)$. We have $a_1 \diamond_{min} a_2$. This excludes the Min criterion from the family characterized by Theorem 2 since condition (2) in this theorem would imply in this case that $a_1 \rhd_{min} a_2$. A similar argument excludes the Max criterion.

Thus, the above two criteria do not satisfy Theorem 2. They satisfy Theorem 1, but they do it in a trivial way in some sense because they only take account for one extremal consequence whereas Theorem 1 shows that the decision maker can take advantage of a criterion which takes account for both of them. We now give two criteria which makes use of this possibility.

Given n binary relations $\trianglerighteq_i$ on $\mathcal{A}$, the associated *lexicographic relation* lex($\trianglerighteq_1, \cdots, \trianglerighteq_n$) is defined by $a_1 \geq_{lex(\trianglerighteq_1, \cdots, \trianglerighteq_n)} a_2$ iff $a_1 \rhd_{lex(\trianglerighteq_1, \cdots, \trianglerighteq_n)} a_2$ or $a_1 \diamond_{lex(\trianglerighteq_1, \cdots, \trianglerighteq_n)} a_2$ with:

- $a_1 \rhd_{lex(\trianglerighteq_1, \cdots, \trianglerighteq_n)} a_2$ iff $\exists j \in [1, n]$ s.t. $\forall i < j \; a_1 \diamond_i a_2$ and $a_1 \rhd_j a_2$.
- $a_1 \diamond_{lex(\trianglerighteq_1, \cdots, \trianglerighteq_n)} a_2$ iff $\forall i \in [1, n] \; a_1 \diamond_i a_2$.

Definition 3 (Minmax - Maxmin) *Let $a_1, a_2 \in \mathcal{A}$.*
$a_1 \trianglerighteq_{minmax} a_2$ *iff* $a_1 \trianglerighteq_{lex(\trianglerighteq_{min}, \trianglerighteq_{max})} a_2$.
$a_1 \trianglerighteq_{maxmin} a_2$ *iff* $a_1 \trianglerighteq_{lex(\trianglerighteq_{max}, \trianglerighteq_{min})} a_2$.

The Minmax criterion is a refinement of the Min criterion, taking the Max into account when the two minimal consequences are equivalent. Dually, the Max criterion can be refined into a lexicographic Maxmin, i.e., refining the Max thanks to the minimal consequences.

Proposition 3 *Minmax and Maxmin satisfy ($\boldsymbol{\alpha}$), ($\boldsymbol{\beta}$) and ($\boldsymbol{\beta'}$).*

Another well-known criterion (actually, a family of criteria) which satisfies our three axioms is Hurwicz's criterion (Hurwicz 1951); it supposes that consequences are viewed as real numbers:

Definition 4 (Hurwicz) *Suppose that $\mathcal{C} \subseteq I\!R$. Consider two actions $a_1, a_2 \in \mathcal{A}$ and $\alpha \in [0, 1]$. $a_1 \trianglerighteq_\alpha a_2$ iff $(\alpha.minR(a_1) + (1 - \alpha).maxR(a_1)) \geq (\alpha.minR(a_2) + (1 - \alpha).maxR(a_2))$.*

Proposition 4 *For each $\alpha \in]0, 1[$, the Hurwicz criterion satisfies ($\boldsymbol{\alpha}$), ($\boldsymbol{\beta}$) and ($\boldsymbol{\beta'}$).*

Finally, note that other decision criteria for decision making under complete ignorance have been axiomatized. For instance (Nitzan and Pattanaik 1984) characterizes the Median criterion, and (Gravel, Marchant, and Sen 2008) characterizes the Uniform Expected Utility criterion (using Laplace's principle of insufficient reason). Clearly enough, none of these criteria belongs to the families characterized by our axioms.

Comparison with Other Axioms

In this section, we consider several axioms pointed out so far for characterizing rational decision making under uncertainty and we compare them to the axioms we proposed in this paper.

(I) $\forall a \in A$, let $x, y \in \mathcal{C}$ s.t. $x \notin R(a)$ and $y \notin R(a)$, and let $a' \in A$ s.t. $R(a') = R(a) \cup \{x\}$ and $a^{''} \in A$ s.t. $R(a^{''}) = R(a) \cup \{y\}$. Then $x \geq y \Rightarrow a' \trianglerighteq a^{''}$.

Axiom (I) was introduced in (Puppe 1995) as a property of context independence. Roughly, (I) states that the preference between two alternatives should not change whichever the context in which it is tested. Although the conclusion of the axiom seems natural, the corresponding family of optimality criteria is so large that it admits counter-intuitive members, for example the Cardinality criterion (which selects the actions that have the greatest number of consequences). (Nehring and Puppe 1996) balances the weakness of (I) by adjoining to it a continuity axiom (C) for characterizing the so-called range dependent criteria family. Axiom (C) states that a "small" change in the set of consequences of a given action should not dramatically change the decision maker's preference towards it. Yet this axiom imposes restricted structures for the set $\mathcal{C}$ of consequences, and is clearly not satisfied in discrete frameworks.

Let us introduce a notation used in the following propositions: $(A + B)$ denotes the conjunctions of axioms A and B, $(A + B) \Rightarrow C$ means that axiom C is a consequence of axioms $(A + B)$, and $A \not\Rightarrow B$ means that B is not a consequence of A (i.e., it is possible to find a criterion satisfying A but not B).

Proposition 5 $(\boldsymbol{\beta}) \Rightarrow (I)$ *and* $(I) \not\Rightarrow (\boldsymbol{\beta})$.

(SI) $\forall a, b \in A$ s.t. $a \trianglerighteq b$, let x $\in \mathcal{C} \setminus (R(a) \cup R(b))$ and let $a', b' \in A$ s.t. $R(a') = R(a) \cup \{x\}$ and $R(b') = R(b) \cup \{x\}$. Then $a' \trianglerighteq b'$.

