

On Logics and Semantics of Indeterminate Causation

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Abstract

We will explore the use of *disjunctive* causal rules for representing indeterminate causation. We provide first a logical formalization of such rules in the form of a disjunctive inference relation, and describe its logical semantics. Then we consider a nonmonotonic semantics for such rules, described in (Turner 1999). It will be shown, however, that, under this semantics, disjunctive causal rules admit a stronger logic in which these rules are reducible to ordinary, singular causal rules. This semantics also tends to give an exclusive interpretation of disjunctive causal effects, and so excludes some reasonable models in particular cases. To overcome these shortcomings, we will introduce an alternative nonmonotonic semantics for disjunctive causal rules, called a covering semantics, that permits an inclusive interpretation of indeterminate causal information. Still, it will be shown that even in this case there exists a systematic procedure, that we will call a normalization, that allows us to capture precisely the covering semantics using only singular causal rules. This normalization procedure can be viewed as a kind of nonmonotonic completion, and it generalizes established ways of representing indeterminate effects in current theories of action.

Introduction

The ability to represent actions with indeterminate effects constitutes an important objective of any general theory of action and change in AI. And indeed, most of these theories provide specific tools and methodologies for representing such actions in their formalisms.

The basic difficulty in representing indeterminate actions boils down to the fact that a plain classical logical description of the overall ‘disjunctive’ effect of such an action is patently insufficient for describing (and predicting) the actual effects of this action in particular action domains; what remains to be specified is what are the particular fluents that can (or cannot) be affected by the action.

This difficulty is quite general, so it transcends the boundaries of specific action theories. Accordingly, most action theories have adopted a two-part approach to representing indeterminate actions which stipulates both the disjunctive effect of an action, and the particular fluents that can be influenced by (or depend on) this action (see, e.g., (Lang, Lin, and Marquis 2003; Castilho, Herzig, and Varzinczak 2002)).

A prominent approach to reasoning about actions consists in using causal representations (see, e.g., (Lin 1995; McCain and Turner 1997; Thielscher 1997; Giunchiglia et al. 2004)). In particular, the causal calculus of (McCain and Turner 1997) makes use of causal rules $A \Rightarrow B$ that express causal relations among propositions. The nonmonotonic semantics of such theories is determined by causally explained interpretations in which every fact is explained by some causal rule.

It has been shown in (McCain and Turner 1997) (see also (Giunchiglia et al. 2004)) that such causal rules are also capable of describing indeterminate causal effects, but also in this case, the corresponding causal description involved two kinds of rules, a rule (constraint) that describes the disjunctive effect of an action, and a number of ‘explanatory’ rules of the form $A \wedge F \Rightarrow F$ saying that a literal F is causally explainable if it holds after action A .

As early as in (Lin 1996), Fangzhen Lin has suggested an alternative approach, according to which an action with indeterminate effects could be fully described using a single disjunctive causal rule of the form:

$$A \Rightarrow B_1, \dots, B_n$$

saying, roughly, that, whenever A holds, one of B_i is caused. This idea has been taken up in (Bochman 2003b), where the formalism of the causal calculus has been generalized to such disjunctive causal rules.

Taking as independent concept, a disjunctive causal rule can be naturally viewed as a qualitative counterpart of probabilistic causation, the only difference being that the latter also assigns probabilities to each particular effect. Accordingly, such rules have much in common with probabilistic causal rules of CP-logic (Vennekens, Denecker, and Bruynooghe 2009) (see also (Bogaerts et al. 2014)).

The appropriateness of rules of this kind for representing indeterminate causation can be justified as follows. Suppose that we have two ordinary causal rules $A \wedge C \Rightarrow B$ and $A \wedge \neg C \Rightarrow \neg B$ ¹. The rules can be interpreted as saying that A causes either B or $\neg B$, depending on whether the additional condition C holds. Suppose now that C is absent from our vocabulary (it is a ‘hidden parameter’). Still,

¹Say, A is ‘The switch is flipped’, B is ‘The switch is down’ and C is ‘The switch is initially up’.

the above description conveys a nontrivial information about the situation in question, namely that when A holds, either B is caused or $\neg B$ is caused, that is, $A \Rightarrow B, \neg B$. Note that this information is completely lost in a singular causal rule $A \Rightarrow B \vee \neg B$, which always holds for causal relations.

A further interesting aspect of the above example is that the use of disjunctive causal rules is still not essential for the above situation: due to the fact that B and $\neg B$ are logically incompatible, the disjunctive rule $A \Rightarrow B, \neg B$ turns out to be logically equivalent to a pair of ordinary causal rules $A \wedge B \Rightarrow B$ and $A \wedge \neg B \Rightarrow \neg B$. Nevertheless, such a reduction is impossible for general disjunctive rules.

From a purely technical point of view, disjunctive causal rules form a natural, though largely unused, fragment of Turner's logic of universal causation (UCL) (Turner 1999), and there are deep reasons for this lack of use. Namely, it has been shown in (Bochman 2003b), that the nonmonotonic semantics of UCL sanctions a straightforward reduction of such rules to ordinary singular causal rules. Furthermore, it has a shortcoming in that it tends to give an exclusive interpretation to disjunctive heads of causal rules. That is why in this study we introduce a new, *covering semantics* for such causal theories that will provide an inclusive interpretation of disjunctive causal information.

It will be shown, however, that in many regular cases a disjunctive causal theory can be 'normalized', that is, transformed into a non-disjunctive causal theory in such a way that the standard nonmonotonic semantics of the latter will coincide with the covering semantics of the source disjunctive theory. Furthermore, this transformation will be shown to be closely related to canonical ways of representing indeterminism in causal theories, employed in (McCain and Turner 1997; Giunchiglia et al. 2004). Accordingly, disjunctive causal theories coupled with the new semantics suggest themselves as a systematic framework for representing indeterminate causal information.

Preliminaries: The Causal Calculus

Based on the ideas from (Geffner 1992), the causal calculus was introduced in (McCain and Turner 1997) as a nonmonotonic formalism purported to serve as a logical basis for reasoning about action and change in AI. A generalization of the causal calculus to the first-order classical language was described in (Lifschitz 1997). This line of research has led to the action description language $\mathcal{C}+$, which is based on this calculus and serves for describing dynamic domains (Giunchiglia et al. 2004). A logical basis of the causal calculus was described in (Bochman 2003a), while (Bochman 2004; 2007) studied its possible uses as a general-purpose nonmonotonic formalism.

