Some Complexity Results on Inconsistency Measurement

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Abstract

We survey a selection of inconsistency measures from the literature and investigate their computational complexity wrt. decision problems related to bounds on the inconsistency value and the functional problem of determining the actual value. Our findings show that those inconsistency measures can be partitioned into three classes related to their complexity. The first class contains measures whose complexity are located on the first level of the polynomial hierarchy, the second class contains measures on the second level of the polynomial hierarchy, and the third class is located beyond the second level of the polynomial hierarchy. We provide membership results for all the investigated problems and completeness results for most of them.

1 Introduction

Inconsistency measurement is about the quantitative assessment of the severity of inconsistencies in knowledge bases. Consider the following two knowledge bases $\mathcal{K}_1$ and $\mathcal{K}_2$ formalised in propositional logic:

$$\mathcal{K}_1 = \{a, b \vee c, \neg a \wedge \neg b, d\} \quad \mathcal{K}_2 = \{a, \neg a, b, \neg b\}$$

Both knowledge bases are classically inconsistent as for $\mathcal{K}_1$ we have $\{a, \neg a \wedge \neg b\} \models \perp$ and for $\mathcal{K}_2$ we have, e.g., $\{a, \neg a\} \models \perp$. These inconsistencies render the knowledge bases useless for reasoning if one wants to use classical reasoning techniques. In order to make the knowledge bases useful again, one can either rely on non-monotonic/paraconsistent reasoning techniques (Makinson 2005; Priest 1979) or one revises the knowledge bases appropriately to make them consistent (Hansson 2001). Looking at the knowledge bases $\mathcal{K}_1$ and $\mathcal{K}_2$ one can observe that the *severity* of their inconsistency is different. In $\mathcal{K}_1$, only two out of four formulas (a and $\neg a \wedge \neg b$) are "participating" in making $\mathcal{K}_1$ inconsistent while for $\mathcal{K}_2$ all formulas contribute to its inconsistency. Furthermore, for $\mathcal{K}_1$ only two propositions (a and b) are conflicting and using e.g. paraconsistent reasoning one could still infer meaningful statements about c and d. For $\mathcal{K}_2$ no such statement can be made. This leads to the assessment that $\mathcal{K}_2$ should be regarded *more* inconsistent than $\mathcal{K}_1$.

Inconsistency measures can be used to analyse inconsistencies and to provide insights on how to repair them. An inconsistency measure $\mathcal{I}$ is a function on knowledge bases, such that the larger the value $\mathcal{I}(\mathcal{K})$ the more severe the inconsistency in $\mathcal{K}$. A lot of different approaches of inconsistency measures have been proposed, mostly for classical propositional logic (Hunter and Konieczny 2008; 2010; Ma et al. 2010; Mu et al. 2011; Xiao and Ma 2012; Grant and Hunter 2011; 2013; McAreavey, Liu, and Miller 2014; Jabbour et al. 2015).

In this paper, we address the computational complexity of inconsistency measurement by investigating a selection of 13 inconsistency measures for propositional logic from the literature mentioned above. Inconsistency measurement is, by definition, a computationally intractable problem as it goes beyond merely detecting inconsistency (which is itself an coNP-complete problem for propositional logic). However, no systematic investigation of the complexity of inconsistency measures—and a comparison of measures wrt. it—has been conducted so far. The only complexity analyses on inconsistency measures we are aware of were presented in (Ma et al. 2010) and (Xiao and Ma 2012) and each focused on a particular inconsistency measure. In (Ma et al. 2010) the complexity of a variant of the contension inconsistency measure $\mathcal{I}_c$ (Grant and Hunter 2011) and in (Xiao and Ma 2012) the complexity of the measure $\mathcal{I}_{mv}$ from (Xiao and Ma 2012) itself are investigated (we will recall the formal definitions of these measures in Sec. 3 and the corresponding results in Sec. 4, respectively). Recently, the algorithmic challenges in computing inconsistency measures have gained some attention (Ma et al. 2010; McAreavey, Liu, and Miller 2014; Thimm 2016b) and therefore calls for a theoretical investigation on the complexity of the involved computational problems. In this paper, we take a first step in this direction by providing a detailed analysis on the computational complexity of 13 measures wrt. three decision problems, namely deciding whether a given value is an upper, resp. lower bound, or is the exact value, as well as the functional problem of determining the inconsistency value. We mainly focus on the decision problems of deciding whether a given value is an upper, or resp. a lower bound, since, as we will see, the complexity classification of these decision problems gives crucial insights into the computational complexity of the inconsistency measure at hand.

The main contributions of this paper are as follows.

- We show that the complexity of the decision problems of the inconsistency measures $\mathcal{I}_d$, $\mathcal{I}_\eta$, $\mathcal{I}_c$, $\mathcal{I}_{hs}$, $\mathcal{I}_{\text{dalal}}^\Sigma$, $\mathcal{I}_{\text{dalal}}^{\max}$, and $\mathcal{I}_{\text{dalal}}^{\text{hit}}$ is located on the first level of the polynomial hierarchy. These results imply that one can compute the exact value with logarithmically many calls to an NP-oracle. This in particular suggests the applicability of maximum satisfiability solvers (Ansótegui, Bonet, and Levy 2013; Morgado et al. 2013) and similar systems for computing these measures.

- We establish completeness for a class in the second level of the polynomial hierarchy for decision problems for measures $\mathcal{I}_p$, $\mathcal{I}_{mv}$, and $\mathcal{I}_{nc}$. Thus, these measures can be computed with logarithmically many calls to a Σ_2^p oracle. Systems capable of dealing with such high complexity are, e. g., answer-set programming solvers (Brewka, Eiter, and Truszczynski 2011).

- We prove that counting problems underlying inconsistency measures $\mathcal{I}_{\mathsf{MI}}$ and $\mathcal{I}_{mc}$ are #·coNP-complete. Under complexity theoretic assumptions, our results imply that (i) these underlying counting problems are computationally more challenging than propositional model counting, a problem itself seen as highly intractable and important (Gomes, Sabharwal, and Selman 2009), and (ii) that decision problems associated with these measures presumably are not contained in a class of the polynomial hierarchy. Algorithms for computing these problems can be built upon systems for enumerating minimal unsatisfiable sets such as (Marques-Silva 2012; McAreavey, Liu, and Miller 2014; Liffiton et al. 2015). Additionally, we show that measures $\mathcal{I}_{\mathsf{MI}}$, $\mathcal{I}_{mc}$ and the related measure $\mathcal{I}_{\mathsf{MIc}}$ can be computed in polynomial space.

Before we give the details of our technical contributions in Section 4, we first provide some necessary preliminaries in Section 2 and introduce the inconsistency measures used in our analysis in Section 3. We conclude with a discussion in Section 5.

2 Preliminaries

In the following, we introduce some necessary preliminaries on propositional logic and computational complexity.

2.1 Propositional Logic

Let At be some fixed propositional signature, i. e., a (possibly infinite) set of propositions, and let $\mathcal{L}(\mathsf{At})$ be the corresponding propositional language constructed using the usual connectives $\wedge$ (*conjunction*), $\vee$ (*disjunction*), $\rightarrow$ (*implication*), and $\neg$ (*negation*). A literal is a proposition p or a negated proposition $\neg p$. A clause is a disjunction of literals. A formula is in conjunctive normal form (CNF) if the formula is a conjunction of clauses.

Definition 1. A knowledge base $\mathcal{K}$ is a finite set of formulas $\mathcal{K} \subseteq \mathcal{L}(\mathsf{At})$. Let $\mathbb{K}$ be the set of all knowledge bases.

If X is a formula or a set of formulas we write $\mathsf{At}(X)$ to denote the set of propositions appearing in X. Semantics to a propositional language is given by *interpretations* and an

interpretation ω on At is a function $\omega : \mathsf{At} \rightarrow \{\mathsf{true}, \mathsf{false}\}$. Let $\Omega(\mathsf{At})$ denote the set of all interpretations for At. An interpretation ω *satisfies* (or is a *model* of) an atom $a \in \mathsf{At}$, denoted by $\omega \models a$, if and only if $\omega(a) = \mathsf{true}$. The satisfaction relation $\models$ is extended to formulas in the usual way.

For $\Phi \subseteq \mathcal{L}(\mathsf{At})$ we also define $\omega \models \Phi$ if and only if $\omega \models \phi$ for every $\phi \in \Phi$. Define furthermore the set of models $\mathsf{Mod}(X) = \{\omega \in \Omega(\mathsf{At}) \mid \omega \models X\}$ for every formula or set of formulas X. If $\mathsf{Mod}(X) = \emptyset$ we also write $X \models\perp$ and say that X is *inconsistent*.

We also make use of the notation $\phi[\omega]$ for a formula ϕ and a (partial) interpretation ω, which denotes the uniform replacement of each proposition $x \in \mathsf{dom}\,\omega$ by $\top$ if $\omega(x) = \mathsf{true}$ and by $\perp$ if $\omega(x) = \mathsf{false}$.[1] If ω is partial, i. e. defined on a subset of variables in ϕ, this results in a formula with reduced vocabulary.