Axiom (SI) states that the addition of a common consequence to the respective consequences sets of two actions should not change the decision maker's preference between these actions. It is widely used in the literature of both decision making under complete ignorance (see e.g. (Bossert 1997)) and freedom of choice interpretation (see e.g. (Bossert, Pattanaik, and Xu 1994), (Klemisch-Ahlert 1993)), which makes it appear as a "natural" axiom. (Nehring and Puppe 1996) proves that (C+SI) characterizes a similar (but smaller w.r.t. set-inclusion) family of criteria than the one characterized by (C+I). This is due to the fact that (SI) strictly implies (I) (i.e., there exist criteria satisfying (I) and not (SI), as the Hurwicz criterion) (Nehring and Puppe 1996).

The following proposition shows that $(\boldsymbol{\beta})$ and (SI) are incomparable w.r.t. entailment:

Proposition 6 $(SI) \not\Rightarrow (\boldsymbol{\beta})$ *and* $(\boldsymbol{\beta}) \not\Rightarrow (SI)$.

(IND) $\forall a, b \in A$, let $x \in \mathcal{C} \setminus (R(a) \cup R(b))$ and let $a' \in A$ s.t. $R(a') = R(a) \cup \{x\}$ and $b' \in A$ s.t. $R(b') = R(b) \cup \{x\}$. If $a \rhd b$ then $a' \trianglerighteq b'$.

A variant of (SI) is (IND) introduced in (Kannai and Peleg 1984). (IND) requires that the addition of a common consequence to two actions should not reverse the preference between them when it is strict. It is easy to see that (SI) implies (IND).

In (Bossert, Pattanaik, and Xu 2000), (IND) is supplemented by the simple dominance axiom (SD) which is a weak version of the forthcoming axiom (GP).

(SD) $\forall x, y \in \mathcal{C}$ s.t. $x > y$, $[x] \rhd [x, y] \rhd [y]$.

Interestingly, the conjunction of (IND) and (SD) leads to the following result, close to our Lemma 1:

Proposition 7 ((Bossert, Pattanaik, and Xu 2000)) *Suppose that* $\trianglerighteq$ *is a complete preorder on* $\mathcal{A}$. *If* $\trianglerighteq$ *satisfies (SD+IND) then* $\forall a \in \mathcal{A}$, $a \diamond [minR(a), maxR(a)]$.

Now, the following result shows that $(\boldsymbol{\beta})$ and (IND) are incomparable w.r.t. entailment:

Proposition 8 $(IND) \not\Rightarrow (\boldsymbol{\beta})$ *and* $(\boldsymbol{\beta}) \not\Rightarrow (IND)$.

Comparing $(\boldsymbol{\beta})$ to (SI) and (IND) is valuable because each of them states in some way that the common consequences of actions are useless for characterizing the optimal actions. Remembering that (SI) implies both (I) and (IND), $(\boldsymbol{\beta})$ appears as a good balance between them. Thus $(\boldsymbol{\beta})$ excludes some criteria which intuitively seem not to fit the complete ignorance context and are nevertheless admitted by these axioms (the Cardinal criterion is one of them). Replacing $(\boldsymbol{\beta})$ by one of these axioms would not lead to the same family of criteria.

(GP) $\forall a \in A$, let $x \in \mathcal{C}$ and let $a' \in A$ such that $R(a') = R(a) \cup \{x\}$:

- If $\forall c \in R(a), x > c$ then $a' \rhd a$,
- If $\forall c \in R(a), c > x$ then $a \rhd a'$.

Axiom (GP) has been presented by (Kannai and Peleg 1984) in the finite setting and called (Dominance) in (Barbera, Bossert, and Pattanaik 2004). (Nehring and Puppe 1996) used it for refining the family of criteria admitted by (C+SI).

Axiom $(\boldsymbol{\beta'})$ has been introduced as a strict counterpart of $(\boldsymbol{\beta})$. Theorem 2 has shown that the satisfaction of $(\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\beta'})$ leads to a refinement of the family of criteria characterized by $(\boldsymbol{\alpha} + \boldsymbol{\beta})$. A similar refinement has been achieved in (Nehring and Puppe 1996) via the introduction of axiom (GP): when the set of consequences $\mathcal{C}$ is a Haussdorf space, (GP) refines the family of criteria admitted by (C+I). It was thus interesting to investigate the logical relationships between (GP) and $(\boldsymbol{\beta'})$:

Proposition 9 *1.* $(GP) \not\Rightarrow (\boldsymbol{\beta})$ *and* $(\boldsymbol{\beta}) \not\Rightarrow (GP)$.

2. $(GP) \not\Rightarrow (\boldsymbol{\beta'})$ *and* $(\boldsymbol{\beta'}) \not\Rightarrow (GP)$.

3. $(\boldsymbol{\alpha} + \boldsymbol{\beta}) \Rightarrow [(GP) \Leftrightarrow (\boldsymbol{\beta'})]$.

Our results show that although (GP) and $(\boldsymbol{\beta'})$ are not equivalent (a consequence of point (2) of the previous proposition), in the presence of $(\boldsymbol{\alpha} + \boldsymbol{\beta})$, they become equivalent so that we could replace $(\boldsymbol{\beta'})$ by (GP) in Theorem 2.

Related Work and Discussion

We have proposed in this paper a characterization of optimality criteria for decision making under complete ignorance. We have shown that the decision criteria satisfying axioms $(\boldsymbol{\alpha})$ and $(\boldsymbol{\beta})$ are exactly the ones which take account for the minimal and maximal consequences of the actions.

The closest work to our own one is (Nehring and Puppe 1996), which characterizes the same family of criteria as the ones captured by $(\boldsymbol{\alpha}+\boldsymbol{\beta})$, but using different axioms. More importantly, (Nehring and Puppe 1996) focuses on decision frames with a continuous set of consequences: in (Nehring and Puppe 1996), $\mathcal{C}$ is supposed to be a Hausdorf space, i.e., a separable arc connected space. This is a rich (and restrictive) structure. The characterization result given in (Nehring and Puppe 1996) also requires a continuity axiom (C), which does not make sense for discrete sets of consequences. The authors justify the specific study of such a topology by the central role played by continuous preferences in the economic field, but this choice makes their characterization results not suited to the case of a discrete set of consequences, unlike ours. Especially, as explained in the previous section, the axioms (I), (SI) and (GP) considered in (Nehring and Puppe 1996) are not sufficient to derive the characterization theorems we have pointed out.