We will assume that our language is an ordinary propositional language with the classical connectives and constants $\{\wedge, \vee, \neg, \rightarrow, \mathbf{t}, \mathbf{f}\}$. \models will stand for the classical entailment, while Th will denote the classical provability operator. In what follows we will identify a propositional interpretation ('world') with the set of propositional formulas that hold in it.

A *causal rule* is an expression of the form $A \Rightarrow B$ (" A causes B "), where A and B are propositional formulas. A

causal theory is a set of causal rules.

A nonmonotonic semantics of a causal theory can be defined as follows.

For a causal theory Δ and a set u of propositions, let $\Delta(u)$ denote the set of propositions that are caused by u in Δ :

$$\Delta(u) = \{B \mid A \Rightarrow B \in \Delta, \text{ for some } A \in u\}$$

Definition 1. A world α is an *exact model* of a causal theory Δ if it is the unique model of $\Delta(\alpha)$. The set of exact models forms a *nonmonotonic semantics* of Δ .

The above nonmonotonic semantics of causal theories is equivalent to the semantics described in (McCain and Turner 1997). It can be verified that exact models of a causal theory are precisely the worlds that satisfy the condition

$$\alpha = \text{Th}(\Delta(\alpha)).$$

Informally speaking, an exact model is a world that is closed with respect to the causal rules and also has the property that any proposition that holds in it is caused (determined) ultimately by other propositions.

The causal calculus can be viewed as a two-layered construction. The nonmonotonic semantics defined above forms its top level. Its bottom level is the monotonic logic of causal rules introduced in (Bochman 2003a; 2004); it constitutes the *causal logic* of the causal calculus.

A *causal inference relation* is a relation \Rightarrow on the set of propositions satisfying the following conditions:

(Strengthening) If $A \models B$ and $B \Rightarrow C$, then $A \Rightarrow C$;

(Weakening) If $A \Rightarrow B$ and $B \models C$, then $A \Rightarrow C$;

(And) If $A \Rightarrow B$ and $A \Rightarrow C$, then $A \Rightarrow B \wedge C$;

(Or) If $A \Rightarrow C$ and $B \Rightarrow C$, then $A \vee B \Rightarrow C$;

(Cut) If $A \Rightarrow B$ and $A \wedge B \Rightarrow C$, then $A \Rightarrow C$;

(Truth) $\mathbf{t} \Rightarrow \mathbf{t}$;

(Falsity) $\mathbf{f} \Rightarrow \mathbf{f}$.

Causal inference relations satisfy almost all the usual postulates of classical inference, except Reflexivity $A \Rightarrow A$. The absence of the latter has turned out to be essential for an adequate representation of causal reasoning.

A possible worlds semantics. A logical semantics of causal inference relations has been given in (Bochman 2004) in terms of possible worlds (Kripke) models.

Definition 2. A causal rule $A \Rightarrow B$ is said to be *valid* in a Kripke model (W, R, V) if, for any worlds α, β such that $R\alpha\beta$, if A holds in α , then B holds in β .

It has been shown that causal inference relations are complete for quasi-reflexive Kripke models, that is, for Kripke models in which the accessibility relation R satisfies the condition that if $R\alpha\beta$, for some β , then $R\alpha\alpha$.

The above semantics sanctions a simple modal representation of causal rules. Namely, the validity of $A \Rightarrow B$ in a possible worlds model is equivalent to validity of the formula $A \rightarrow \Box B$, where \Box is the standard modal operator. In fact, this modal representation has been used in many other approaches to formalizing causation in action theories (see, e.g., (Geffner 1990; Turner 1999; Giordano, Martelli, and Schwind 2000; Zhang and Foo 2001)).

Strong equivalence. It has been shown in (Bochman 2003a) that if \Rightarrow_Δ is the least causal inference relation that includes a causal theory Δ , then \Rightarrow_Δ has the same non-monotonic semantics as Δ . This has shown that the rules of causal inference are adequate for reasoning with respect to the nonmonotonic semantics. Moreover, as a consequence of a corresponding *strong equivalence* theorem, it was shown that the above causal inference relations constitute a maximal such logic.

Definition 3. Causal theories Γ and Δ are called

- *objectively equivalent* if they have the same nonmonotonic semantics;
- *strongly equivalent* if, for any set Φ of causal rules, $\Delta \cup \Phi$ is objectively equivalent to $\Gamma \cup \Phi$;
- *causally equivalent* if $\Rightarrow_\Delta = \Rightarrow_\Gamma$.

Two causal theories are causally equivalent if each theory can be obtained from the other using the inference postulates of causal relations. Then the following result has been proved in (Bochman 2004):

Proposition 1 (Strong equivalence). *Causal theories are strongly equivalent iff they are causally equivalent.*

Disjunctive Causal Relations

Now we are going to generalize the causal calculus to indeterminate causal rules that involve multiple heads.

In what follows a, b, \dots will denote finite sets of propositions, while u, v, \dots will denote arbitrary such sets. For a finite set of propositions a , we will denote by $\bigwedge a$ the conjunction of all propositions from a ; as a special case, $\bigwedge \emptyset$ will denote \mathbf{t} . For any set u of propositions, we will denote by \bar{u} the complement of u in the set of all propositions, and by $\neg u$ the set $\{\neg A \mid A \in u\}$.

We will consider disjunctive causal rules as rules holding primarily between finite sets of propositions: $a \Rightarrow b$ will be taken to mean that if all the propositions in a hold, then one of the propositions in b is caused. We will use also an ordinary notation for premise and conclusion sets in causal rules. Thus, $a, b \Rightarrow c$ will stand for $a \cup b \Rightarrow c$, and $a, A \Rightarrow$ will mean $a \cup \{A\} \Rightarrow \emptyset$, etc.