2.2 Computational Complexity

We assume familiarity with the complexity classes P, NP, and coNP. We also make use of the polynomial hierarchy, that can be defined (using oracle Turing machines) as follows: $\Sigma_0^p = \Delta_0^p = \mathsf{P}$, $\Sigma_{i+1}^p = \mathsf{NP}^{\Sigma_i^p}$, $\Delta_{i+1}^p = \mathsf{P}^{\Sigma_i^p}$ for $i \geq 0$. Here, $\mathcal{C}^{\mathcal{D}}$ denotes the class of decision problems solvable in class $\mathcal{C}$ with access to an oracle for some problem complete in $\mathcal{D}$. A language is in D_i^p iff it is the intersection of a language in Σ_i^p and a language in Π_i^p. Further, $\mathsf{P}^{C[\log]}$ contains all problems that can be solved with a deterministic polynomial-time algorithm that may make logarithmically many calls to a C oracle. The class PSPACE contains all problems that can be solved in polynomial space. We also make use of the functional complexity classes $\mathsf{FP}^{\mathsf{NP}[\log n]}$ and $\mathsf{FP}^{\Sigma_2^p[\log n]}$, i. e. classes containing problems whose solutions can be computed with a polynomial-time algorithm that may make a logarithmically bounded number of oracle calls to an NP, resp. Σ_2^p, oracle. The class FPSPACE is the class of function problems whose solutions can be computed in polynomial space. For function complexity classes we utilize metric reductions to show hardness. A functional problem A reduces to a functional problem B if there exist polynomial-time computable functions f and g s. t. for input x for A it holds that $g(x, y)$ is a correct solution for problem A iff y is a correct solution for input $f(x)$ for problem B.

Some of the inconsistency measures inherently count certain semantical structures, making counting complexity (see (Valiant 1979b; 1979a)) a natural tool for our analysis. While decision problems typically ask whether at least one solution for a problem exists, counting problems ask for the number of solutions. The most well-known counting complexity class is #P, the class containing problems asking for the number of accepting paths in a non-deterministic polynomial-time Turing machine. The prototypical #P-complete problem is #SAT, the problem of finding the number of models of a given formula. In this paper we use the class #·coNP from the counting complexity class hierarchy defined in (Hemaspaandra and Vollmer 1995). Towards the definition of this class we first define counting problems

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[1] dom f denotes the domain of a function f.

which are in turn defined via witness functions w, which assign to a string from an input alphabet Σ a finite set of strings from an alphabet Γ. An instance for a counting problem consists of a given input string x from alphabet Σ and the task is to return $|w(x)|$, i.e. the cardinality of witnesses defined by witness function w associated with the counting problem. If C is a complexity class of decision problems, then $\#\cdot C$ is the class of all counting problems for whose witness function w it holds that

- for every input string x, every $y \in w(x)$ is polynomially bounded by x; and

- the decision problem of deciding $y \in w(x)$ for given strings x and y is in C.

For example, for #SAT the witness function is $\mathsf{Mod}(\phi)$ for input strings ϕ corresponding to formulas. It holds that $\#\cdot\mathsf{P} = \#\mathsf{P}$ and $\#\cdot\mathsf{P} \subseteq \#\cdot\mathsf{coNP}$. The main type of reduction used for classes like $\#\cdot\mathsf{coNP}$ are subtractive reductions (Durand, Hermann, and Kolaitis 2005), since Turing reductions do not preserve counting complexity classes $\#\cdot\Pi_i^p$. Let $\#A$ and $\#B$ be counting problems. We denote their witness sets by $A(x)$ and $B(y)$ for input strings x and y. The counting problem $\#A$ reduces to $\#B$ via a strong subtractive reduction if there exist polynomial-time computable functions f and g s.t. for each input string x we have $B(f(x)) \subseteq B(g(x))$ and $|A(x)| = |B(g(x))| - |B(f(x))|$. Intuitively, we may overcount the solutions and carefully subtract what we overcounted. A strong subtractive reduction is called parsimonious if $B(f(x)) = \emptyset$ for all input strings x, i.e. $|A(x)| = |B(g(x))|$. Subtractive reductions are the transitive closure of strong subtractive reductions.

The class $\#\cdot\mathsf{coNP}$ contains several natural counting problems complete for this class, including problems in the field of knowledge representation and reasoning, e.g. counting the number of explanations in the context of abduction (Hermann and Pichler 2010).

3 Inconsistency Measures

Inconsistency measures are functions $\mathcal{I} : \mathbb{K} \to [0, \infty]$ that aim at assessing the severity of the inconsistency in a knowledge base $\mathcal{K}$, cf. (Grant and Hunter 2011). The basic idea is that the larger the inconsistency in $\mathcal{K}$ the larger the value $\mathcal{I}(\mathcal{K})$. However, inconsistency is a concept that is not easily quantified and there have been quite a number of proposals for inconsistency measures so far, in particular for classical propositional logic, see e.g. (McAreavey, Liu, and Miller 2014; Jabbour et al. 2015) for some recent works. Formally, we define inconsistency measures as follows, cf. e.g. (Hunter and Konieczny 2008).

Definition 2. An inconsistency measure $\mathcal{I}$ is a function $\mathcal{I} : \mathbb{K} \to [0, \infty]$ satisfying $\mathcal{I}(\mathcal{K}) = 0$ if and only if $\mathcal{K}$ is consistent, for all $\mathcal{K} \in \mathbb{K}$.

Here, we select a representative set of 13 inconsistency measures from the literature in order to conduct our analysis on computational complexity, taken from (Hunter and Konieczny 2010; Grant and Hunter 2011; Knight 2002; Thimm 2016b; Grant and Hunter 2013; Xiao and Ma 2012; Doder et al. 2010). We briefly introduce these measures in

$$\mathcal{I}_d(\mathcal{K}) = \begin{cases} 1 & \text{if } \mathcal{K} \models \bot \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{I}_{\mathsf{MI}}(\mathcal{K}) = |\mathsf{MI}(\mathcal{K})|$$

$$\mathcal{I}_{\mathsf{MI}^c}(\mathcal{K}) = \sum_{M \in \mathsf{MI}(\mathcal{K})} \frac{1}{|M|}$$

$$\mathcal{I}_\eta(\mathcal{K}) = 1 - \max\{\xi \mid \exists P \in \mathcal{P}(\mathsf{At}) : \forall \alpha \in \mathcal{K} : P(\alpha) \geq \xi\}$$

$$\mathcal{I}_c(\mathcal{K}) = \min\{|v^{-1}(B)| \mid v \models^3 \mathcal{K}\}$$

$$\mathcal{I}_{mc}(\mathcal{K}) = |\mathsf{MC}(\mathcal{K})| + |\mathsf{SC}(\mathcal{K})| - 1$$

$$\mathcal{I}_p(\mathcal{K}) = |\bigcup_{M \in \mathsf{MI}(\mathcal{K})} M|$$

$$\mathcal{I}_{hs}(\mathcal{K}) = \min\{|H| \mid H \text{ is a hitting set of } \mathcal{K}\} - 1$$

$$\mathcal{I}_{\text{dalal}}^\Sigma(\mathcal{K}) = \min\{\sum_{\alpha \in \mathcal{K}} d_{\mathsf{d}}(\mathsf{Mod}(\alpha), \omega) \mid \omega \in \Omega(\mathsf{At})\}$$

$$\mathcal{I}_{\text{dalal}}^{\max}(\mathcal{K}) = \min\{\max_{\alpha \in \mathcal{K}} d_{\mathsf{d}}(\mathsf{Mod}(\alpha), \omega) \mid \omega \in \Omega(\mathsf{At})\}$$

$$\mathcal{I}_{\text{dalal}}^{\text{hit}}(\mathcal{K}) = \min\{|\{\alpha \in \mathcal{K} \mid d_{\mathsf{d}}(\mathsf{Mod}(\alpha), \omega) > 0\}| \mid \omega \in \Omega(\mathsf{At})\}$$

$$\mathcal{I}_{mv}(\mathcal{K}) = \frac{|\bigcup_{M \in \mathsf{MI}(\mathcal{K})} \mathsf{At}(M)|}{|\mathsf{At}(\mathcal{K})|}$$

$$\mathcal{I}_{nc}(\mathcal{K}) = |\mathcal{K}| - \max\{n \mid \forall \mathcal{K}' \subseteq \mathcal{K} : |\mathcal{K}'| = n \Rightarrow \mathcal{K}' \not\models \bot\}$$

Figure 1: Definitions of the considered inconsistency measures

this section for the sake of completeness, but we refer for a detailed explanation to the corresponding original papers.

The formal definitions of the considered inconsistency measures can be found in Figure 1, while the necessary notation for understanding these measures follows below.

The measure $\mathcal{I}_d(\mathcal{K})$ (Hunter and Konieczny 2008) is usually referred to as a baseline for inconsistency measures as it only distinguishes between consistent and inconsistent knowledge bases. The measures $\mathcal{I}_{\mathsf{MI}}(\mathcal{K})$ (Hunter and Konieczny 2008), $\mathcal{I}_{\mathsf{MI}^c}(\mathcal{K})$ (Hunter and Konieczny 2008), $\mathcal{I}_p$ (Grant and Hunter 2011), and $\mathcal{I}_{mv}$ (Xiao and Ma 2012) are defined using minimal inconsistent subsets. A set $M \subseteq \mathcal{K}$ is called *minimal inconsistent subset* (MI) of $\mathcal{K}$ if $M \models \bot$ and there is no $M' \subset M$ with $M' \models \bot$. Let $\mathsf{MI}(\mathcal{K})$ be the set of all MIs of $\mathcal{K}$. For $\mathcal{I}_{mc}$ (Grant and Hunter 2011), let furthermore $\mathsf{MC}(\mathcal{K})$ be the set of maximal consistent subsets of $\mathcal{K}$, i.e., $\mathsf{MC}(\mathcal{K}) = \{\mathcal{K}' \subseteq \mathcal{K} \mid \mathcal{K}' \not\models \bot \land \forall \mathcal{K}'' \supsetneq \mathcal{K}' : \mathcal{K}'' \models \bot\}$, and let $\mathsf{SC}(\mathcal{K})$ be the set of self-contradictory formulas of $\mathcal{K}$, i.e., $\mathsf{SC}(\mathcal{K}) = \{\phi \in \mathcal{K} \mid \phi \models \bot\}$. Note also that $\mathcal{I}_{nc}$ (Doder et al. 2010) uses the concept of maximal consistency in its formal definition, but in a slightly different manner. The measure $\mathcal{I}_\eta$ (Knight 2002) considers probability functions P of the form $P : \Omega(\mathsf{At}) \to [0, 1]$ with $\sum_{\omega \in \Omega(\mathsf{At})} P(\omega) = 1$. Let $\mathcal{P}(\mathsf{At})$ be the set of all those probability functions and for a given probability function $P \in \mathcal{P}(\mathsf{At})$ define the probability of an arbitrary for-