Our characterization results are also close to the ones reported in (Arrow and Hurwicz 1977). Indeed, the authors characterize the same set of decision criteria as we did but in the setting of decision making under strict uncertainty. Accordingly, our work can be considered as an extension of (Arrow and Hurwicz 1977) to the case of decision making under complete ignorance. Finally, in (Maskin 1979) several extensions of the results of (Arrow and Hurwicz 1977) have been presented (still in the setting of decision making under strict uncertainty). A perspective for further work is to determine whether these results can be lifted to the case of decision making under complete ignorance.

Proofs

Proof of Lemma 1: Let $a \in \mathcal{A}$ and let us define a_1, a_n, a'_1, a'_n by $R(a_1) = \{minR(a)\}$, $R(a_n) = \{maxR(a)\}$, $R(a'_1) = R(a) \setminus \{minR(a)\}$, $R(a'_n) = R(a) \setminus \{maxR(a)\}$. Then we have $\forall c \in R(a_1)$, $\forall c' \in R(a'_n)$, $c' \geqslant c$. Since $R([minR(a), maxR(a)]) = R(a_1) \cup \{maxR(a)\}$ and $R(a) = R(a'_n) \cup \{maxR(a)\}$, axiom $(\boldsymbol{\beta})$ gives $a \trianglerighteq [minR(a), maxR(a)]$. Besides, we also have $\forall c \in R(a_n), \forall c' \in R(a'_1), c \geqslant c'$. Since $R([minR(a), maxR(a)]) = R(a_n) \cup \{minR(a)\}$ and $R(a) = R(a'_1) \cup \{minR(a)\}$, axiom $(\boldsymbol{\beta})$ gives $[minR(a), maxR(a)] \trianglerighteq a$. So we can conclude $a \diamond [minR(a), maxR(a)]$. □

Proof of Lemma 2: Let us first prove point 1. If a_1, a_2 are not optimally equivalent with respect to A, suppose w.l.o.g. $a_1 \in \hat{A}$, $a_2 \in A \setminus \hat{A}$. Since $\{a_1, a_2\}$ intersects $\hat{A}$, $(\boldsymbol{\alpha})$ imposes that the optimal set for $\{a_1, a_2\}$ consists of a_1. That contradicts the hypothesis $a_1 \diamond a_2$. Points 2 and 3 follow by similar applications of axiom $(\boldsymbol{\alpha})$. □

Proof of Proposition 1: Straighforwardly, if an optimality criterion satisfies $(\boldsymbol{\beta}^*)$ then it satisfies $(\boldsymbol{\beta})$. Let us show the other direction. Let A be a decision problem, $a, a' \in A$ such that $\forall c \in R(a), \forall c' \in R(a'), c \geqslant c'$. Consider $C \subseteq \mathcal{C}$ and $a_1, a'_1 \in A$ s.t. $R(a_1) = R(a) \cup C$ and $R(a'_1) = R(a') \cup C$. Consider the 4 following consequences $m = min(minR(a), minC)$, $m' = min(minR(a'), minC)$, $M = max(maxR(a), maxC)$ and $M' = max(maxR(a'), maxC)$. Lemma 1 gives $a_1 \diamond [m, M]$ and $a_1' \diamond [m', M']$. By construction, we always have $m \geq m'$ and $M \geq M'$; Since we assume that $(\boldsymbol{\alpha})$ and $(\boldsymbol{\beta})$ hold, Theorem 1 implies that $a_1 \trianglerighteq a'_1$. □

Proof of Theorem 1: To prove the only-if direction, suppose that axioms $(\boldsymbol{\alpha})$ and $(\boldsymbol{\beta})$ are satisfied by the criterion. We want to prove that there exists a complete preorder $\succcurlyeq$ which satisfies (1) and (2).

We define $\succcurlyeq$ on pairs of consequences (m, M) as follows:

$$(m_1, M_1) \succcurlyeq (m_2, M_2) \text{ iff } [m_1, M_1] \trianglerighteq [m_2, M_2].$$

Let us first prove that the relation $\succcurlyeq$ is a complete preorder, i.e., a reflexive, transitive and complete relation.

As to reflexivity, we must prove that for every pair (m, M) we have $(m, M) \succcurlyeq (m, M)$. By definition of the relation $\succcurlyeq$, this is equivalent to show that if we consider two actions a_1 and a_2 such that $R(a_1) = R(a_2) = \{m, M\}$ then $a_1 \trianglerighteq a_2$.

From Corollary 1, if $R(a_1) = R(a_2)$ we have $a_1 \diamond a_2$. Let A be the decision problem which actions are a_1 and a_2. Since from Hypothesis (A) $\hat{A} \neq \emptyset$ then by definition of $\diamond$, we have $\hat{A} = A$ with $A = \{a_1, a_2\}$. Therefore $a_1 \in \hat{A}$. Then by definition of $\trianglerighteq$, $a_1 \trianglerighteq a_2$. For proving transitivity, let us suppose $(m_1, M_1) \succcurlyeq (m_2, M_2)$ and $(m_2, M_2) \succcurlyeq (m_3, M_3)$. Let A be the decision problem which actions are $[m_1, M_1], [m_2, M_2], [m_3, M_3]$. Following Hypothesis (A), we have $\hat{A} \neq \emptyset$.

Suppose that $[m_3, M_3]$ is the only optimal action, i.e., $\hat{A} = \{[m_3, M_3]\}$, then $[m_3, M_3] \triangleright [m_2, M_2][A]$.

By Lemma 1, we have that $[m_3, M_3] \triangleright [m_2, M_2]$, hence $[m_3, M_3] = \hat{A}'$ where $A' = \{[m_2, M_2], [m_3, M_3]\}$. But by hypothesis $(m_2, M_2) \succcurlyeq (m_3, M_3)$ which gives by definition $[m_2, M_2] \trianglerighteq [m_3, M_3]$ and then $[m_2, M_2] \in \hat{A}'$. Contradiction.

Suppose now that $[m_2, M_2] \in \hat{A}$ but $[m_1, M_1] \notin \hat{A}$.