Definition 4. A *disjunctive causal relation* is a binary relation \Rightarrow on finite sets of classical propositions satisfying the following conditions:

- (Left Monotonicity) If $a \Rightarrow b$, then $A, a \Rightarrow b$;
- (Right Monotonicity) If $a \Rightarrow b$, then $a \Rightarrow b, A$;
- (Cut) If $a \Rightarrow b, A$ and $A, a \Rightarrow b$, then $a \Rightarrow b$;
- (Strengthening) If $A \models B$ and $a, B \Rightarrow b$, then $a, A \Rightarrow b$;
- (Weakening) If $A \models B$ and $a \Rightarrow b, A$, then $a \Rightarrow b, B$;
- (Left And) If $a, A \wedge B \Rightarrow b$, then $a, A, B \Rightarrow b$;
- (And) If $a \Rightarrow b, A$ and $a \Rightarrow b, B$, then $a \Rightarrow b, A \wedge B$;
- (Or) If $A, a \Rightarrow b$ and $B, a \Rightarrow b$, then $A \vee B, a \Rightarrow b$;
- (Falsity) $\mathbf{f} \Rightarrow$;
- (Truth) $\Rightarrow \mathbf{t}$.

Similarly to ordinary consequence relations, a disjunctive causal relation can be extended to arbitrary sets of propositions by requiring *compactness*:

(Comp) $u \Rightarrow v$ iff $a \Rightarrow b$, for some finite $a \subseteq u, b \subseteq v$.

As can be seen, a disjunctive causal relation forms a subsystem of the classical sequent calculus. However, the former is only an applied logical formalism that is not purported to give meaning to the logical connectives. In fact, the *classical* meaning of these connectives is secured by the use of the classical entailment in the above postulates. Note, in particular, that Strengthening and Weakening imply that classically equivalent propositions are interchangeable both in bodies and heads of the causal rules. Some further ‘classical’ properties are determined by the other postulates. Thus, it is straightforward to show that the postulates imply the following two general rules:

(Logical Strengthening) If $c \models A$ and $A, a \Rightarrow b$, then $c, a \Rightarrow b$.

(Logical Weakening) If $c \models A$ and $a \Rightarrow b, C$, for every $C \in c$, then $a \Rightarrow b, A$.

The first rule implies that a finite set of premises in a causal rule can be replaced by their conjunction:

$$a \Rightarrow b \text{ iff } \bigwedge a \Rightarrow b$$

However, the heads of disjunctive causal rules cannot be replaced with their classical disjunctions; we have only that $a \Rightarrow b$ always implies $a \Rightarrow \bigvee b$, though not vice versa. Also, only the following structural rule for negation holds for disjunctive causal relations:

(Reduction) If $a \Rightarrow b, A$, then $a, \neg A \Rightarrow b$.

As before, an arbitrary set of causal rules will be called a *causal theory*. Any causal theory Δ determines a unique least disjunctive causal relation that includes Δ ; it will be denoted by \Rightarrow_Δ .

Bitheories

Bitheories, introduced below, will play the same role as ordinary theories for consequence relations.

Definition 5. • A *bitheory* is a pair (α, u) , where u is a deductively closed set and α a world (maximal consistent set) such that $u \subseteq \alpha$.

- A *bitheory of a disjunctive causal relation* \Rightarrow is any bitheory (α, u) such that $\alpha \not\models \bar{u}$.
- Bitheories of \Rightarrow_Δ will also be called *bitheories of a causal theory* Δ .

Bitheories of a causal relation can be seen as bitheories that are closed with respect to all its causal rules; this understanding is justified by the following simple result:

Lemma 2. A *bitheory* (α, u) is a *bitheory of a causal relation* \Rightarrow if and only if $b \cap u \neq \emptyset$, for any causal rule $a \Rightarrow b$ from \Rightarrow such that $a \subseteq \alpha$.

Similarly, bitheories of a causal theory Δ are precisely bitheories that are closed with respect to the rules from Δ .

As follows directly from the definition, if (α, u) is a bitheory of \Rightarrow , and v an arbitrary deductively closed set such that $u \subseteq v \subseteq \alpha$, then (α, v) will also be a bitheory of \Rightarrow . This indicates that the set of bitheories is determined, in effect, by its inclusion minimal elements; we will call them *minimal* bitheories in what follows.

Logical semantics of disjunctive causal inference

We will describe now a logical (monotonic) semantics of disjunctive causal relations.

An ‘official’ semantic interpretation of disjunctive causal relations could be given in terms of possible worlds models of the form (i, W) , where i is a propositional interpretation, while W is a set of propositional interpretations that contains i (see (Turner 1999)). However, any pair (i, W) determines a unique bitheory (α, u) , where α is the world corresponding to the interpretation i , while u is a theory containing the propositions that hold in all interpretations from W . Accordingly, by a *causal semantics* we will mean a set of bitheories. The validity of disjunctive causal rules with respect to this semantics is defined as follows.

Definition 6. A causal rule $a \Rightarrow b$ will be said to be *valid* in a causal semantics \mathcal{B} if, for any bitheory (α, u) from \mathcal{B} , $a \subseteq \alpha$ implies $b \cap u \neq \emptyset$.

We will denote by $\Rightarrow_{\mathcal{B}}$ the set of all causal rules that are valid in a causal semantics \mathcal{B} . It can be easily verified that this set is closed with respect to the postulates for disjunctive causal relations, and hence $\Rightarrow_{\mathcal{B}}$ forms a disjunctive causal relation. In order to prove completeness, for any disjunctive causal relation \Rightarrow , we can consider its *canonical causal semantics* defined as the set of all minimal bitheories of \Rightarrow . Then it can be shown that this semantics is *strongly complete* for the source disjunctive causal relation. As a result, we obtain the following theorem (stated in (Bochman 2003b)):

Theorem 3. A relation on sets of propositions is a disjunctive causal relation if and only if it is determined by a causal semantics.

Due to the direct correspondence between the causal semantics and the possible worlds semantics from (Turner 1999), disjunctive causal relations correspond to a subsystem of Turner’s universal causal logic (UCL). Namely, a causal rule $a \Rightarrow B_1, \dots, B_n$ corresponds to the modal UCL formula $\bigwedge a \rightarrow \mathbf{C}B_1 \vee \dots \vee \mathbf{C}B_n$. Thus, disjunctive causal relations cover Turner’s causal theories that involve modal formulas with only positive occurrences of the causal operator \mathbf{C} (see also (Lin 1996)).

Singular Causal Inference

Disjunctive causal rules having a single proposition in their heads will be called *singular* in what follows. Such causal rules have the same meaning as the causal rules of the causal calculus. In this section we will give a precise characterization of disjunctive causal relations that are generated by singular causal rules.