mula ϕ via $P(\phi) = \sum_{\omega \models \phi} P(\omega)$. The measure $\mathcal{I}_c$ (Grant and Hunter 2011) utilizes three-valued interpretations for propositional logic (Priest 1979).[2] A three-valued interpretation υ on At is a function $\upsilon : \mathsf{At} \rightarrow \{T, F, B\}$ where the values T and F correspond to the classical true and false, respectively. The additional truth value B stands for *both* and is meant to represent a conflicting truth value for a proposition. Taking into account the *truth order* $\prec$ defined via $T \prec B \prec F$, an interpretation υ is extended to arbitrary formulas via $\upsilon(\phi_1 \wedge \phi_2) = \min_\prec(\upsilon(\phi_1), \upsilon(\phi_2))$, $\upsilon(\phi_1 \vee \phi_2) = \max_\prec(\upsilon(\phi_1), \upsilon(\phi_2))$, and $\upsilon(\neg T) = F$, $\upsilon(\neg F) = T$, $\upsilon(\neg B) = B$. An interpretation υ satisfies a formula α, denoted by $\upsilon \models^3 \alpha$ if either $\upsilon(\alpha) = T$ or $\upsilon(\alpha) = B$. For $\mathcal{I}_{hs}$ (Thimm 2016b), a subset $H \subseteq \Omega(\mathsf{At})$ is called a *hitting set* of $\mathcal{K}$ if for every $\phi \in \mathcal{K}$ there is $\omega \in H$ with $\omega \models \phi$. The *Dalal distance* d_d is a distance function for interpretations in $\Omega(\mathsf{At})$ and is defined as $d(\omega, \omega') = |\{a \in \mathsf{At} \mid \omega(a) \neq \omega'(a)\}|$ for all $\omega, \omega' \in \Omega(\mathsf{At})$. If $X \subseteq \Omega(\mathsf{At})$ is a set of interpretations we define $d_\mathrm{d}(X, \omega) = \min_{\omega' \in X} d_\mathrm{d}(\omega', \omega)$ (if $X = \emptyset$ we define $d_\mathrm{d}(X, \omega) = \infty$). We consider the inconsistency measures $\mathcal{I}_\mathrm{dalal}^\Sigma$, $\mathcal{I}_\mathrm{dalal}^\mathrm{max}$, and $\mathcal{I}_\mathrm{dalal}^\mathrm{hit}$ from (Grant and Hunter 2013) but only for the Dalal distance. Note that in (Grant and Hunter 2013) these measures were considered for arbitrary distances and that we use a slightly different but equivalent definition of these measures.

We conclude this section with a small example illustrating the behavior of the considered inconsistency measures on the example knowledge bases from the introduction.

Example 1. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be given as

$$\mathcal{K}_1 = \{a, b \vee c, \neg a \wedge \neg b, d\} \quad \mathcal{K}_2 = \{a, \neg a, b, \neg b\}$$

Then

$$
\begin{aligned}
\mathcal{I}_d(\mathcal{K}_1) &= 1 & \mathcal{I}_d(\mathcal{K}_2) &= 1 \\
\mathcal{I}_\mathsf{MI}(\mathcal{K}_1) &= 1 & \mathcal{I}_\mathsf{MI}(\mathcal{K}_2) &= 2 \\
\mathcal{I}_{\mathsf{MI}^c}(\mathcal{K}_1) &= 1/2 & \mathcal{I}_{\mathsf{MI}^c}(\mathcal{K}_2) &= 1 \\
\mathcal{I}_\eta(\mathcal{K}_1) &= 1/2 & \mathcal{I}_\eta(\mathcal{K}_2) &= 1/2 \\
\mathcal{I}_c(\mathcal{K}_1) &= 1 & \mathcal{I}_c(\mathcal{K}_2) &= 2 \\
\mathcal{I}_{mc}(\mathcal{K}_1) &= 1 & \mathcal{I}_{mc}(\mathcal{K}_2) &= 3 \\
\mathcal{I}_p(\mathcal{K}_1) &= 2 & \mathcal{I}_p(\mathcal{K}_2) &= 4 \\
\mathcal{I}_{hs}(\mathcal{K}_1) &= 1 & \mathcal{I}_{hs}(\mathcal{K}_2) &= 1 \\
\mathcal{I}_\mathrm{dalal}^\Sigma(\mathcal{K}_1) &= 1 & \mathcal{I}_\mathrm{dalal}^\Sigma(\mathcal{K}_2) &= 2 \\
\mathcal{I}_\mathrm{dalal}^\mathrm{max}(\mathcal{K}_1) &= 1 & \mathcal{I}_\mathrm{dalal}^\mathrm{max}(\mathcal{K}_2) &= 1 \\
\mathcal{I}_\mathrm{dalal}^\mathrm{hit}(\mathcal{K}_1) &= 1 & \mathcal{I}_\mathrm{dalal}^\mathrm{hit}(\mathcal{K}_2) &= 2 \\
\mathcal{I}_{mv}(\mathcal{K}_1) &= 1/2 & \mathcal{I}_{mv}(\mathcal{K}_2) &= 1 \\
\mathcal{I}_{nc}(\mathcal{K}_1) &= 3 & \mathcal{I}_{nc}(\mathcal{K}_2) &= 3
\end{aligned}
$$

4 Analysis of Computational Complexity

In this paper, we consider the following three decision problems for our investigation of the computational complexity

of inconsistency measurement. Let $\mathcal{I}$ be some inconsistency measure.

$\text{EXACT}_\mathcal{I}$ **Input**: $\mathcal{K} \in \mathbb{K}$, $x \in [0, \infty]$
Output: TRUE iff $\mathcal{I}(\mathcal{K}) = x$

$\text{UPPER}_\mathcal{I}$ **Input**: $\mathcal{K} \in \mathbb{K}$, $x \in [0, \infty]$
Output: TRUE iff $\mathcal{I}(\mathcal{K}) \leq x$

$\text{LOWER}_\mathcal{I}$ **Input**: $\mathcal{K} \in \mathbb{K}$, $x \in (0, \infty]$
Output: TRUE iff $\mathcal{I}(\mathcal{K}) \geq x$

Note that for any inconsistency measure $\mathcal{I}$ according to Definition 2 the decision problems $\text{EXACT}_\mathcal{I}$ and $\text{UPPER}_\mathcal{I}$ are at least NP-hard as deciding whether $\mathcal{I}(\mathcal{K}) = 0$ is equivalent to deciding whether $\mathcal{K}$ is consistent, which itself is equivalent to the satisfiability problem SAT. Similarly, the problem $\text{LOWER}_\mathcal{I}$ is at least coNP-hard as deciding whether $\mathcal{I}(\mathcal{K}) \geq x$ for some $x > 0$ entails that $\mathcal{K}$ is inconsistent[3], which itself is equivalent to the unsatisfiability problem UNSAT. Furthermore, we consider the following natural function problem for our investigation:

$\text{VALUE}_\mathcal{I}$ **Input**: $\mathcal{K} \in \mathbb{K}$
Output: The value of $\mathcal{I}(\mathcal{K})$

Table 1 gives an overview on the technical results shown in the remainder of this paper. As can be seen, most measures fall into the first level of the polynomial hierarchy ($\mathcal{I}_d, \mathcal{I}_\eta, \mathcal{I}_c, \mathcal{I}_{hs}, \mathcal{I}_\mathrm{dalal}^\Sigma, \mathcal{I}_\mathrm{dalal}^\mathrm{max}, \mathcal{I}_\mathrm{dalal}^\mathrm{hit}$), where the decision problems $\text{UPPER}_\mathcal{I}$ and $\text{LOWER}_\mathcal{I}$ can be shown to be NP-complete and coNP-complete, respectively, and thus not computationally harder than SAT and UNSAT problems, respectively. The remaining measures are either on the second level of the polynomial hierarchy ($\mathcal{I}_p, \mathcal{I}_{mv}, \mathcal{I}_{nc}$) or involve counting (sub)problems whose complexity goes beyond the second level of the polynomial hierarchy ($\mathcal{I}_\mathsf{MI}, \mathcal{I}_{\mathsf{MI}^c}, \mathcal{I}_{mc}$).

Before we continue with the details of the technical results, we make some general observations first. In particular, in order to provide insights into the computational complexity of the problem $\text{VALUE}_\mathcal{I}$ it is useful to investigate the number of values an inconsistency measure can attain for knowledge bases of a given size, cf. (Thimm 2016a) for a more detailed discussion of this topic.

Definition 3. Let ϕ be a formula. The *length* $\mathtt{len}(\phi)$ of ϕ is recursively defined as

$$
\mathtt{len}(\phi) = \begin{cases}
1 & \text{if } \phi \in \mathsf{At} \\
1 + \mathtt{len}(\phi') & \text{if } \phi = \neg\phi' \\
1 + \mathtt{len}(\phi_1) + \mathtt{len}(\phi_2) & \text{if } \phi = \phi_1 \wedge \phi_2 \\
1 + \mathtt{len}(\phi_1) + \mathtt{len}(\phi_2) & \text{if } \phi = \phi_1 \vee \phi_2
\end{cases}
$$

Define the *length* $\mathtt{len}(\mathcal{K})$ of a knowledge base $\mathcal{K}$ via $\mathtt{len}(\mathcal{K}) = \sum_{\phi \in \mathcal{K}} \mathtt{len}(\phi)$.