We have by hypothesis $(m_1, M_1) \succcurlyeq (m_2, M_2)$. Which by definition gives $[m_1, M_1] \trianglerighteq [m_2, M_2]$ and then $[m_2, M_2] \in \hat{A}$ implies $[m_1, M_1] \in \hat{A}$. Contradiction.

Hence, $[m_3, M_3]$ cannot be the unique element of $\hat{A}$ and if $[m_2, M_2]$ is optimal then so is $[m_1, M_1]$.

We must conclude that $[m_1, M_1] \in \hat{A}$, which gives $[m_1, M_1] \trianglerighteq [m_3, M_3][A]$. By Lemma 2 $[m_1, M_1] \trianglerighteq [m_3, M_3]$. Then by definition of $\succcurlyeq$, $(m_1, M_1) \succcurlyeq (m_3, M_3)$. The relation $\succcurlyeq$ is therefore transitive.

Following Hypothesis (A), since every decision problem $\{[m_1, M_1], [m_2, M_2]\}$ contains at least one optimal element, we have $(m_1, M_1) \succcurlyeq (m_2, M_2)$ or $(m_2, M_2) \succcurlyeq (m_1, M_1)$, which means that the relation $\succcurlyeq$ is total.

In order to prove that $\succcurlyeq$ satisfies condition (1) of the theorem, suppose now that $m_1 \geqslant m_2, M_1 \geqslant M_2$ and let us show that $(m_1, M_1) \succcurlyeq (m_2, M_2)$. If $m_1 \geq m_2$ then axiom $(\boldsymbol{\beta})$ allows us to conclude by adding M_2, that: $[m_1, M_2] \trianglerighteq [m_2, M_2]$.

Moreover, since $M_1 \geq M_2$, axiom $(\boldsymbol{\beta})$ allows us to conclude by adding m_1, that: $[m_1, M_1] \trianglerighteq [m_1, M_2]$. Then by definition of the preorder $(m_1, M_1) \succcurlyeq (m_1, M_2)$ and $(m_1, M_2) \succcurlyeq (m_2, M_2)$. The transitivity of the preorder implies that $(m_1, M_1) \succcurlyeq (m_2, M_2)$.

Let us now prove that condition (2) of the theorem is also satisfied. Suppose that $a \in \hat{A}$. Then $a \trianglerighteq a'[A]$ for every $a' \in A$, and by Lemma 2 we have $a \trianglerighteq a'$ for every $a' \in A$. By Lemma 1 we also have $[minR(a), maxR(a)] \diamond a$ and $[minR(a'), maxR(a')] \diamond a'$. Now consider the problem $A_1 = \{a, a', [minR(a), maxR(a)], [minR(a'), maxR(a')]\}$. By Lemma 2 (point 1) we have $[minR(a), maxR(a)] \diamond a[A_1]$ and $[minR(a'), maxR(a')] \diamond a'[A_1]$.

If $a' \triangleright a[A_1]$ then by Lemma 2 (point 3) $a' \triangleright a$, which is false by hypothesis. Then $\hat{A}_1$ matches one of the following cases:

- $a \in \hat{A}'$ and $a' \in \hat{A}'$.
- $a \in \hat{A}'$ and $a' \notin \hat{A}'$.
- $a \notin \hat{A}'$ and $a' \notin \hat{A}'$. But in this case, $[minR(a), maxR(a)] \diamond a[A_1]$ and $[minR(a'), maxR(a')] \diamond a'[A_1]$ implies $[minR(a), maxR(a)] \notin \hat{A}'$ and $[minR(a'), maxR(a')] \notin \hat{A}'$. Then $\hat{A}' = \emptyset$ which is impossible.

We can see that in all the remaining cases $a \in \hat{A}_1$ and then, since $[minR(a), maxR(a)] \diamond a[A_1]$, we can deduce that $[minR(a), maxR(a)] \in \hat{A}_1$. By definition of $\trianglerighteq$, we get:

$[minR(a), maxR(a)] \trianglerighteq [minR(a'), maxR(a')][A_1]$ and by Lemma 2 (point 2) $[minR(a), maxR(a)] \trianglerighteq [minR(a'), maxR(a')]$. Then by definition of the preorder $\succcurlyeq$ we have $(minR(a), maxR(a)) \succcurlyeq (minR(a'), maxR(a'))$. Suppose now $a \in A \setminus \hat{A}$. From Hypothesis (A), there exists an element a' in $\hat{A}$ such that $a' \triangleright a[A]$. Therefore by Lemma 2 (point 3), we also have $a' \triangleright a$. By Lemma 1 we have $[minR(a), maxR(a)] \diamond a$ and $[minR(a'), maxR(a')] \diamond a'$. We can conclude using a similar argument that: $(minR(a'), maxR(a')) \succ (minR(a), maxR(a))$ for some $a' \in A$.

Let us now prove the if direction of the theorem. Suppose that we have a total preorder on pairs of elements of $\mathcal{C}$ and an optimality criterion so that conditions (1) and (2) of the theorem are satisfied.

Let us show that the optimality criterion satisfies axiom $(\boldsymbol{\alpha})$. Suppose we have $A_1 \subseteq A_2$ and $A_1 \cap \hat{A}_2$ is non empty. Let $a \in A_1 \cap \hat{A}_2$. Since $a \in \hat{A}_2$ by condition (2) of the theorem we have $\forall a' \in A_2, (minR(a), maxR(a)) \succcurlyeq (minR(a'), maxR(a'))$. Since $A_1 \subseteq A_2$ and $a \in A_1 \cap \hat{A}_2$ implies $a \in \hat{A}_1$, we have $(minR(a), maxR(a)) \succcurlyeq (minR(a'), maxR(a')), \forall a' \in A_1$. Hence $a \in \hat{A}_1$. So $A_1 \cap \hat{A}_2 \subseteq \hat{A}_1$. Now let $a \in \hat{A}_1$. By condition (2) of the theorem, we have $(minR(a), maxR(a)) \succcurlyeq (minR(a'), maxR(a')), \forall a' \in A_1$. And especially for the elements of $A_1 \cap \hat{A}_2$. Let $a'' \in A_1 \cap \hat{A}_2$. $a'' \in \hat{A}_2$ implies by definition of $\succcurlyeq$ that $(minR(a''), maxR(a'')) \succcurlyeq (minR(a'''), maxR(a''')), \forall a''' \in A_2$ which implies by transitivity of $\succcurlyeq$ that $(minR(a), maxR(a)) \succcurlyeq (minR(a'''), maxR(a''')), \forall a''' \in A_2$. In other words we have $a \in \hat{A}_2$. We can then conclude that $a \in A_1 \cap \hat{A}_2$. Hence $\hat{A}_1 \subseteq A_1 \cap \hat{A}_2$. Finally we conclude that $\hat{A}_1 = A_1 \cap \hat{A}_2$.