To begin with, the following fact can be easily verified:

Lemma 4. The set of singular rules belonging to a disjunctive causal relation forms a causal inference relation.

In what follows, the above causal inference relation will be called the *normal subrelation* of a disjunctive causal relation (by analogy with normal logic programs). Actually, there is a quite simple and modular recipe how such a subrelation could be obtained from a given set of disjunctive causal rules.

For any set of disjunctive causal rules Δ , let us consider the following set of singular rules:

$$N(\Delta) = \{a, \neg b \Rightarrow \bigvee c \mid a \Rightarrow b, c \in \Delta\}$$

Let $\Rightarrow_{N(\Delta)}^n$ denote the least causal inference relation that includes $N(\Delta)$. Then the following result shows that the set $N(\Delta)$ captures the ‘singular content’ of Δ .

Theorem 5. $\Rightarrow_{N(\Delta)}^n$ is the normal subrelation of \Rightarrow_{Δ} .

Let us say that a disjunctive causal relation is *singular* if it is a least disjunctive causal relation containing some set of singular causal rules. In what follows, we are going to give more instructive descriptions of such disjunctive causal relations.

A causal semantics \mathcal{B} will be called *functional*, if for any world α there is no more than one theory u such that $(\alpha, u) \in \mathcal{B}$. Then we have

Theorem 6. A disjunctive causal relation is singular if and only if its canonical causal semantics is functional.

The next result states that, for singular causal relations, worlds produce only determinate effects.

Lemma 7. A disjunctive causal relation is singular iff, for any world α , $\alpha \Rightarrow b, c$ only if either $\alpha \Rightarrow b$ or $\alpha \Rightarrow c$.

Finally, the following theorem shows that, for singular causal relations, disjunctive effects are always ‘separable’ by adding some further assumptions to the premises.

Theorem 8. A disjunctive causal relation is singular if and only if it satisfies the following condition:

If $a \Rightarrow b, c$, then $a, A \Rightarrow b$ and $a, \neg A \Rightarrow c$, for some proposition A .

The above theorem amounts to saying that indeterminate effects arise in singular disjunctive causal relations only due to ‘forgetting’ of some relevant parameters. This characteristic property is even more vivid in the following corollary:

Corollary 9. A disjunctive causal relation is singular if and only if it satisfies the following condition:

$a \Rightarrow B_1, \dots, B_n$ iff there are pairwise incompatible propositions A_1, \dots, A_n such that their disjunction is a tautology, and $a, A_i \Rightarrow B_i$, for any $i \leq n$.

As a last result in this section, we will show that any disjunctive causal relation can be viewed as a language restriction of some singular causal relation. We provide a constructive proof of this claim by transforming any set Δ of causal rules in a language \mathcal{L} into a set of singular rules Δ_s in some extended language as follows.

For any causal rule $r = a \Rightarrow B_1, \dots, B_n$ from Δ , we will introduce n new propositional atoms r_1, \dots, r_n , and consider the following set of singular rules:

$$\begin{aligned} & a, \neg r_1, \dots, \neg r_n \Rightarrow \mathbf{f} \\ & a, r_i \Rightarrow B_i, \text{ for all } i \leq n \end{aligned}$$

Δ_s will denote the set of all singular rules obtained in this way from Δ . Then it can be shown that the generated causal relation \Rightarrow_{Δ} is exactly the restriction of \Rightarrow_{Δ_s} to \mathcal{L} . As a result, we obtain

Theorem 10. *For any disjunctive causal relation \Rightarrow in a language \mathcal{L} , there exists a singular causal relation \Rightarrow_s in an extended language $\mathcal{L}_s \supseteq \mathcal{L}$ such that \Rightarrow coincides with the restriction of \Rightarrow_s to the language \mathcal{L} .*

The above theorem shows that any disjunctive causal relation could be seen as a singular causal relation in which some of the relevant factors are ‘hidden’.

It is important to note also that a number of further causal rules involving the new propositional atoms can be added to the translation without affecting the proofs. For example, we may require further that the new atoms are mutually incompatible by adding the following causal constraints

$$r_i, r_j \Rightarrow \mathbf{f}$$

for any $i \neq j$. Moreover, each new atom could be made exogenous by adding the rules:

$$r_i \Rightarrow r_i \quad \neg r_i \Rightarrow \neg r_i$$

Remark. The above result lends, in effect, a formal logical support to the claim that deterministic causation (and determinism in general) provides a comprehensive basis for a general theory of causation - see, e.g., (Pearl 2000).

The Stable Nonmonotonic Semantics

Now we will turn to the main subject of interest in causal theories, namely their nonmonotonic semantics.

The nonmonotonic semantics of causal theories arises from a general requirement that adequate models of such theories should satisfy two conditions. First, they should be closed with respect to the causal rules. Second, they should be *causally explainable*: propositions that hold in such models should have justifications in the sense of being produced (caused) by the causal rules that are ‘active’ in the model.

A nonmonotonic semantics that satisfies the above requirement has been suggested in (McCain and Turner 1997) for singular causal theories, and it has been generalized to arbitrary theories of universal causal logic (UCL) in (Turner 1999). Since disjunctive causal relations form a subsystem of UCL, this semantics can be immediately translated into our framework.

Definition 7. A world α will be said to be *exact* with respect to a disjunctive causal relation \Rightarrow if (α, α) is a minimal bitheory of \Rightarrow . The set of all exact worlds will be said to form a *stable nonmonotonic semantics* of \Rightarrow .

The above definition is equivalent to the definition of causally explained interpretations, given in (Turner 1999).

The above stable semantics appears as a natural, and even almost inevitable, extension of the nonmonotonic semantics for singular causal theories to the disjunctive case. This impression can also be supported by the fact, established in (Turner 1999), that in the general correspondence between UCL and disjunctive default logic, causally explained interpretations correspond to extensions of disjunctive default theories. Moreover, by the same correspondence, simple disjunctive theories can be translated into disjunctive logic programs, and then causally explained interpretations will exactly correspond to answer sets of such programs. Summing up, there is a tight correspondence between the above

semantics and respectable semantics of nonmonotonic formalisms and disjunctive logic programming.