Definition 4. For an inconsistency measure $\mathcal{I}$ and $n \in \mathbb{N}$ define $\mathcal{C}_\mathcal{I}(n) = \{\mathcal{I}(\mathcal{K}) \mid \mathtt{len}(\mathcal{K}) \leq n\}$, i.e., $\mathcal{C}_\mathcal{I}(n)$ is the set of different inconsistency values that can be attained by $\mathcal{I}$ on knowledge bases of maximal length n.

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[2]Note that slightly different formalizations of this idea have been given in (Hunter and Konieczny 2010; Ma, Qi, and Hitzler 2011).

[3]Note that determining the first possible positive value for every considered inconsistency measure is straightforward; most measures are integer-valued, so the first possible positive value is 1, for $\mathcal{I}_{\mathsf{MI}^c}$ it is $1/|\mathcal{K}|$, for $\mathcal{I}_{mv}$ it is $1/|\mathsf{At}(\mathcal{K})|$, and for $\mathcal{I}_\eta$ it is $1/2^{|\mathsf{At}(\mathcal{K})|}$ (the latter is due to combinatorial considerations, we omit the formal proof due to space restrictions).

	$\text{EXACT}_{\mathcal{I}}$	$\text{UPPER}_{\mathcal{I}}$	$\text{LOWER}_{\mathcal{I}}$	$\text{VALUE}_{\mathcal{I}}$
$\mathcal{I}_d$	D_1^p	NP-c	coNP-c	$\mathsf{FP}^{\mathsf{NP}}$
$\mathcal{I}_{\text{MI}}$	PSPACE	PSPACE	PSPACE	$\#\cdot\mathsf{coNP}$-c
$\mathcal{I}_{\text{MI}^c}$	PSPACE	PSPACE	PSPACE	FPSPACE
$\mathcal{I}_\eta$	D_1^p	NP-c	coNP-c	$\mathsf{FP}^{\mathsf{NP}[\log n]}$
$\mathcal{I}_c$	D_1^p	NP-c	coNP-c	$\mathsf{FP}^{\mathsf{NP}[\log n]}$
$\mathcal{I}_{mc}$	PSPACE	PSPACE	PSPACE	$\mathsf{FPSPACE}^\dagger$
$\mathcal{I}_p$	D_2^p	Π_2^p-c	Σ_2^p-c	$\mathsf{FP}^{\Sigma_2^p[\log n]}$
$\mathcal{I}_{hs}$	D_1^p	NP-c	coNP-c	$\mathsf{FP}^{\mathsf{NP}[\log n]}$
$\mathcal{I}_{\text{dalal}}^\Sigma$	D_1^p	NP-c	coNP-c	$\mathsf{FP}^{\mathsf{NP}[\log n]}$-c
$\mathcal{I}_{\text{dalal}}^{\max}$	D_1^p	NP-c	coNP-c	$\mathsf{FP}^{\mathsf{NP}[\log n]}$
$\mathcal{I}_{\text{dalal}}^{hit}$	D_1^p	NP-c	coNP-c	$\mathsf{FP}^{\mathsf{NP}[\log n]}$-c
$\mathcal{I}_{mv}$	D_2^p-c	Π_2^p-c	Σ_2^p-c	$\mathsf{FP}^{\Sigma_2^p[\log n]}$
$\mathcal{I}_{nc}$	D_2^p	Π_2^p-c	Σ_2^p-c	$\mathsf{FP}^{\Sigma_2^p[\log n]}$

Table 1: Computational complexity of the considered inconsistency measures (all statements are membership statements, an additionally attached "-c" also indicates completeness for the class); under complexity-theoretic assumptions, if $\text{VALUE}_{\mathcal{I}}$ is $\#\cdot\mathsf{coNP}$ hard then the corresponding decision problems for $\mathcal{I}$ are presumably not contained in a class of the polynomial hierarchy; †decomposes into a $\#\cdot\mathsf{coNP}$-complete and an $\mathsf{FP}^{\mathsf{NP}[\log n]}$-complete problem.

Lemma 1. *For $\mathcal{I} \in \{\mathcal{I}_d, \mathcal{I}_\eta, \mathcal{I}_c, \mathcal{I}_p, \mathcal{I}_{hs}, \mathcal{I}_{dalal}^\Sigma, \mathcal{I}_{dalal}^{\max}, \mathcal{I}_{dalal}^{hit}, \mathcal{I}_{mv}, \mathcal{I}_{nc}\}$ there is $k \in \mathbb{N}$ such that $|\mathcal{C}_{\mathcal{I}}(n)| \in O(n^k)$.*

Proof. We omit the proof for $\mathcal{I}_\eta$ due to space restrictions. For $\mathcal{I}_d$ we trivially have $|\mathcal{C}_{\mathcal{I}_d}(n)| \in O(1)$ as $\mathcal{I}_d$ only has two different values. For $\mathcal{I}_c$ observe that a knowledge base $\mathcal{K}$ with $\text{len}(\mathcal{K}) \leq n$ cannot mention more than n different propositions. Therefore $\mathcal{I}_c(\mathcal{K}) \leq n$ for every $\mathcal{K}$ with $\text{len}(\mathcal{K}) \leq n$ and $|\mathcal{C}_{\mathcal{I}_c}(n)| \in O(n)$. For $\mathcal{I}_p$ note that $\mathcal{I}_p(\mathcal{K}) \leq |\mathcal{K}| \leq n$ and therefore $|\mathcal{C}_{\mathcal{I}_p}(n)| \in O(n)$. For $\mathcal{I}_{hs}$, observe that in the worst-case when every formula of $\mathcal{K}$ is pair-wise inconsistent with each other, there is an H with $|H| = |\mathcal{K}|$ that is a minimal hitting set of $\mathcal{K}$. So either $\mathcal{I}_{hs}(\mathcal{K}) \leq |\mathcal{K}| \leq n$ or $\mathcal{I}_{hs}(\mathcal{K}) = \infty$ (if there is a contradictory formula) and therefore $|\mathcal{C}_{\mathcal{I}_{hs}}(n)| \in O(n)$. For $\mathcal{I}_{dalal}^\Sigma$ observe that both the number of formulas and the number of propositions in $\mathcal{K}$ with $\text{len}(\mathcal{K}) \leq n$ are bounded by n. Then $d_{\mathsf{d}}(\text{Mod}(\alpha), \omega) \leq n$ for every $\alpha \in \mathcal{K}$ and $\omega \in \Omega(\text{At})$ and therefore

$$\mathcal{I}_{\text{dalal}}^\Sigma(\mathcal{K}) = \min\{\sum_{\alpha \in \mathcal{K}} d_{\mathsf{d}}(\text{Mod}(\alpha), \omega) \mid \omega \in \Omega(\text{At})\}$$
$$\leq \sum_{\alpha \in \mathcal{K}} n = n^2$$

and hence $|\mathcal{C}_{\mathcal{I}_{\text{dalal}}^\Sigma}(n)| \in O(n^2)$. It also follows $|\mathcal{C}_{\mathcal{I}_{\text{dalal}}^{\max}}(n)| \in O(n)$ and $|\mathcal{C}_{\mathcal{I}_{\text{dalal}}^{hit}}(n)| \in O(n)$. For $\mathcal{I}_{mv}$ observe that $|\bigcup_{M \in \text{MI}(\mathcal{K})} \text{At}(M)| \leq n$ for $\mathcal{K}$ with $\text{len}(\mathcal{K}) \leq n$ as $\mathcal{K}$ cannot mention more than n propositions. It follows $|\mathcal{C}_{\mathcal{I}_{mv}}(n)| \in O(n)$. Finally, observe $\mathcal{I}_{nc}(\mathcal{K}) \in \{0, \ldots, |\mathcal{K}|\}$ and therefore $|\mathcal{C}_{\mathcal{I}_{nc}}(n)| \in O(n)$. $\square$

The above lemma basically states that the number of different values most of the investigated inconsistency measures can attain on knowledge bases up to a certain size, is polynomially bounded by this size. Note that the statement is not true in general for $\mathcal{I}_{\text{MI}}, \mathcal{I}_{\text{MI}^c}$, and $\mathcal{I}_{mc}$ (a knowledge base may have an exponential number of minimal (in-)consistent subsets).

Lemma 1 is in particular useful in combination with (exact) complexity bounds for problems $\text{UPPER}_{\mathcal{I}}$ and $\text{LOWER}_{\mathcal{I}}$. If, e.g., $\text{UPPER}_{\mathcal{I}}$ is in complexity class C for a measure $\mathcal{I}$ for which it holds that $|\mathcal{C}_{\mathcal{I}}(n)| \in O(n^k)$, we can find the exact value of $\mathcal{I}(\mathcal{K})$ for a knowledge base $\mathcal{K}$ with binary search on the possible values requiring thus to solve just a logarithmic number of consecutive problems in C. These considerations are summarized in the following result.

Lemma 2. *Let $\mathcal{I}$ be some inconsistency measure and $i > 0$ an integer. If $\text{UPPER}_{\mathcal{I}}$ is in Σ_i^p or in Π_i^p, and $|\mathcal{C}_{\mathcal{I}}(n)| \in O(n^k)$ for some $k \in \mathbb{N}$, then $\text{VALUE}_{\mathcal{I}}$ is in $\mathsf{FP}^{\Sigma_i^p[\log n]}$.*

The decision problems $\text{EXACT}_{\mathcal{I}}$, $\text{UPPER}_{\mathcal{I}}$, and $\text{LOWER}_{\mathcal{I}}$ are also related to each other. $\text{UPPER}_{\mathcal{I}}$ and $\text{LOWER}_{\mathcal{I}}$ are complementary to each and $\text{EXACT}_{\mathcal{I}}$ is the combination of both. However, we need another condition on inconsistency measures to see this.