Let us now show that the optimality criterion satisfies axiom $(\boldsymbol{\beta})$. Let $a, a' \in A$ such that $\forall c \in R(a), \forall c' \in R(a'), c \geq c'$. And let $c'' \in \mathcal{C}$. Let a_1, a'_1 be such that $R(a_1) = R(a) \cup \{c''\}$ and $R(a'_1) = R(a') \cup \{c''\}$ Let us show that $a_1 \trianglerighteq a'_1$.

- If $c'' \geq maxR(a)$, then $minR(a_1) \geq minR(a'_1)$ and $maxR(a_1) = maxR(a'_1)$.
- If $minR(a') \geq c''$, then $minR(a_1) = minR(a'_1)$ and $maxR(a_1) \geq maxR(a'_1)$.
- If $maxR(a) \geq c'' \geq minR(a')$, then $min(c'', minR(a)) \geq minR(a')$ and $maxR(a) \geq max(maxR(a'), c'')$. Therefore $minR(a_1) \geq minR(a'_1)$ and $maxR(a_1) \geq maxR(a'_1)$.

In each of the three cases, we have $minR(a_1) \geq minR(a'_1)$ and $maxR(a_1) \geq maxR(a'_1)$, which implies by condition (1) that $(minR(a_1), maxR(a_1)) \succcurlyeq (minR(a'_1), maxR(a'_1))$. □

Proof of Theorem 2: Given Theorem 1, it is enough to prove that adding c ondition (1b) amounts to adding axiom $(\boldsymbol{\beta'})$.

As in Theorem 1, let us define:

$(m_1, M_1) \succcurlyeq (m_2, M_2)$ if and only if $[m_1, M_1] \trianglerighteq [m_2, M_2]$

We now define in the natural way the associated strict relation:

$(m_1, M_1) \succ (m_2, M_2)$ if and only if $[m_1, M_1] \triangleright [m_2, M_2]$

Let us first prove that adding axiom $(\boldsymbol{\beta'})$ implies condition (1b). There are two cases: $m_1 > m_2$ or $M_1 > M_2$.

- Let m_1, m_2, M_1, M_2 be such that $m_1 > m_2$ and $M_1 \geq M_2$. Since $m_1 > m_2$, $(\boldsymbol{\beta'})$ enables us to conclude by adding M_2 that $[m_1, M_2] \triangleright [m_2, M_2]$. Since $M_1 \geq M_2$, $(\boldsymbol{\beta})$ also enables us to conclude by adding m_1 that $[m_1, M_1] \trianglerighteq [m_1, M_2]$. Finally by transitivity we get that $[m_1, M_1] \triangleright [m_2, M_2]$, showing that $(m_1, M_1) \succ (m_2, M_2)$.
- Let m_1, m_2, M_1, M_2 be such that $m_1 \geq m_2$ and $M_1 > M_2$. Since $m_1 \geq m_2$, $(\boldsymbol{\beta})$ enables us to conclude by

adding M_2 that $[m_1, M_2] \trianglerighteq [m_2, M_2]$. Since $M_1 > M_2$, $(\boldsymbol{\beta'})$ also enables us to conclude by adding m_1 that $[m_1, M_1] \triangleright [m_1, M_2]$. Finally by transitivity we get $[m_1, M_1] \triangleright [m_2, M_2]$, showing that $(m_1, M_1) \succ (m_2, M_2)$.

The other way around, assume that condition (1b) holds; let us prove that $(\boldsymbol{\beta'})$ holds as well. Let a, a' be such that $\forall c \in R(a), c' \in R(a'), c > c'$. Let $c'' \in \mathcal{C}$. Let a_1, a'_1 such that $R(a_1) = R(a) \cup \{c''\}$ and $R(a'_1) = R(a') \cup \{c''\}$:

- If $c'' \geq maxR(a)$, then $minR(a_1) > minR(a'_1)$ and $maxR(a_1) = maxR(a'_1)$.
- If $minR(a') \geq c''$, then $minR(a_1) = minR(a'_1)$ and $maxR(a_1) > maxR(a'_1)$.
- If $maxR(a) > c'' > minR(a')$, then we have $maxR(a_1) = maxR(a) > maxR(a'_1) = max(c", maxR(a'))$ and $minR(a_1) = min(c", minR(a)) > minR(a'_1) = minR(a')$.

In each case, using condition (1b), we conclude that $a_1 \triangleright a'_1$. □

Proof of Proposition 2: Let us first show that Min satisfies $(\boldsymbol{\alpha})$. Consider a decision problem $A \subseteq \mathcal{A}$. Let $a \in \hat{A}$ be an action and $A_1 \subseteq A$ be a decision problem such that $a \in A_1$.

If one uses the Min criterion, then $a \in \hat{A}$ implies $a \trianglerighteq_{min} a', \forall a' \in A$, and by definition $minR(a) \geq minR(a'), \forall a' \in A$. In particular $minR(a) \geq minR(a'), \forall a' \in A_1$. So $a \in \hat{A}_1$, showing that Min satisfies $(\boldsymbol{\alpha})$.

Let us now show that the Min criterion satisfies $(\boldsymbol{\beta})$. Consider two actions $a, a' \in \mathcal{A}$ such that $\forall c \in R(a), c' \in R(a')$ $c \geq c'$. Then we have $minR(a) \geq minR(a')$. Now consider $c^{''} \in \mathcal{C}$ and $a_1, a_1^{'} \in \mathcal{A}$ such that $R(a_1) = R(a) \cup \{c^{''}\}$ and $R(a_1^{'}) = R(a') \cup \{c^{''}\}$. Then we are in one of the following cases:

- If $c'' \geq maxR(a)$, then $minR(a_1) \geq minR(a'_1)$.
- If $minR(a') \geq c''$, then $minR(a_1) = minR(a'_1)$.
- If $maxR(a) > c'' > minR(a')$, then $minR(a_1) = min(c", minR(a)) > minR(a'_1)$.