Despite the above impressive support, it has been noticed in the literature that semantics of this kind are problematic in their treatment of indeterminate information. Namely, due to minimization that is involved in their definitions, these semantics tend to give an exclusive interpretation to disjunctions. It turns out that the same shortcoming is preserved by the above stable semantics for disjunctive causal theories. The following example has been suggested by Ray Reiter:

Example 1. Suppose that we drop a pin on a board painted black and white. As a result, the pin lands on the board in such a way that it touches either black or white area, *or both*. Moreover, if the pin is large (or black and white areas are small), then a most probable effect is that the pin touches both black and white areas.

A natural representation of this situation could be given using the main disjunctive rule

$$Drop \Rightarrow Black, White$$

and a couple of auxiliary causal assertions that need not bother us for now. Unfortunately, the stable semantics of the resulting causal theory does not explain the world in which the pin touches both black and white areas, though it readily justifies worlds in which the pin touches only one of them.

Furthermore, it has been shown in (Bochman 2003b) that, under the stable semantics, disjunctive causal rules can be considered as an inessential ‘syntactic sugar’ that does not change our representation capabilities. More precisely, the following result has been shown:

Theorem 11. *The stable semantics of a disjunctive causal relation coincides with the nonmonotonic semantics of its normal subrelation.*

In accordance with the above theorem, in order to compute a stable semantics of a disjunctive causal theory Δ , we can transform Δ into a set of singular causal rules $N(\Delta)$, as described earlier, and then compute the nonmonotonic semantics for the latter singular causal theory. In this sense, any disjunctive causal rule $a \Rightarrow b$ can be directly replaced by a set of singular causal rules.

Example 2. Returning to Reiter’s example, the rule $Drop \Rightarrow Black, White$ implies the following three singular rules:

$$\begin{aligned} Drop, \neg Black &\Rightarrow White & Drop, \neg White &\Rightarrow Black \\ Drop &\Rightarrow Black \vee White \end{aligned}$$

Since the last rule is not determinate, only the first two rules determine the nonmonotonic semantics. Note that these two rules cannot be applied simultaneously, due to the constraint $Drop \wedge \neg Black \wedge \neg White \Rightarrow \mathbf{f}$ that is implied by each of them. As a result, such rules cannot give an explanation for the possible fact $Black \wedge White$.

Extending these results, we are going to show now that they are actually a by-product of a deeper, *logical* reduction of disjunctive causal rules in an appropriate causal logic that is sanctioned by the stable nonmonotonic semantics.

Stable disjunctive causal relations

It turns out that, unlike the causal calculus, disjunctive causal relations do not form a maximal causal logic that is adequate for the stable semantics. Instead, the latter corresponds to a rather peculiar stronger logic described in the next definition.

Definition 8. A disjunctive causal relation will be called *stable* if it satisfies the following postulate:

(Saturation) $A, B \Rightarrow A, A \rightarrow B$

A logical semantics of stable disjunctive causal relations can be obtained by restricting bitheories to saturated ones, described in the following definition.

Definition 9. A bitheory (α, u) will be called *saturated* if $u = \alpha \cap \beta$, for some world β . A causal semantics will be called *saturated*, if it contains only saturated bitheories.

A bitheory (α, u) is saturated if u either coincides with α , or is a maximal sub-theory of α . The following result shows that stable causal relations are sound and complete for saturated causal semantics.

Theorem 12. A disjunctive causal relation is stable if and only if it is generated by a saturated causal semantics.

The following lemma gives a number of equivalent conditions of Saturation.

Lemma 13. A disjunctive causal relation is stable if and only if it satisfies one of the following conditions:

1. $A \Rightarrow B \vee A, \neg B \vee A$;
2. If $a, \neg b_1 \Rightarrow \bigvee b_2$, for any partition (b_1, b_2) of b , then $a \Rightarrow b$.

The second condition above is especially interesting, since it shows, in effect, that any disjunctive rule is equivalent to the set of its singular consequences with respect to a stable causal relation. Indeed, we have seen earlier that the set of singular causal rules derived from a given disjunctive causal rule $a \Rightarrow b$ amounts to $\{a, \neg b_1 \Rightarrow \bigvee b_2\}$, while the condition (2) says that the latter are sufficient for deriving $a \Rightarrow b$ in stable causal relations. As a result, we obtain

Corollary 14. For stable causal relations, any disjunctive causal theory Δ is logically equivalent to its singular reduction $N(\Delta)$.

The above result shows that stable disjunctive causal relations are essentially equivalent to singular causal production relations. More exactly, for stable disjunctive causal relations, any disjunctive rule $a \Rightarrow b$ can be seen as a shortcut for the set of singular rules $\{a, \neg b_1 \Rightarrow \bigvee b_2\}$.

Finally, we are going to show that stable causal relations constitute a strongest causal logic that is adequate for the stable semantics. To this end, we need to define first the corresponding notions of equivalence.

In the definition below, $\Rightarrow_{\Delta}^{st}$ denotes the least stable disjunctive causal relation that includes a causal theory Δ .

Definition 10. Causal theories Δ and Γ will be called

- *stably equivalent* if $\Rightarrow_{\Delta}^{st}$ coincides with $\Rightarrow_{\Gamma}^{st}$;
- *nonmonotonically equivalent* if they determine the same stable semantics;

- *strongly equivalent* if, for any set Φ of causal rules, $\Delta \cup \Phi$ is nonmonotonically equivalent to $\Gamma \cup \Phi$.

Now, as a consequence of the preceding result, we obtain

Corollary 15. Causal theories Δ and Γ are stably equivalent if and only if $N(\Delta)$ is causally equivalent to $N(\Gamma)$.

Second, we have

Lemma 16. Δ is strongly equivalent to $N(\Delta)$.

These results immediately imply the following

Theorem 17. Two disjunctive causal theories are strongly equivalent if and only if they are stably equivalent.

The Covering Semantics

As we already mentioned, Reiter's example presents difficulties not only for causal reasoning, but also for disjunctive logic programming, default logic and PMA theory of updates. Briefly put, since the intended models of all these formalisms are required to be minimal, they usually enforce an exclusive interpretation of disjunctive information. However, in the history of nonmonotonic reasoning there have been a number of attempts to define nonmonotonic semantics that would preserve the usual inclusive understanding of logical disjunctions, most prominent suggestions being Minker's Weak GCWA (see (Lobo, Minker, and Rajasekar 1992)) and Sakama's possible model semantics (Sakama 1989) in logic programming, an alternative semantics for circumscription, suggested in (Eiter, Gottlob, and Gurevich 1993), and a theory of disjunctive updates suggested in (Zhang and Foo 1996). The covering semantics, described below, will belong to the same class.