Definition 5. *An inconsistency measure $\mathcal{I}$ is called well-serializable if the following two problems are in P:*

1. Given $n \in \mathbb{N}$ and $x \in \mathcal{C}_{\mathcal{I}}(n)$, determine $y \in \mathcal{C}_{\mathcal{I}}(n)$ such that $y > x$ and there is no $y' \in \mathcal{C}_{\mathcal{I}}(n)$ with $y > y' > x$.
2. Given $n \in \mathbb{N}$ and $x \in \mathcal{C}_{\mathcal{I}}(n)$, determine $y \in \mathcal{C}_{\mathcal{I}}(n)$ such that $y < x$ and there is no $y' \in \mathcal{C}_{\mathcal{I}}(n)$ with $y < y' < x$.

In other words, a measure is called well-serializable if the immediate successor and predecessor of a value of $\mathcal{I}$ can be efficiently determined. Note that all considered measures satisfy this property[4].

Lemma 3. *Let $\mathcal{I}$ be some well-serializable inconsistency measure and $i > 0$ an integer. Let $C \in \{\Sigma_i^p, \Pi_i^p\}$.*

- *$\text{UPPER}_{\mathcal{I}}$ is C-complete iff $\text{LOWER}_{\mathcal{I}}$ is co-C-complete;*
- *if $\text{UPPER}_{\mathcal{I}}$ or $\text{LOWER}_{\mathcal{I}}$ is in C, then $\text{EXACT}_{\mathcal{I}}$ is in D_i^p.*

Proof. Let $\mathcal{K}$ and x be an instance of $\text{UPPER}_{\mathcal{I}}$. This is a yes instance iff $\mathcal{K}$ together with y—where y is the immediate successor of x in $\mathcal{C}_{\mathcal{I}}(n)$—is a no instance of $\text{LOWER}_{\mathcal{I}}$. If $\text{UPPER}_{\mathcal{I}}$ is Σ_i^p-complete (Π_i^p-complete) then $\text{LOWER}_{\mathcal{I}}$ is in Π_i^p-complete (Σ_i^p-complete). For the second item note that $\mathcal{I}(\mathcal{K}) = x$ holds iff $\mathcal{I}(\mathcal{K}) \geq x$ and $\mathcal{I}(\mathcal{K}) \leq x$. $\square$

Lemma 2 and Lemma 3 taken together imply that showing the complexity of either $\text{UPPER}_{\mathcal{I}}$ or $\text{LOWER}_{\mathcal{I}}$ gives crucial insights into the computation of measure $\mathcal{I}$.

In the following, we give the details on the technical contributions summarized in Table 1. We structure our presentation by first discussing the problems on the first level of the polynomial hierarchy (Section 4.1), then those on the second level (Section 4.2), and finally those beyond the second level of the polynomial hierarchy (Section 4.3).

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[4]The argumentation is similar as for Footnote 3.

4.1 Problems on the first level of the polynomial hierarchy

In this section we discuss the measures $\mathcal{I}_d, \mathcal{I}_\eta, \mathcal{I}_c, \mathcal{I}_{hs}, \mathcal{I}_{\text{dalal}}^\Sigma, \mathcal{I}_{\text{dalal}}^{\max}$, and $\mathcal{I}_{\text{dalal}}^{\text{hit}}$ and show that the corresponding decision and function problems reside on the first level of the polynomial hierarchy. For all these measures, we start by showing that $\text{UPPER}_\mathcal{I}$ is NP-complete and then utilize Lemmas 2 and 3 to gain insights on the remaining problems.

The first measure we investigate is the baseline inconsistency measure, $\mathcal{I}_d$, which is equal to 0 if the given knowledge base is consistent and 1 otherwise, making the problem $\text{UPPER}_{\mathcal{I}_d}$ obviously NP-complete.

Proposition 1. $\text{UPPER}_{\mathcal{I}_d}$ *is NP-complete.*

As one can compute the value for $\mathcal{I}_d$ by one call to a SAT-solver we also have that $\text{VALUE}_{\mathcal{I}_d}$ is in FP^{NP}.

Proposition 2. $\text{UPPER}_{\mathcal{I}_\eta}$ *is NP-complete.*

Proof. (Sketch) Note that the problem to compute $\mathcal{I}_\eta(\mathcal{K})$ can be represented as a linear program over an exponential number of variables (the possible worlds) and a linear number of equalities and inequalities (Knight 2002). Any solution to this problem is nonnegative and due to the small-model-property of linear programs (Chvátal 1983), there is a solution where only a polynomial number of variables receive a non-zero value. We can therefore guess a set of polynomial many variables, set the objective function to the given upper bound x, and solve the corresponding program using a polynomial-time algorithm (as linear programming is in P). If it is feasible, x is indeed an upper bound. Completeness for NP follows from the fact that we can reduce SAT to $\text{UPPER}_{\mathcal{I}_\eta}$ with $x = 0$. $\qquad\square$

Proposition 3. $\text{UPPER}_{\mathcal{I}_c}$ *is NP-complete.*

Proposition 4. $\text{UPPER}_{\mathcal{I}_{hs}}$ *is NP-complete.*

The proofs of Propositions 3 and 4 are omitted due to space restrictions, but the statements can be shown using simple guess-and-check algorithms. For Proposition 3 see also (Ma et al. 2010) where the result has been shown for a variant of $\mathcal{I}_c$.

We move on to the measures involving the distance measures (Grant and Hunter 2013). Membership in NP for $\text{UPPER}_\mathcal{I}$ with $\mathcal{I} \in \{\mathcal{I}_{\text{dalal}}^{\text{hit}}, \mathcal{I}_{\text{dalal}}^\Sigma, \mathcal{I}_{\text{dalal}}^{\max}\}$ relies on the fact that we can non-deterministically guess multiple interpretations and in polynomial time verify whether these interpretations satisfy the given formulas and, additionally, compute the (Dalal) distance between these interpretations in polynomial time. For the measures $\mathcal{I}_{\text{dalal}}^{\text{hit}}$ and $\mathcal{I}_{\text{dalal}}^\Sigma$ we also give exact complexity bounds for their functional problems $\text{VALUE}_\mathcal{I}$.

Proposition 5. $\text{UPPER}_{\mathcal{I}_{\text{dalal}}^{\text{hit}}}$ *is NP-complete and* $\text{VALUE}_{\mathcal{I}_{\text{dalal}}^{\text{hit}}}$ *is $\text{FP}^{\text{NP}[\log n]}$-complete.*

Proof. We begin with the hardness proof for $\text{VALUE}_{\mathcal{I}_{\text{dalal}}^{\text{hit}}}$. Let $\phi_s = \{c_1, \ldots, c_n\}$ be an instance of the $\text{FP}^{\text{NP}[\log n]}$-complete problem MaxSAT Size, where the task is to find the maximum number of clauses c_i of ϕ_s that can be simultaneously satisfied. We show that for $\phi_s = \mathcal{K}$ we have

$\mathcal{I}_{\text{dalal}}^{\text{hit}}(\mathcal{K}) = n - k$ with k the maximum number of clauses that can be simultaneously satisfied in ϕ_s, i.e. k is the solution to ϕ_s in the MaxSAT Size problem.

$$n - \max\{|C| \mid C \subseteq \mathcal{K}, \bigwedge_{c \in C} \not\models \perp\}$$
$$= n - \max\{|\{\alpha \in \mathcal{K} \mid d_{\text{d}}(\text{Mod}(\alpha), \omega) = 0\}| \mid \omega \in \Omega(\text{At})\}$$
$$= \min\{|\{\alpha \in \mathcal{K} \mid d_{\text{d}}(\text{Mod}(\alpha), \omega) > 0\}| \mid \omega \in \Omega(\text{At})\}$$
$$= \mathcal{I}_{\text{dalal}}^{\text{hit}}(\mathcal{K})$$

Thus, we have reduced MaxSAT Size to $\text{VALUE}_{\mathcal{I}_{\text{dalal}}^{\text{hit}}}$.

Hardness for $\text{UPPER}_{\mathcal{I}_{\text{dalal}}^{\text{hit}}}$ follows since we can reduce SAT to $\text{UPPER}_{\mathcal{I}_{\text{dalal}}^{\text{hit}}}$ if we set bound $x = 0$.