In each case we have $minR(a_1) \geq minR(a'_1)$, showing that $a_1 \trianglerighteq_{min} a'_1$. So Min satisfies $(\boldsymbol{\beta})$.

The proof for Max is similar. □

Proof of Proposition 3: It is enough to prove that the conditions (1a), (1b) and (2) of Theorem 2 are satisfied. Let the preorder $\succcurlyeq$ over the pairs (m, M) of $\mathcal{C} \times \mathcal{C}$ such that $m \leq M$ defined by $(m_1, M_1) \succcurlyeq (m_2, M_2)$ iff $m_1 > m_2$ or ($m_1 = m_2$ and $M_1 \geq M_2$). Clearly, $\succcurlyeq$ satisfies both conditions (1a) and (1b). Furthermore, for $a, b \in \mathcal{A}$, we have $a \trianglerighteq_{minmax} b$ iff $(minR(a), maxR(a)) \succcurlyeq (minR(b), max(R(b))$, which shows that condition (2) is satisfied, and concludes the proof.

The proof for Maxmin is similar. □

Proof of Proposition 4: As in the proof of Proposition 3, we take advantage of Theorem 2. Let the preorder $\succcurlyeq$ over the pairs (m, M) of $\mathcal{C} \times \mathcal{C}$ such that $m \leq M$ defined by $(m_1, M_1) \succcurlyeq (m_2, M_2)$ iff $\alpha * m_1 + (1 - \alpha) * M_1 > \alpha * m_2 + (1 - \alpha) * M_2$. Clearly, $\succcurlyeq$ satisfies both conditions (1a) and (1b). Furthermore, for $a, b \in \mathcal{A}$, we have $a \triangleright_\alpha b$ iff $(minR(a), maxR(a)) \succcurlyeq (minR(b), max(R(b))$, which shows that condition (2) is satisfied, and concludes the proof. □

Proof of Proposition 5:

Let us first show that $(\boldsymbol{\beta}) \Rightarrow (I)$.

Consider $a \in A$, and $x, y \in \mathcal{C} \setminus R(a)$ s.t. $x \geq y$. Then take $a_x, a_y \in A$ s.t. $R(a_x) = R(a) \cup \{x\}$ and $R(a_y) = R(a) \cup \{y\}$. We have $y \geq minR(a_y)$ and $minR(a) \geq minR(a_y)$. Let $t \in R(a_x)$. If $t = x$, then by transitivity of $\geq$, we get $t \geq minR(a_y)$. If $t \in R(a)$, then we have $t \geq minR(a)$, hence by transitivity of $\geq$, we also get $t \geq minR(a_y)$. Then we have $\forall t \in R(a_x)$, $t \geq minR(a_y)$ and especially $minR(a_x) \geq minR(a_y)$. A similar argument gives $maxR(a_x) \geq maxR(a_y)$. From axiom $(\boldsymbol{\beta})$, we get that that $[minR(a_x), maxR(a_y)] \trianglerighteq [minR(a_y), maxR(a_y)]$, and that $[minR(a_x), maxR(a_x)] \trianglerighteq [minR(a_x), maxR(a_y)]$. Then we can conclude by transitivity of the preorder $\trianglerighteq$ that $[minR(a_x), maxR(a_x)] \trianglerighteq [minR(a_y), maxR(a_y)]$. Finally Lemma 1 gives $a_x \trianglerighteq a_y$, showing that axiom (I) is satisfied.

To show that $(I) \not\Rightarrow (\boldsymbol{\beta})$, it is enough to exhibit a criterion satisfying (I) without satisfying $(\boldsymbol{\beta})$. This is the case of the Leximin criterion, defined as follows. For any subset C of $(\mathcal{C})$, let $C\uparrow$ be the list obtained by sorting C in increasing order w.r.t. $\leq$. For $a, b \in \mathcal{A}$, let $R(a)\uparrow = \langle c_1, \ldots, c_m \rangle$ and $R(b)\uparrow = \langle c'_1, \ldots, c'_n \rangle$. We have $a \trianglerighteq_{lex} b$ iff $a \diamond_{lex} b$ or $a \triangleright_{lex} b$, where $a \diamond_{lex} b$ iff $R(a) = R(b)$ and $a \triangleright_{lex} b$ iff $\exists i$ s.t. $0 \leq i < m$, $\forall j$ s.t. $1 \leq j \leq i$, $c_j = c'_j$ and ($c_{i+1} > c'_{i+1}$ or $m > n = j$).

- Leximin satisfies (I).
  Let $a \in A, x, y \in \mathcal{C} \setminus R(a)$ s.t. $x \geq y$. Let $a_x, a_y \in A$ s.t. $R(a_x) = R(a) \cup \{x\}$ and $R(a_y) = R(a) \cup \{y\}$.
  - If $x = y$ then $a_x = a_y$ and $a_x \diamond_{lex} a_y$.
  - If $x > y$ then let i be the rank of y in $R(a_y)\uparrow$; c_{xi}, the element of $R(a_x)\uparrow$ at rank i is such that $y < c_{xi}$, showing that $a_x \triangleright_{lex} a_y$.

  So if $x \geq y$ then $a_x \trianglerighteq_{lex} a_y$.
- Leximin does not satisfy $(\boldsymbol{\beta})$.
  Let $x, y, z, t \in \mathcal{C}$ such that $x > y > z > t$. Let A be a decision problem, and $a, a', a_1, a'_1 \in A$ such that $R(a) = \{x, z, t\}, R(a') = \{t\}, R(a_1) = R(a) \cup \{y\}$ and $R(a'_1) = R(a') \cup \{y\}$. So $\forall c \in R(a), \forall c' \in R(a')$, $c \geq c'$, but $a'_1 \triangleright_{lex} a_1$ whereas $(\boldsymbol{\beta})$ demands that $a_1 \trianglerighteq_{lex} a'_1$.