Basically, our desiderata will be the same as those described above for the stable nonmonotonic semantics. Namely, we want to single out models that are closed with respect to the causal rules, and such that any fact that holds in the model is explainable, ultimately, by the causal rules that hold in the model. Our point of departure, however, will be that the notion of explainability determined by disjunctive rules could be given a different interpretation.

A disjunctive rule $p \Rightarrow q, r$ says that p causes either q or r . In the deterministic setting this *normally* means that there are additional facts, say q_0 and r_0 , such that $p \wedge q_0$ causes q , $p \wedge r_0$ causes r , and, in addition, either q_0 , or r_0 holds in any situation in which p holds. This gives us a way of explaining why q or r occurs, provided we know that p holds. Note, however, that there are no a priori constraints on the compatibility of q_0 and r_0 , so they could hold simultaneously, in which case both q and r are caused (and hence explainable).

Our qualification about normality above indicates, however, that the above interpretation may sometimes lead us astray. Thus, it may well be that the disjunctive rule $p \Rightarrow q, r$ is simply a logical consequence of a valid singular causal rule $p \Rightarrow q$, in which case it should not give us an explanation for an occurrence of r . This means that the disjunctive rule should at least be *irreducible* in order to serve as an explanation for its indeterminate conclusions.

Unfortunately, even this qualification is not sufficient. A more complex, but equally plausible situation might be that, though $p \Rightarrow q, r$ is an irreducible rule, q is actually caused

by another rule, say $q_0 \Rightarrow q$, in which case the presence of p cannot serve as a proper explanation for r .

The above considerations suggest that indeterminate causal rules can provide explanations for their conclusions only in situations when they are not superseded by other active causal rules that bring about more determinate effects. Only in such cases the explanation provided by the indeterminate rules will be the only explanation possible. The definition of causally covered models, given below, is an attempt to make this idea precise.

If (α, u) is a minimal bitheory of a disjunctive causal relation \Rightarrow , we will say that u is a *causal projection* of α with respect to \Rightarrow . A world is *causally consistent* if it has a causal projection in this sense, but it may have a number of causal projections. Still, there are worlds that are uniquely determined by their causal projections. Such worlds will constitute the alternative nonmonotonic semantics of disjunctive causal theories.

Definition 11. A world α will be said to be *causally covered* with respect to a disjunctive causal relation \Rightarrow if it is the unique world that includes all its causal projections. The set of all causally covered worlds will constitute a nonmonotonic *covering semantics* of \Rightarrow .

Yet another description of causally covered worlds is given in the next lemma.

Lemma 18. A world α is causally covered if and only if

$$\alpha = \text{Th}(\bigcup \{u \mid u \text{ is a causal projection of } \alpha\})$$

The above lemma gives a precise meaning to the requirement that facts holding in intended interpretations of a causal theory should be explained by the causal rules of the theory. Namely, any proposition belonging to a causally covered world should be a logical consequence of the facts that have a causal explanation in this world.

The following results describe the relationship between the covering and stable semantics. Thus, the next lemma shows that the stable semantics is stronger than the covering semantics.

Lemma 19. Any exact world is also causally covered.

The next result shows that both semantics coincide for singular causal relations.

Theorem 20. If \Rightarrow is a singular causal relation, then its stable semantics coincides with its covering semantics.

Thus, the difference between the two semantics can show itself only for genuinely disjunctive causal relations.

Example 3. Let us consider the following disjunctive causal theory:

$$p \Rightarrow q, r; \quad p \Rightarrow p; \quad \neg p \Rightarrow \neg p; \quad \neg q \Rightarrow \neg q; \quad \neg r \Rightarrow \neg r.$$

The world α that is determined by the literals $\{p, q, r\}$ is not exact, so it is not included in the stable semantics. However, it has two causal projections, namely $\text{Th}(p, q)$ and $\text{Th}(p, r)$. Consequently, it is causally covered with respect to this theory. As a result, the stable semantics for this example validates an intuitively implausible conclusion $\neg(q \wedge r)$, though it is not valid in the covering semantics.

As a last result in this section, we will show that disjunctive causal relations constitute a maximal logic that is adequate for the covering semantics.

Definition 12. Disjunctive causal theories Γ and Δ will be called

- *causally equivalent*, if they determine the same disjunctive causal relation;
- *c-equivalent*, if they determine the same covering semantics.
- *strongly c-equivalent*, if, for any set Φ of causal rules, $\Gamma \cup \Phi$ is c-equivalent to $\Delta \cup \Phi$.

Disjunctive theories are causally equivalent if each can be obtained from the other using the postulates for disjunctive causal relations, while strongly c-equivalent theories are interchangeable in any larger causal theory without changing the covering semantics. In other words, strong c-equivalence can be seen as a maximal logical (monotonic) equivalence that preserves the covering semantics.

Theorem 21. Disjunctive causal theories are strongly c-equivalent if and only if they are causally equivalent.

The above theorem says, in effect, that, unlike the case of the stable semantics, any logical distinction between disjunctive causal theories, allowed by the formalism of disjunctive causal relations, is relevant for determining the covering semantics either for these theories themselves, or for their extensions.

Normalization

In this final section we are going to show that in regular circumstances, a disjunctive causal relation can be systematically extended to a singular causal inference relation in such a way that the covering semantics of the former will coincide the nonmonotonic semantics of the latter.

To begin with, that there are many cases when disjunctive causal rules are *logically* reducible to singular rules even in the general logic of disjunctive causal inference. For example, the rule $A \Rightarrow B, \neg B$ is equivalent to a pair of singular causal rules $A, B \Rightarrow B$ and $A, \neg B \Rightarrow \neg B$. This is a consequence of the following general fact:

Lemma 22. A rule $a \Rightarrow B_1, \dots, B_n$ is reducible to the set of singular rules $\{a, B_i \Rightarrow B_i \mid i = 1, \dots, n\}$ and a constraint $a, \neg B_1, \dots, \neg B_n \Rightarrow \mathbf{f}$ if and only if $a, B_i, B_j \Rightarrow B_i$, for any $i \neq j$.