Further, $\text{UPPER}_{\mathcal{I}_{\text{dalal}}^{\text{hit}}}$ is in NP, since we can non-deterministically guess an $\omega \in \Omega(\text{At})$ and interpretations for each $\alpha \in \mathcal{K}$ for a given $\mathcal{K}$, and verify that the d_{d}-distance is 0 for at least $\lfloor x \rfloor$ many elements in $\mathcal{K}$ for a given real x. This implies that $\text{UPPER}_{\mathcal{I}_{\text{dalal}}^{\text{hit}}}$ is NP-complete and $\text{VALUE}_{\mathcal{I}_{\text{dalal}}^{\text{hit}}}$ is $\text{FP}^{\text{NP}[\log n]}$-complete due to Lemma 2. $\qquad\square$

Proposition 6. $\text{UPPER}_{\mathcal{I}_{\text{dalal}}^\Sigma}$ *is NP-complete and* $\text{VALUE}_{\mathcal{I}_{\text{dalal}}^\Sigma}$ *is $\text{FP}^{\text{NP}[\log n]}$-complete.*

Proof. We again start showing hardness for $\text{VALUE}_{\mathcal{I}_{\text{dalal}}^\Sigma}$ by reduction from the functional problem MaxSAT Size. Let $\phi_s = \{c_1, \ldots, c_n\}$ be again an instance of MaxSAT Size with ϕ_s over variables $\{x_1, \ldots, x_m\}$. We construct $\mathcal{K} = \{\alpha_1, \alpha_2\}$ with $\alpha_1 = \bigwedge_{1 \le i \le n}(c_i \vee (\neg y_i))$ and $\alpha_2 = \bigwedge_{1 \le i \le n} y_i$ with fresh variables y_i. We now show that for k the maximum number of clauses in ϕ_s that can be simultaneously satisfied, $n - k = \mathcal{I}_{\text{dalal}}^\Sigma(\mathcal{K})$. First, we prove that $\forall \omega \in \Omega(\text{At})$

$$d_{\text{d}}(\text{Mod}(\alpha_1), \omega) + d_{\text{d}}(\text{Mod}(\alpha_2), \omega) \ge n - k. \quad (1)$$

Define shorthands $d_{\text{d}}(\text{Mod}(\alpha_1), \omega) = a(\omega)$ and $d_{\text{d}}(\text{Mod}(\alpha_2), \omega) = b(\omega)$. Suppose the contrary, i.e. $\exists \omega \in \Omega(\text{At}), a(\omega) + b(\omega) < n - k$. The interpretation ω assigns $b(\omega)$ many y_i variables to false and $n - b(\omega)$ many to true. By presumption, there must exist a model ω_1 of α_1 s.t. $d_{\text{d}}(\omega_1, \omega) < n - k - b(\omega)$. Consider now the maximum number c of y_i variables that ω_1 assigns to false. Under the previous constraint on the symmetric difference, ω_1 can assign all y_i variables to false that are also assigned to false by ω ($b(\omega)$ many), and additionally less than $n - k - b(\omega)$ (remainder of symmetric difference). Thus we can bound c by $c < b(\omega) + n - k - b(\omega)$ and in turn by $c < n - k$. This implies that ω_1 satisfies at least $n - c$ clauses of ϕ_s, i.e. strictly more than k clauses, a contradiction. Thus, Equation (1) holds. There always exists an $\omega_2 \in \Omega(\text{At})$ that assigns all y_i to true s.t. Equation 1 holds with equality. This implies our claim, i.e. $n - k = \mathcal{I}_{\text{dalal}}^\Sigma(\mathcal{K})$.

By similar reasoning as for $\mathcal{I}_{\text{dalal}}^{\text{hit}}$, we conclude that $\text{UPPER}_{\mathcal{I}_{\text{dalal}}^\Sigma}$ is NP-complete and $\text{VALUE}_{\mathcal{I}_{\text{dalal}}^\Sigma}$ is $\text{FP}^{\text{NP}[\log n]}$-complete. $\qquad\square$

Proposition 7. $\text{UPPER}_{\mathcal{I}_{\text{dalal}}^{\max}}$ *is NP-complete.*

Proof. Membership follows from considering the following algorithm. For $\mathcal{K} = \{\alpha_1, \ldots, \alpha_n\}$ guess $(\omega_0, \omega_1, \ldots, \omega_n) \in \Omega(\mathsf{At})^{n+1}$. For each α_i, $i = 1, \ldots, n$, check whether $\omega_i \models \alpha_i$ (this a polynomial test). Then compute $x = \max_{i=1,\ldots,n} d_{\mathrm{d}}(\omega_i, \omega_0)$, also in polynomial time. It is easy to see that x is an upper bound for $\mathcal{I}_{\mathrm{dalal}}^{\max}(\mathcal{K})$. NP-hardness follows from the fact that SAT can be reduced to $\mathrm{UPPER}_{\mathcal{I}_{\mathrm{dalal}}^{\max}}$ with $x = 0$. $\qquad\square$

Utilizing Lemma 3, we directly obtain the following statements regarding $\mathrm{LOWER}_{\mathcal{I}}$ and $\mathrm{EXACT}_{\mathcal{I}}$ and the inconsistency measures from above.

Corollary 1. *It holds that*

- *problems* $\mathrm{LOWER}_{\mathcal{I}_d}$, $\mathrm{LOWER}_{\mathcal{I}_\eta}$, $\mathrm{LOWER}_{\mathcal{I}_c}$, $\mathrm{LOWER}_{\mathcal{I}_{hs}}$, $\mathrm{LOWER}_{\mathcal{I}_{\mathrm{dalal}}^\Sigma}$, $\mathrm{LOWER}_{\mathcal{I}_{\mathrm{dalal}}^{\max}}$, *and* $\mathrm{LOWER}_{\mathcal{I}_{\mathrm{dalal}}^{hit}}$ *are* coNP-*complete; and*
- *problems* $\mathrm{EXACT}_{\mathcal{I}_d}$, $\mathrm{EXACT}_{\mathcal{I}_\eta}$, $\mathrm{EXACT}_{\mathcal{I}_c}$, $\mathrm{EXACT}_{\mathcal{I}_{hs}}$, $\mathrm{EXACT}_{\mathcal{I}_{\mathrm{dalal}}^\Sigma}$, $\mathrm{EXACT}_{\mathcal{I}_{\mathrm{dalal}}^{\max}}$, *and* $\mathrm{EXACT}_{\mathcal{I}_{\mathrm{dalal}}^{hit}}$ *are in* D_1^p.

Regarding the functional problem $\mathrm{VALUE}_{\mathcal{I}}$ and by combining lemmas 1 and 2 we obtain the following.

Corollary 2. *It holds that* $\mathrm{VALUE}_{\mathcal{I}_\eta}$, $\mathrm{VALUE}_{\mathcal{I}_c}$, $\mathrm{VALUE}_{\mathcal{I}_{hs}}$, *and* $\mathrm{VALUE}_{\mathcal{I}_{\mathrm{dalal}}^{\max}}$ *are in* $FP^{NP[\log n]}$.

Note that in Proposition 5 and Proposition 6 we already showed $FP^{NP[\log n]}$-completeness for $\mathrm{VALUE}_{\mathcal{I}_{\mathrm{dalal}}^{hit}}$ and $\mathrm{VALUE}_{\mathcal{I}_{\mathrm{dalal}}^\Sigma}$, respectively.

4.2 Problems on the second level of the polynomial hierarchy

We now turn to inconsistency measures which involve problems on the second level of the polynomial hierarchy. We first recall a result regarding $\mathcal{I}_{mv}$ from (Xiao and Ma 2012) which is given without proof.

Proposition 8. $\mathrm{EXACT}_{\mathcal{I}_{mv}}$ *is* D_2^p-*complete,* $\mathrm{UPPER}_{\mathcal{I}_{mv}}$ *is* Π_2^p-*complete,* $\mathrm{LOWER}_{\mathcal{I}_{mv}}$ *is* Σ_2^p-*complete, and* $\mathrm{VALUE}_{\mathcal{I}_{mv}}$ *is in* $FP^{\Sigma_2^p[\log n]}$.

We now continue with two novel results regarding the inconsistency measures $\mathcal{I}_{nc}$ and $\mathcal{I}_p$. For both we provide direct proofs of Σ_2^p-completeness for the problem $\mathrm{LOWER}_{\mathcal{I}}$ and utilize again Lemmas 2 and 3 to gain insights on the remaining problems. Intuitively, the increase in terms of complexity of measures in this section, compared to measures discussed in previous Sec. 4.1 is due to the fact that to verify a lower bound we non-deterministically guess a witness that guarantees the bound, but checking the witness itself is a coNP-hard problem. This can be seen in the crucial observation for the first measure we study in this section, $\mathcal{I}_{nc}$, where the lower bounds depends on the size of unsatisfiable subsets of $\mathcal{K}$.

Proposition 9. $\mathrm{LOWER}_{\mathcal{I}_{nc}}$ *is* Σ_2^p-*complete.*

Proof. Observe

$$\mathcal{I}_{nc}(\mathcal{K}) = |\mathcal{K}| - \max\{n \mid \forall \mathcal{K}' \subseteq \mathcal{K} : |\mathcal{K}'| = n \Rightarrow \mathcal{K}' \not\models \bot\}$$
$$= |\mathcal{K}| - \min\{m \mid \exists \mathcal{K}' \subseteq \mathcal{K} : |\mathcal{K}'| = m \wedge \mathcal{K}' \models \bot\} + 1$$

We non-deterministically guess a set $\mathcal{K}' \subseteq \mathcal{K}$ with $|\mathcal{K}'| = k$ and ask an NP-oracle whether $\mathcal{K}'$ is inconsistent. If it is inconsistent then k is an upper bound for

$$\min\{m \mid \exists \mathcal{K}' \subseteq \mathcal{K} : |\mathcal{K}'| = m \wedge \mathcal{K}' \models \bot\}$$

and thus $|\mathcal{K}| - k + 1$ is a lower bound for $\mathcal{I}_{nc}(\mathcal{K})$.