□

Proof of Proposition 6:

- Let the Cardinality criterion be defined by $\forall a, b \in \mathcal{A}$, $a \trianglerighteq_{card} b$ iff $|R(a)| \geq |R(b)|$.

Let us show that the Cardinality criterion satisfies (SI) but does not satisfy $(\boldsymbol{\beta})$.

On the one hand, let $a, b \in A$ s.t. $a \trianglerighteq_{card} b$ and $x \in \mathcal{C} \setminus (R(a) \cup R(b))$. Let $a', b' \in A$ s.t. $R(a') = R(a) \cup \{x\}$ and $R(b') = R(b) \cup \{x\}$. Obviously, if $a \trianglerighteq_{card} b$, then we also have $|R(a) \cup \{x\}| \geq |R(b) \cup \{x\}|$, showing that $a' \trianglerighteq_{card} b'$. Hence, the Cardinality criterion satisfies (SI).

On the other hand, consider $a, b \in A$ s.t. $R(a) = \{y\}$ and $R(b) = \{z, t\}$ and $y > z > t$. Then take $x \in \mathcal{C} \setminus (R(a) \cup R(b))$. Let $a', b' \in A$ s.t. $R(a') = R(a) \cup \{x\}$ and $R(b') = R(b) \cup \{x\}$. Then $b' \triangleright_{card} a'$ whereas $(\boldsymbol{\beta})$ requires that $a' \trianglerighteq_{card} b'$.

- Minmax satisfies $(\boldsymbol{\beta})$ (cf. Proposition 3) but it does not satisfy (SI). Indeed, it does not satisfy (IND) – cf. the proof of Proposition 8 –, and (IND) is a consequence of (SI).

□

Proof of Proposition 8:

- The Cardinality criterion satisfies (IND) since it satisfies (SI) and (SI) implies (IND), and it does not satisfy $(\boldsymbol{\beta})$ (cf. Proposition 6).
- The Minmax criterion satisfies $(\boldsymbol{\beta})$ (cf. Proposition 3), but it but does not satisfy (IND). Indeed, let $a, b \in A$ s.t. $minR(a) > minR(b)$ and $maxR(b) > maxR(a)$. Clearly $a \triangleright_{minmax} b$. Let $c \in \mathcal{C}$ s.t. $minR(b) > c$ and $a', b' \in A$ s.t. $R(a') = R(a) \cup \{c\}$ and $R(b') = R(b) \cup \{c\}$. We have $minR(a') = minR(b') = c$ and $maxR(b') = maxR(b) > maxR(a) = maxR(a')$, showing that $b' \triangleright_{minmax} a'$ whereas (IND) requires that $a' \trianglerighteq_{minmax} b'$.

□

Proof of Proposition 9:

Let us first demonstrate the point (1) of the proposition. To show that $(\boldsymbol{\beta}) \not\Rightarrow (GP)$, consider the Min criterion. We have already shown that Min satisfies $(\boldsymbol{\beta})$ (cf. Proposition 2). Let us show that it does not satisfy (GP). Let $a \in A$ be an action, and $x \in \mathcal{C}$ be a consequence such that $\forall c \in R(a), x > c$. Let $a' \in A$ such that $R(a') = R(a) \cup \{x\}$. Then we have $a \diamond_{min} a'$ whereas (GP) would demand that $a' \triangleright_{min} a$.

Let us show now that $(GP) \not\Rightarrow (\boldsymbol{\beta})$. We have already shown that Leximin does not satisfy $(\boldsymbol{\beta})$ (cf. the proof of Proposition 5). Let us show that Leximin satisfies (GP).
Let $a \in A$ and $x \in \mathcal{C}$. Then take $a' \in A$ s.t. $R(a') = R(a) \cup \{x\}$,

- If $\forall c \in R(a)\ x > c$ and $R(a)\uparrow = \langle c_1, \ldots, c_m \rangle$, then we have $R(a)\uparrow = \langle c_1, \ldots, c_m, x \rangle$, showing that $a' \triangleright_{lex} a$, as expected.
- If $\forall c \in R(a)\ c > x$, then in particular $minR(a) > x$, which gives $a \triangleright_{lex} a'$.

Let us now demonstrate the point (2) of the proposition. Let us first show that $(\boldsymbol{\beta'}) \not\Rightarrow (GP)$ with the Min2 criterion (comparison w.r.t. the "second minimal consequence" in case of equality of the "first minimal consequence"), that can be defined as follows: for any $a, b \in \mathcal{A}$, if at least one of $R(a)$ or $R(b)$ is a singleton, then Min2 is Min; otherwise, for any action a, let $minR2(a) = minR(a2)$ where $R(a2) = R(a) \setminus \{minR(a)\}$. Finally, $a \trianglerighteq_{min2} b$ iff $minR(a) > minR(b)$ or ($minR(a) = minR(b)$ and $minR2(a) \geq minR2(b)$) .

We first show that Min2 satisfies $(\boldsymbol{\beta'})$.

Let $a, a' \in A$ s.t. $\forall c \in R(a), c' \in R(a'), c > c'$. Let $c'' \in \mathcal{C}$, then choose $R(a_1) = R(a) \cup \{c''\}$ and $R(a'_1) = R(a') \cup \{c"\}$. There are two possible cases:

- $c'' > minR(a')$. Then $minR(a_1) = min(c", minR(a)) > minR(a'_1) = c"$. This implies that $a_1 \triangleright_{min2} a'_1$.
- $minR(a') \geq c"$. So $minR(a_1) = minR(a'_1)$. Then we have $minR2(a_1) = minR2(a) > minR2(a'_1)$, showing that $a_1 \triangleright_{min2} a'_1$.

In the two cases $a_1 \triangleright_{min2} a'_1$, showing that $(\boldsymbol{\beta'})$ is satisfied.

In order to show that $min2$ does not satisfy (GP), let us consider the following example:

Let $a \in A$ such that $R(a) = \{y, z\}$ with $y > z$, and let $x \in \mathcal{C}$ such that $x > y$. Let $a' \in A$ is such that $R(a') = R(a) \cup \{x\}$. We have $a \diamond_{min2} a'$ whereas (GP) would demand $a' \triangleright_{min2} a$.