Note that the above characteristic condition for reduction holds, in particular, when all B_i are incompatible, given a , that is, when $a, B_i, B_j \Rightarrow \mathbf{f}$. For example (cf. (Lin 1996)), a disjunctive rule $a \Rightarrow B \wedge C, B \wedge \neg C, \neg B \wedge C$ is reducible to the following set of singular rules:

$$\begin{aligned} a, B \Rightarrow B \quad a, \neg B \Rightarrow \neg B \quad a, C \Rightarrow C \\ a, \neg C \Rightarrow \neg C \quad a, \neg B, \neg C \Rightarrow \mathbf{f} \end{aligned}$$

The above result shows, in effect, that irreducibly disjunctive rules arise only in cases when their heads contain mutually compatible propositions. For such rules, we need a different strategy.

Let \mathcal{B} be a causal semantics. For a world α , we will denote by u_α the least theory containing all the causal projections of α in \mathcal{B} , that is

$$u_\alpha = \text{Th}(\bigcup \{u \mid u \text{ is a causal projection of } \alpha\})$$

Now, let $n(\mathcal{B})$ denote the set of all bitheories of the form (α, u_α) . Clearly, $n(\mathcal{B})$ is a functional causal semantics, so it will produce a singular causal relation. We will call $n(\mathcal{B})$ the *normalization* of \mathcal{B} .

Using the above notions, the basic idea behind the constructions below can be described quite simply. Suppose that \mathcal{B} is the canonical semantics of some disjunctive causal relation \Rightarrow . Then it is easy to verify that a world α is causally covered in \mathcal{B} if and only if it is an exact world in its normalization $n(\mathcal{B})$ (cf. Lemma 18). Consequently, if $n(\mathcal{B})$ is a canonical semantics of some singular causal relation \Rightarrow^n , then the stable semantics of \Rightarrow^n will coincide with the covering semantics of \Rightarrow .

Unfortunately, the above construction works only under severe finiteness constraints. Accordingly, in order to simplify our discussion at this stage, we will occasionally restrict our attention to finite causal theories and causal relations in some finite propositional language.

We will begin with the following notions:

Definition 13. Let \mathcal{B} be the canonical semantics of a disjunctive causal relation \Rightarrow , and $n(\mathcal{B})$ its normalization. The (singular) causal inference relation generated by $n(\mathcal{B})$ will be called the *normalization* of \Rightarrow , and it will be denoted by \Rightarrow^n .

Then the following result can be shown to hold:

Theorem 23. *If \Rightarrow is a disjunctive causal relation in a finite language, then its covering semantics coincides with the stable semantics of its normalization.*

Thus, for a broad class of disjunctive causal relations, the covering semantics coincides with the stable semantics of their normalizations. It should be kept in mind, however, that this normalization involves, in effect, an extension of the source disjunctive causal relation with new singular rules that are not derivable logically from it.

Normalization as a nonmonotonic completion. The normalization can be viewed as a (presumably unusual) kind of *nonmonotonic completion* of the source disjunctive causal relation, which is obtained by restricting its logical semantics to functional bitheories. As such, it is similar to other non-standard forms of completion that exist in the AI literature, such as the Markov assumption (see (Bochman 2013)). Thus, as for the latter, this semantic restriction does not lead to a change of the underlying logic of disjunctive causal inference; this follows immediately from the fact that any disjunctive causal relation is a language restriction of some singular causal relation - see Theorem 10. Still, the restriction of the set of models means that more causal inferences become valid, so more information is obtained in particular cases.

Causal covers

We still need to provide a more constructive description of normalization. To this end, we introduce the following

Definition 14. A proposition C is a *causal cover* of a set of propositions b in a disjunctive causal relation, if $C \Rightarrow b$ and $D \wedge \neg C \Rightarrow$, for any D such that $D \Rightarrow b$.

The causal cover of a set of propositions is a weakest causally sufficient condition for this set. Notice, in particular, that falsity \mathbf{f} is a cover of the empty set \emptyset .

An alternative characterization of covers is given in the next lemma.

Lemma 24. *C is a causal cover of b iff, for any causally consistent world α , $\alpha \Rightarrow b$ if and only if $C \in \alpha$.*

Based on this lemma, it can be easily verified that in the finite case any set of propositions has a causal cover.

The following consequence of the lemma connects causal covers with normalization:

Corollary 25. *If C is a causal cover of b , and $a \Rightarrow b$, A , then $a, \neg C \Rightarrow^n A$.*

The notion of a causal cover allows us to provide a relatively simple description of normalization.

Given a finite disjunctive causal theory Δ , we define Δ^s as the set of all singular causal rules of the form:

$$a, \neg C_b \Rightarrow A$$

such that $a \Rightarrow b$, $A \in \Delta$, and C_b is a causal cover of b . Using this set, we will construct the following singular causal theory:

$$\Delta^n = \Delta^s \cup \{a \Rightarrow \mathbf{f} \mid a \Rightarrow \emptyset \in \Delta\}.$$

The next result shows that the above singular causal theory generates precisely the normalization of Δ .

Theorem 26. *If Δ is finite, then \Rightarrow_Δ^n coincides with \Rightarrow_{Δ^n} .*

In accordance with the above result, the covering semantics of Δ will coincide with the standard nonmonotonic semantics of a singular causal theory Δ^n . However, this description of normalization is still not fully constructive, since it depends on determining causal covers.

Construction of covers. As a final step in our description, we describe algorithms of constructing covers for finite causal theories. We consider first a simplest case of causal rules that do not involve logical connectives, and then generalize the method to arbitrary causal theories.

A causal rule $a \Rightarrow b$ will be called *flat* if a and b are sets of literals. A causal theory will be called *flat* if it contains only flat causal rules. Finally, a disjunctive causal relation will be called *flat* if it is generated by a flat causal theory. It should be noted that most examples of indeterminate causation appearing in the literature are representable by causal theories of this kind.

By a *reduction* of a causal rule $a \Rightarrow b$ we will mean any rule $a, \neg b_0 \Rightarrow b_1$ such that $b = b_0 \cup b_1$ and $b_0 \cap b_1 = \emptyset$. Δ^\neg will denote the set of all reductions of the rules from Δ .