Regarding hardness, we provide a reduction from the Σ_2^p-complete problem of checking whether a given closed quantified Boolean formula $\phi = \exists X \forall Y \psi$ in prenex normal form (PNF) is satisfiable. Let $X = \{x_1, \ldots, x_n\}$. Construct an instance of $\mathrm{LOWER}_{\mathcal{I}_{nc}}$ as follows.

$$\chi_i = p_i \wedge (d \to x_i) \quad \overline{\chi_i} = p_i \wedge (d \to \neg x_i)$$
$$\chi = \bigwedge_{1 \le i \le n} p_i \to (d \wedge \neg \psi)$$

Then let the knowledge base be $\mathcal{K} = \bigcup_{1 \le i \le n}\{\chi_i, \overline{\chi_i}\} \cup \{\chi\}$. Further set bound $x = |\mathcal{K}| - (n+1) + 1 = 2 \cdot (n+1) - n = n + 1$. Knowledge base $\mathcal{K}$ can be constructed in polynomial time. We claim that ϕ is true iff $\mathcal{I}_{nc}(\mathcal{K}) \ge n + 1$. We start with the following observation: it holds that any $\mathcal{K}' \subseteq \mathcal{K}$ is satisfiable if (i) $\chi \notin \mathcal{K}'$, or (ii) $|\mathcal{K}'| < n + 1$. If $\chi \notin \mathcal{K}'$ then there is a model of $\mathcal{K}'$ assigning d to false. If $|\mathcal{K}'| < n + 1$ then $\mathcal{K}'$ is satisfiable, since either $\chi \notin \mathcal{K}'$ or for one i with $1 \le i \le n$ neither χ_i nor $\overline{\chi_i} \in \mathcal{K}'$ (p_i can be assigned to false). It holds that

$\mathcal{I}_{nc}(\mathcal{K}) \ge n + 1$

iff $\exists \mathcal{K}' \subseteq \mathcal{K}$ s.t. $\mathcal{K}' \models \bot$ and $|\mathcal{K}'| = n + 1$

iff $\exists \mathcal{K}' \subseteq \mathcal{K}$ s.t.

$\mathcal{K}' \models \bot, \chi \in \mathcal{K}'$, and $|\mathcal{K}' \cap \{\chi_i, \overline{\chi_i}\}| = 1 \; \forall i \; 1 \le i \le n$

iff $\exists \omega$ defined on X s.t. $\neg\psi[\omega] \models \bot$

iff $\exists \omega$ defined on X s.t. $\top \models \psi[\omega]$

iff ϕ is true. $\qquad\square$

The next inconsistency measure computes the union of all MIs of the knowledge base. Guessing non-deterministically a subset of propositions and verifying for each whether they are contained in one MI (which is in Σ_2^p) establishes the following membership result.

Proposition 10. $\mathrm{LOWER}_{\mathcal{I}_p}$ *is* Σ_2^p-*complete.*

Proof. Let knowledge base $\mathcal{K}$ together with x be an arbitrary instance of $\mathrm{LOWER}_{\mathcal{I}_p}$. Eiter and Gottlob (1992) have shown that checking whether some $\phi \in \mathcal{K}$ is contained in any MI is in Σ_2^p. For membership of $\mathrm{LOWER}_{\mathcal{I}_p}$ in Σ_2^p, we guess $\mathcal{K}' \subseteq \mathcal{K}$ with $|\mathcal{K}'| = \lfloor x \rfloor$ and use the non-deterministic algorithm utilizing a coNP oracle given by (Eiter and Gottlob 1992) to verify that each $\phi \in \mathcal{K}'$ is contained in an MI of $\mathcal{K}$.

For hardness, we utilize a similar, but simpler, reduction as in Proposition 9. Let $\phi = \exists X \forall Y \psi$ be a closed QBF in PNF with $X = \{x_1, \ldots, x_n\}$. Construct $\mathcal{K} = \bigcup_{1 \le i \le n}\{x_i, \neg x_i\} \cup \{\neg\psi\}$. We now claim that ϕ is true iff $\mathcal{I}_p(\mathcal{K}) = 2 \cdot n + 1 = |\mathcal{K}|$. That is, $2 \cdot n + 1$ is a lower bound iff ϕ is true. It is immediate that for any ϕ we have $\mathcal{I}_p(\mathcal{K}) \ge 2 \cdot n$, since for any i with $1 \le i \le n$ it holds that

$\{x_i, \neg x_i\}$ is an MI. It holds that

$$\mathcal{I}(\mathcal{K}) \geq 2 \cdot n + 1$$

$$\text{iff } \exists M \in \mathsf{MI}(\mathcal{K}) \text{ s.t. } \neg\psi \in M$$

$$\text{iff } \exists \mathcal{K}' \subseteq \mathcal{K} \text{ s.t.}$$

$$\mathcal{K}' \subseteq (\mathcal{K} \setminus \{\neg\psi\}) \not\models \bot \text{ and } \mathcal{K}' \cup \{\neg\psi\} \models \bot$$

$$\text{iff } \exists \omega \text{ defined on } X \text{ s.t. } \neg\psi[\omega] \models \bot$$

$$\text{iff } \phi \text{ is true.} \qquad \square$$

Utilizing Lemma 3, we directly obtain the following statements regarding $\text{LOWER}_\mathcal{I}$ and $\text{EXACT}_\mathcal{I}$ and the inconsistency measures from above.

Corollary 3. *It holds that*

- $\text{UPPER}_{\mathcal{I}_{nc}}$ *and* $\text{UPPER}_{\mathcal{I}_p}$ *are* Π_2^p-*complete; and*
- $\text{EXACT}_{\mathcal{I}_{nc}}$ *and* $\text{EXACT}_{\mathcal{I}_p}$ *are in* D_2^p.

Regarding the functional problem $\text{VALUE}_\mathcal{I}$ and by combining lemmas 1 and 2 we obtain the following.

Corollary 4. $\text{VALUE}_{\mathcal{I}_{nc}}$ *and* $\text{VALUE}_{\mathcal{I}_p}$ *are in* $\mathsf{FP}^{\Sigma_2^p[\log n]}$.

4.3 Problems beyond the second level of the polynomial hierarchy

In this section we study complexity of measures $\mathcal{I}_{\mathsf{MI}}$, $\mathcal{I}_{mc}$, and $\mathcal{I}_{\mathsf{MI^c}}$. The main results in this section are that measures $\mathcal{I}_{\mathsf{MI}}$ and $\mathcal{I}_{mc}$ contain (sub)problems whose counting complexity is higher than for propositional model counting. In particular, we show #·coNP-completeness of the problems of counting all MIs and also of the problem of counting all MCs. We prove #·coNP hardness via subtractive reductions (Durand, Hermann, and Kolaitis 2005) (see Sec. 2 for the definition). This, presumably drastic, jump in complexity compared to other measures considered in this paper can be intuitively explained by the fact that both the problems of verifying if a given subset is an MI or if it is an MC are D_1^p-complete and, additionally, these measures admit exponentially many possible values for a knowledge base $\mathcal{K}$ wrt. the size of $\mathcal{K}$.

Proposition 11. $\text{VALUE}_{\mathcal{I}_{\mathsf{MI}}}$ *is* #·*coNP-complete via subtractive reductions.*

Proof. Regarding membership, we use the fact that #·coNP = #·Δ_2^p (Hemaspaandra and Vollmer 1995, Theorem 1.5) and further that it holds that verifying whether a given subset is an MI is a D_1^p-complete problem (Papadimitriou and Wolfe 1988). This means $\text{VALUE}_{\mathcal{I}_{\mathsf{MI}}}$ is in #·coNP, since MI is the witness function producing finite subsets for a given knowledge base, all such subsets are polynomially bounded in size of the given knowledge base, and checking whether such a set is indeed an MI is in Δ_2^p.

For hardness, let $\chi(X) = \forall Y \phi(X, Y)$ with $X = \{x_1, \ldots, x_n\}$ be an arbitrary instance of the #·coNP-complete problem #Π_1SAT. In this problem we have to compute the number of assignments on X that satisfy χ, which contains also variables over set Y. We define

$$\phi_i = p_i \wedge (\bigwedge_{j \neq i} p_j \rightarrow x_i), \quad \overline{\phi_i} = p_i \wedge (\bigwedge_{j \neq i} p_j \rightarrow \neg x_i),$$

$$\psi = \bigwedge_{1 \leq i \leq n} p_i \rightarrow \neg\phi(X, Y).$$

We construct the following knowledge bases. Let $P_1 = \bigcup_{1 \leq i \leq n}\{\phi_i\}$ and $\overline{P_1} = \bigcup_{1 \leq i \leq n}\{\overline{\phi_i}\}$. Finally, let $P_2 = \{\psi\} \cup P_1 \cup \overline{P_1}$. We now claim that the number of truth assignments over X that satisfy χ is $|\mathsf{MI}(P_2)| - |\mathsf{MI}(P_1 \cup \overline{P_1})|$, and further that it holds that $\mathsf{MI}(P_1 \cup \overline{P_1}) \subseteq \mathsf{MI}(P_2)$, i.e. that this a subtractive reduction. The latter claim follows from monotonicity of MI, i.e. if $\mathcal{K}_1 \subseteq \mathcal{K}_2$ then $\mathsf{MI}(\mathcal{K}_1) \subseteq \mathsf{MI}(\mathcal{K}_2)$.

Let $M \in \mathsf{MI}(P_2)$. It holds that for $1 \leq i \leq n$ that $\phi_i \in M$ or $\overline{\phi_i} \in M$. Suppose the contrary, i.e. there exists an i s.t. neither ϕ_i nor $\overline{\phi_i}$ is in M. Then a truth assignment assigning all p_j with $j \neq i$ to true and p_i to false satisfies all formulas in M. This is a contradiction to $M \in \mathsf{MI}(P_2)$.

Now assume that $M \in \mathsf{MI}(P_2)$ s.t. $\exists i$ with both $\phi_i \in M$ and $\overline{\phi_i} \in M$. Then $\psi \notin M$, since by observations above we have for each $1 \leq j \leq n$ with $j \neq i$ that $\phi_j \in M$ or $\overline{\phi_j} \in M$ and both for formulas for i. These formulas together are inconsistent, and thus adding a further formula (such as ψ) would not be minimal anymore. This means if $M \in \mathsf{MI}(P_2)$ s.t. $\exists i$ with both $\phi_i \in M$ and $\overline{\phi_i} \in M$, then $M \in \mathsf{MI}(P_1 \cup \overline{P_1})$. Further, if $M \in \mathsf{MI}(P_2)$ and $\nexists i$ with both $\phi_i \in M$ and $\overline{\phi_i} \in M$, then $\psi \in M$ (if one of ϕ_i or $\overline{\phi_i}$ is missing and also ψ is not present, then the set of formulas is satisfiable).