Finally, the fact that $(GP) \not\Rightarrow (\boldsymbol{\beta'})$ comes directly from the fact that $(GP) \not\Rightarrow (\boldsymbol{\beta})$, since $(\boldsymbol{\beta'}) \Rightarrow (\boldsymbol{\beta})$.

Let us now demonstrate the point (3) of the proposition. Let us first suppose that $(\boldsymbol{\beta'})$ is satisfied.

Let $a \in A$ and $x \in \mathcal{C}$. Let $a' \in A$ such that $R(a') = R(a) \cup \{x\}$.
By Lemma 1 we get $a \diamond [minR(a), maxR(a)]$ and $a' \diamond [minR(a'), maxR(a')]$.

- If $\forall c \in R(a), x > c$ then $minR(a') = minR(a)$ and $maxR(a') > maxR(a)$. In this case, $(\boldsymbol{\beta'})$ implies that $[minR(a'), maxR(a')] \triangleright [minR(a), maxR(a)]$, or equivalently $a' \triangleright a$.
- If $\forall c \in R(a), c > x$ then $minR(a') < minR(a)$ and $maxR(a') = maxR(a)$. In this case, $(\boldsymbol{\beta'})$ implies that $[minR(a), maxR(a)] \triangleright [minR(a'), maxR(a')]$, or equivalently $a \triangleright a'$.

So (GP) is satisfied.

Let us suppose now that (GP) is satisfied.

Let A be a decision problem, and $a, a', a_1, a'_1 \in A$ such that $\forall c \in R(a), \forall c' \in R(a'), c > c'$.

Consider $c'' \in \mathcal{C}$, if $R(a_1) = R(a) \cup \{c''\}$ and $R(a'_1) = R(a') \cup \{c''\}$, then we are in one of the following cases:

- $c'' \geq maxR(a)$. In this case $minR(a_1) > minR(a'_1)$ and $maxR(a_1) = maxR(a'_1)$. Axiom (GP) implies that $[minR(a_1), maxR(a'_1)] \triangleright [minR(a_1), minR(a'_1), maxR(a'_1)]$.

Hence $[minR(a_1), maxR(a_1)] \rhd [minR(a_1), minR(a'_1), maxR(a'_1)]$ because $maxR(a_1) = maxR(a'_1)$. By Lemma 1 $[minR(a'_1), maxR(a'_1)] \diamond [minR(a_1), minR(a'_1), maxR(a'_1)]$. So we finally get $[minR(a_1), maxR(a_1)] \rhd [minR(a'_1), maxR(a'_1)]$; by Lemma 1 we get that $a_1 \rhd a'_1$.

- $minR(a) \geq c''$. In this case $minR(a_1) = minR(a'_1)$ and $maxR(a_1) > maxR(a'_1)$. Axiom (GP) implies that $[minR(a_1), maxR(a'_1), maxR(a_1)] > [minR(a_1), maxR(a'_1)]$. Hence $[minR(a_1), maxR(a_1), maxR(a'_1)] \rhd [minR(a'_1), maxR(a'_1)]$ since $minR(a_1) = minR(a'_1)$. Lemma 1 gives $[minR(a_1), maxR(a_1)] \diamond [minR(a_1), maxR(a'_1), maxR(a_1)]$. Then we have $[minR(a_1), maxR(a_1)] \rhd [minR(a'_1), maxR(a'_1)]$; by Lemma 1 we get that $a_1 \rhd a'_1$.
- $maxR(a) > c'' > minR(a')$. In this case $minR(a_1) > minR(a'_1)$ and $maxR(a_1) > maxR(a'_1)$. Depending of the relative position of $maxR(a'_1)$ and $minR(a_1)$, we have one of the following cases:
  - $maxR(a'_1) > minR(a_1)$. Axiom (GP) implies that $[minR(a_1), maxR(a'_1)] \rhd [minR(a'_1), minR(a_1), maxR(a'_1)]$. Lemma 1 gives $[minR(a_1), maxR(a'_1)] \rhd [minR(a'_1), maxR(a'_1)]$. Furthermore, axiom (GP) implies that $[minR(a_1), maxR(a'_1), maxR(a_1)] \rhd [minR(a_1), maxR(a'_1)]$ and by Lemma 1 $[minR(a_1), maxR(a_1)] \rhd [minR(a_1), maxR(a'_1)]$. By transitivity of the preorder we get that $[minR(a_1), maxR(a_1)] \rhd [minR(a'_1), maxR(a'_1)]$. By Lemma 1, we get $a_1 \rhd a'_1$.
  - $maxR(a_1) > minR(a'_1)$. Axiom (GP) implies that $[minR(a_1), maxR(a'_1)] \rhd [minR(a_1), maxR(a'_1), minR(a'_1)]$. Axiom (GP) also implies that $[minR(a_1), maxR(a'_1), minR(a'_1)] \rhd [minR(a'_1), maxR(a'_1)]$. Then by transitivity $[minR(a_1), maxR(a'_1)] \rhd [minR(a'_1), maxR(a'_1)]$. A similar argument gives $[minR(a_1), maxR(a_1)] \rhd [minR(a_1), maxR(a'_1)]$. By transitivity of the preorder we get $[minR(a_1), maxR(a_1)] \rhd [minR(a'_1), maxR(a'_1)]$ and Lemma 1 gives $a_1 \rhd a'_1$.
  - $maxR(a'_1) = minR(a_1)$. Axiom (GP) implies that $[minR(a_1), maxR(a'_1)] \rhd [minR(a'_1), minR(a_1), maxR(a_1)]$.And $[minR(a_1), maxR(a_1), minR(a'_1)] = [minR(a'_1), maxR(a'_1), maxR(a_1)]$. (GP) implies that $[minR(a'_1), maxR(a'_1), maxR(a_1)] \rhd [minR(a'_1), maxR(a'_1)]$. By Lemma 1, we get $a_1 \rhd a'_1$.

In each case we have $a_1 \rhd a'_1$. So (β') is satisfied. □

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