Note that all the rules from Δ^\neg are derivable from Δ (by Reduction). It should be clear also that if Δ is a flat theory, then Δ^\neg will also be flat.

Now, given a finite flat causal theory Δ , we can define a causal cover for a finite set of literals b as follows:

$$C_b = \bigvee \{ \wedge a \mid a \Rightarrow d \in \Delta^- \text{ \& } d \subseteq b \}$$

Lemma 27. *If Δ is flat, then C_b is a causal cover of b in \Rightarrow_Δ .*

Recall that for a finite causal theory Δ , the normalization can be constructed using the set of singular rules $a, \neg C_b \Rightarrow A$, for every disjunctive rule $a \Rightarrow b, A$. Now, the above construction can be used for determining all causal covers required for defining the normalization of a flat causal theory. Note also that if Δ is a flat causal theory, then its normalization will be a determinate causal theory, that is it will contain only causal rules of the form $a \Rightarrow l$, where l is a literal. As has been shown already in (McCain and Turner 1997), the nonmonotonic semantics of such theories coincides with the set of models of their classical completion.

The following examples show that normalization of flat disjunctive causal theories corresponds to established ways of describing non-determinate information by singular causal rules.

Example 4. Let us return to the disjunctive causal theory:

$$p \Rightarrow q, r; \quad p \Rightarrow p; \quad \neg p \Rightarrow \neg p; \quad \neg q \Rightarrow \neg q; \quad \neg r \Rightarrow \neg r.$$

In order to construct the normalization, the rule $p \Rightarrow q, r$ should be replaced by two singular rules $p, \neg C_{\{q\}} \Rightarrow r$ and $p, \neg C_{\{r\}} \Rightarrow q$. The relevant causal covers can be calculated by the above formula for C_b , and we immediately obtain $C_{\{q\}} = p \wedge \neg r$ and $C_{\{r\}} = p \wedge \neg q$. As a result, the above singular rules are reduced, respectively, to $p \wedge r \Rightarrow r$ and $p \wedge q \Rightarrow q$. As can be seen, the latter rules correspond exactly to the way of representing indeterminate information suggested in (McCain and Turner 1997; Giunchiglia et al. 2004).

Example 5. Suppose that we add a rule $s \Rightarrow q$ to the causal theory in the preceding example. Then the causal cover $C_{\{q\}}$ for $\{q\}$ changes to $(p \wedge \neg r) \vee s$. As a result, the first singular rule of the normalization will now be $p \wedge r \wedge \neg s \Rightarrow r$. In other words, the explainability of r will depend now not only on p , but also on the absence of s ; this is because s causes q and thereby supersedes the indeterminate rule $p \Rightarrow q, r$.

The next example shows that the normalization procedure gives us a reasonable solution also in more complex situations than what can be handled intuitively.

Example 6. Consider the following theory:

$$p \Rightarrow q, r; \quad p \Rightarrow s, \neg r.$$

The relevant causal covers are: $C_{\{q\}} = p \wedge \neg r$, $C_{\{r\}} = p \wedge \neg q$, $C_{\{s\}} = p \wedge r$, $C_{\{\neg r\}} = p \wedge \neg s$. As a result, the normalization of the above causal theory is

$$p \wedge q \Rightarrow q; \quad p \wedge r \Rightarrow r; \quad p \wedge s \Rightarrow s; \quad p \wedge \neg r \Rightarrow \neg r.$$

Finally, we will briefly describe a construction of causal covers for arbitrary finite causal theories.

To begin with, we will introduce a special notation for dealing with ‘disjunctive’ sets of propositions. For two sets of propositions a, b , we will denote by $a \& b$ the set $\{A_i \wedge B_j \mid$

$A_i \in a, B_j \in b\}$. In addition, we will use a ‘disjunctive’ entailment relation \models^\vee on sets of propositions, defined as follows: $a \models^\vee b$ holds iff, for any $A \in a$ there exists $B \in b$ such that $A \models B$.

By a *join* of two causal rules $a_1 \Rightarrow b_1$ and $a_2 \Rightarrow b_2$ we will mean a rule $a_1, a_2 \Rightarrow b_1 \& b_2$. $\Delta^\&$ will denote the set of all finite joins of the rules from Δ , while Δ^\oplus will denote $(\Delta^\&)^-$, that is, the set of all reductions of the rules from $\Delta^\&$.

A causal cover C_b of a set of propositions b can now be described as follows:

$$C_b = \bigvee \{ \wedge a \mid a \Rightarrow d \in \Delta^\oplus \text{ \& } d \models^\vee b \}$$

Theorem 28. *C_b is a causal cover of b in \Rightarrow_Δ .*

As a by-product of the above construction, we immediately obtain a constructive proof of the fact that any finite disjunctive causal theory is normalizable.

Summary and Conclusions

The main results of this study are two-fold. On the one hand, confirming the initial idea of (Lin 1996), it has been shown that disjunctive causal rules provide a highly expressive and versatile tool for representing indeterminate causation. Moreover, viewed under a more appropriate, covering nonmonotonic semantics, such rules are not reducible logically to ordinary, singular causal rules, so they contain more information. On the other hand, however, it has been shown that even in the latter case this additional information can be expressed using ordinary causal rules by way of *systematic* completion of the source causal theory with new singular causal rules that provide the required information about causes (or excuses) for particular causal effects. As we have seen in the examples, in simple cases such additional rules correspond to existing ways of describing indeterminate actions, given in the literature. However, the described normalization procedure is fully general, so it allows us to automate an important and non-trivial part of representing nondeterminate actions.

Speaking generally, the above results suggest a certain primacy of ordinary, singular rules for representing causation². More precisely, they show that the restriction to ordinary, ‘deterministic’ causal rules can be sustained as a convenient and useful representation decision, or *representation assumption*. Nevertheless, this general assumption does not deprave truly disjunctive causal rules of their potentially useful role in representing indeterminate causation. Thus, it has been shown that the corresponding ‘translation’ to singular causal rules is in general far from being trivial. It remains to be seen whether disjunctive causal rules can fulfill this representation role in describing complex causal or action domains that go beyond toy examples.

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²In this sense, the situation is quite similar to the use of normal program rules in logic programming.

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