Let $\mathcal{K}^* \subseteq 2^\mathcal{K}$ be the set of subsets of $\mathcal{K}$ s.t. each $\mathcal{K}' \in \mathcal{K}^*$ contains for each i with $1 \leq i \leq n$ exactly one of ϕ_i or $\overline{\phi_i}$ but not both, and in addition $\mathcal{K}'$ contains ψ. We define a bijection f from $\mathcal{K}' \in \mathcal{K}^*$ to an interpretation over X by

$$f(\mathcal{K}')(x_i) = \begin{cases} \text{true if } \phi_i \in \mathcal{K}' \\ \text{false if } \overline{\phi_i} \in \mathcal{K}'. \end{cases}$$

For $f(M) = \omega_M$ and by the observations above it holds that

$$M \in (\mathsf{MI}(P_2) \setminus \mathsf{MI}(P_1 \cup \overline{P_1}))$$

$$\text{iff } M \in \mathsf{MI}(P_2), \psi \in M, \text{ and}$$

$$|M \cap \{\phi_i, \overline{\phi_i}\}| = 1 \, \forall i \text{ with } 1 \leq i \leq n$$

$$\text{iff } \top \models \phi(X, Y)[\omega_M]$$

$$\text{iff } w_M \text{ satisfies } \forall Y \phi(X, Y).$$

Thus, there is a bijection between $\mathsf{MI}(P_2 \setminus (P_1 \cup \overline{P_1}))$ and the set of satisfying assignments defined on X of $\chi(X)$. $\square$

We move on to the complexity of $\mathcal{I}_{mc}$. This measure has two components, which we analyze separately. First, we show the complexity of counting all maximal consistent subsets of a knowledge base. For this, we introduce an auxiliary problem which counts the number of subset-maximal models of a propositional formula wrt. to the propositions assigned to true. For that, we define the ordering $<_t$ over interpretations by $\omega <_t \omega'$ iff $\{p \mid \omega(p) = \text{true}\} \subset \{p \mid \omega'(p) = \text{true}\}$.

#MaxModels **Input**: formula ϕ in CNF
Output: $|\{\omega \mid \nexists\omega' \text{ s.t. } \omega <_t \omega'\}|$

Durand, Hermann, and Kolaitis (2005) have shown (Theorem 5.1) that the problem #CIRCUMSCRIPTION—which is basically the dual of the problem #MaxModels as it counts the subset-minimal models—is #·coNP-complete

(via subtractive reductions). We provide a corollary showing that also counting the number of subset-maximal models is #·coNP-complete (proof is omitted due to space restrictions).

Corollary 5. #MaxModels *is #·coNP-complete via subtractive reductions.*

We are now prepared to show that counting the number of maximal consistent subsets has the same complexity as counting the number of minimal inconsistent subsets.

Proposition 12. *The problem of counting all maximal consistent subsets of a given knowledge base is #·coNP-complete via subtractive reductions.*

Proof. Membership follows from the fact that verifying whether a subset of a knowledge base is a maximal consistent subset is in D_1^p. We show hardness by the following reduction from #MaxModels (#·coNP-completeness proved in Corollary 5). Let ϕ be an instance of #MaxModels with $\{x_1, \ldots, x_n\}$ the vocabulary of ϕ. Construct $\mathcal{K} = \{(x_i \wedge \phi) \mid 1 \leq i \leq n\}$. We claim that $|\mathsf{MC}(\mathcal{K})|$ is equal to the number of subset maximal models.

$$M \in \mathsf{MC}(\mathcal{K})$$

iff $M \not\models \bot$ and $\nexists M'$ s.t. $M \subset M'$ and $M' \not\models \bot$

iff $\phi \wedge \bigwedge_{(x_i \wedge \phi) \in M} x_i \not\models \bot$ and

$\phi \wedge (\bigwedge_{(x_i \wedge \phi) \in M} x_i) \wedge \bigvee_{(x_i \wedge \phi) \in \mathcal{K} \setminus M} x_i \models \bot$

iff $\omega_M \models \phi$ with $\omega(x_i) = \mathsf{true}$ iff $(x_i \wedge \phi) \in M$ and

$\omega' \not\models \phi \; \forall \omega'$ with $w_M <_t w'$ ☐

The other component of $\mathcal{I}_{mc}$, the number of self-conflicting formulas, is arguably easier to compute, the functional problem is $\mathsf{FP}^{\mathsf{NP}[\log n]}$-complete.

Proposition 13. *The problem of counting unsatisfiable formulas in a given knowledge base is $\mathsf{FP}^{\mathsf{NP}[\log n]}$-complete.*

Proof. Membership follows from posing logarithmically many queries asking whether in a subset of size k every formula is satisfiable (guessing the set together with interpretations for each). Hardness follows from a reduction from MaxSAT Size. Let $\phi = c_1 \wedge \cdots \wedge c_n$ be an arbitrary instance of MaxSAT Size. Construct $\mathcal{K} = \{(\mathsf{EXACT}(i, Y) \wedge \phi' \mid 1 \leq i \leq n\}$ with $\phi' = (c_1 \vee y_1) \wedge \cdots \wedge (c_n \vee y_n)$, $Y = \{y_1, \ldots, y_n\}$ fresh variables, and $\mathsf{EXACT}(i, Y)$ a formula that evaluates to true under an assignment iff that assignment assigns exactly i many variables of Y to true. The formula $\mathsf{EXACT}(i, Y)$ can be constructed in polynomial time wrt. the size of Y (Roussel and Manquinho 2009, Section 22.2.3.). It follows immediately from construction that $(\mathsf{EXACT}(i, Y) \wedge \phi')$ is satisfiable iff i many clauses of ϕ can be satisfied simultaneously. ☐

Although we have not given tight results for $\mathcal{I}_{\mathsf{MI^c}}$, we think this measure is not easier than $\mathcal{I}_{\mathsf{MI}}$, as the former is defined by the cardinality of each MI instead of only the number of MIs. Nevertheless, we show that the values of $\mathcal{I}_{\mathsf{MI}}$, $\mathcal{I}_{mc}$, and $\mathcal{I}_{\mathsf{MI^c}}$ can be computed in polynomial space.

Proposition 14. *For $\mathcal{K} \in \mathbb{K}$ the values $\mathcal{I}_{\mathsf{MI}}(\mathcal{K})$, $\mathcal{I}_{\mathsf{MI^c}}(\mathcal{K})$, and $\mathcal{I}_{mc}(\mathcal{K})$ can be computed in polynomial space.*

Proof. For all problems we enumerate all subsets $\mathcal{K}' \subseteq \mathcal{K}$ and verify depending on the problem whether $\mathcal{K}' \in \mathsf{MI}(\mathcal{K})$, $\mathcal{K}' \in \mathsf{MC}(\mathcal{K})$, or $\mathcal{K}' \in \mathsf{SC}(\mathcal{K})$ via enumeration of all subsets, resp. supersets, and interpretations. ☐

From the above proposition it follows that the corresponding decision problems are in PSPACE. Furthermore, it is unlikely that $\mathrm{UPPER}_{\mathcal{I}}$ and $\mathrm{LOWER}_{\mathcal{I}}$ for $\mathcal{I} \in \{\mathcal{I}_{\mathsf{MI}}, \mathcal{I}_{mc}\}$ are contained in one finite level of the polynomial hierarchy e.g. Σ_i^p for $i \geq 0$, as this would imply that the number of MIs (or the number of models of a propositional formula) can be computed via binary search with a deterministic polynomial-time algorithm that has access to a Σ_i^p oracle.

5 Discussion and Summary

The contributions of this paper provide new insights into the challenge of measuring inconsistency and allow for a broader comparison of existing measures in terms of their complexity. One of the key insights in this paper is the partitioning of inconsistency measures in three classes categories of complexity, i.e., measures on the first level of the polynomial hierarchy, measures on the second level, and those beyond the second level. This also shows that inconsistency measurement is sometimes not computationally harder than solving the classical satisfiability problem SAT (for the measures residing on the first level of the polynomial hierarchy, see Section 4.1). However, our results also show that inconsistency measurement can be computationally demanding, as shown in Sections 4.2 and 4.3.

It is also interesting to note that the computational complexity of inconsistency measures does not necessarily correlate with their "logical" complexity. For example, the measure $\mathcal{I}_{\mathsf{MI}}$ was one of the first inconsistency measures presented and follows a simple idea to measure inconsistency, i.e., simply taking the number of minimal inconsistent subsets. Although we did not provide completeness results for the corresponding decisions problems, both the #·coNP-completeness result of the function problem and also the results of (Papadimitriou and Wolfe 1988) related to identifying minimal inconsistent sets, show that $\mathcal{I}_{\mathsf{MI}}$ belongs to the computationally most complex inconsistency measures. Compare this to e.g. the measure $\mathcal{I}_{\mathrm{dalal}}^{\Sigma}$ which features a quite complex definition, involving distances between propositional interpretations, but belongs to the easiest class of measures.

This paper is a first step towards a complete picture of the computational complexity landscape of inconsistency measurement. Current work is about complementing the results of this paper by providing the missing completeness results and investigating the computational complexity of further approaches such as the measures presented in (Mu et al. 2011; Jabbour et al. 2015; Ammoura et al. 2015).

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