An Analysis of the Equational Properties of the Well-Founded Fixed Point

Arnaud Carayol
Université Paris Est and CNRS, France
arnaud.carayol@univ-mlv.fr

Zoltan Ésik*
University of Szeged, Hungary
ze@inf.u-szeged.hu

Abstract
We study the logical properties of the (parametric) well-founded fixed point operation. We show that the operation satisfies several, but not all of the equational properties of fixed point operations described by the axioms of iteration theories.

Introduction
Fixed points and fixed point operations have been used in just about all areas of computer science. There has been a tremendous amount of work on the existence, construction and logic of fixed point operations. It has been shown that most fixed point operations, including the least (or greatest) fixed point operation on monotonic functions over complete lattices, satisfy the same equational properties. These equational properties are captured by the notion of iteration theories, or iteration categories, cf. (Bloom and Ésik 1993) or the recent survey (Ésik 2015a).

In this paper, we study the equational properties of the well-founded fixed point operation as defined in (Denecker, Marek, and Truszczyński 2000; 2004; Vennekens, Gilis, and Denecker 2006) with the aim of relating well-founded fixed points to iteration categories. We extend the well-founded fixed point operation to a parametric fixed point (or dagger) operation (Bloom and Ésik 1993; 1996) over the cartesian category of approximation function pairs between complete bilattices and offer an initial analysis of its equational properties. Our main results show that several identities of iteration theories hold for the well-founded fixed point operation, but some others fail. For proofs, we refer to (Carayol and Ésik 2015).

Preliminaries
Recall that a complete lattice (Davey and Priestley 1990) is a partially ordered set $L$, ordered by $\leq$, such that each $X \subseteq L$ has a supremum $\bigvee X$ and hence also an infimum $\bigwedge X$. In particular, each complete lattice has a least and a greatest element, respectively denoted either $\bot$ and $\top$, or 0 and 1.

We say that a function $f : L \to L$ over a complete lattice $L$ is monotonic (anti-monotonic, resp.) if for all $x, y \in L$, if $x \leq y$ then $f(x) \leq f(y)$ ($f(x) \geq f(y)$, resp.).

A complete bilattice$^1$ (Fitting 2002) $(B, \leq_p, \leq_t)$ is equipped with two partial orders, $\leq_p$ and $\leq_t$, both giving rise to a complete lattice. We will denote the $\leq_p$-least and greatest elements of a complete bilattice by $\bot$ and $\top$, and the $\leq_t$-least and greatest elements by 0 and 1, respectively.

An example of a complete bilattice is $\text{FOUR}_1$ (Fitting 2002) $(B, \leq_p, \leq_t)$, which has 4 elements, $\bot, \top, 0, 1$. The nontrivial order relations are given by $\bot \leq_p 0, 1 \leq_p \top$ and $0 \leq_t \bot, \top \leq_t 1$.

There are two closely related constructions of a complete bilattice from a complete lattice. Here we recall one of them. Suppose that $L = (L, \leq)$ is a complete lattice with extremal (i.e., least and greatest) elements 0 and 1. Then define the partial orders $\leq_p$ and $\leq_t$ on $L \times L$ as follows:

$$(x, x') \leq_p (y, y') \iff x \leq y \land x' \geq y'$$

$$(x, x') \leq_t (y, y') \iff x \leq y \land x' \leq y'.$$

Then $L \times L$ is a complete bilattice with $\leq_p$-extremal elements $\bot = (0, 1)$ and $\top = (1, 0)$, and $\leq_t$-extremal elements $0 = (0, 0)$ and $1 = (1, 1)$. Note that when $L$ is the 2-element lattice $\mathbb{2} = \{0 \leq 1\}$, then $L \times L$ is isomorphic to $\text{FOUR}_1$.

In this paper, we will mainly be concerned with the ordering $\leq_p$.

In any category, we usually denote the composition of morphisms $f : A \to B$ and $g : B \to C$ by $g \circ f$ and the identity morphisms by $\text{id}_A$. We let SET and CL respectively denote the category of sets and functions on the category of complete lattices and monotonic functions. Both categories are cartesian categories with the usual direct product $A_1 \times \cdots \times A_n$ (equipped with the pointwise order in CL) serving as categorical product. The categorical projection morphisms $\pi^i : A_1 \times \cdots \times A_n \to A_i, i \in [n] = \{1, \ldots, n\}$, are the usual projection functions.

Products give rise to a tupling operation. Suppose that $f_i : C \to A_i, i \in [n]$ in a cartesian category. Then there is a unique $f : C \to A_1 \times \cdots \times A_n$ with $\pi^i \circ f = f_i$ for all $i \in [n]$. We denote this unique morphism $f$ by $(f_1, \ldots, f_n)$ and call it (the target) tupling of the $f_i$ (or pairing, when $n = 2$). And when $f : C \to A$ and $g : D \to B$,

$^1$Sometimes bilattices are equipped with a negation operation and the bilattices as defined here are called pre-bilattices.
then we define \( f \times g \) as the unique morphism \( h : C \times D \to A \times B \) with \( \pi_1 \times B \circ h = f \circ \pi_1 \times D \) and \( \pi_2 \times B \circ h = g \circ \pi_2 \times D \).

When \( m, n \geq 0 \), \( \rho \) is a function \([m] \to [n]\) and \( A_1, \ldots, A_n \) is a sequence of objects in a cartesian category, we associate with \( \rho \) and \( (A_1, \ldots, A_n) \) the morphism
\[
\rho A_1 \times \cdots \times A_n = \langle \pi_{\rho(1)} A_1 \times \cdots \times A_n, \ldots, \pi_{\rho(m)} A_1 \times \cdots \times A_n \rangle
\]
from \( A_1 \times \cdots \times A_n \) to \( A_{\rho(1)} \times \cdots \times A_{\rho(m)} \). (Note that in SET and CL, \( \rho A_1 \times \cdots \times A_n \) maps \((x_1, \ldots, x_n) \in A_1 \times \cdots \times A_n\) to \((x_{\rho(1)}, \ldots, x_{\rho(m)}) \in A_{\rho(1)} \times \cdots \times A_{\rho(m)}\).) With a slight abuse of notation, we usually let \( \rho \) denote this morphism as well. Morphisms of this form are sometimes called base morphisms.

The category CL

The objects of CL are complete lattices. Suppose that \( A, B \) are complete lattices. A morphism from \( A \to B \) in CL, denoted \( f : A \to B \), is a \( \leq \)-monotone function \( f : A \times A \to B \times B \), where \( A \times A \) and \( B \times B \) are the complete bilattices determined by \( A \) and \( B \). Thus, \( f = \langle f_1, f_2 \rangle \) such that \( f_1 : A \times A \to B \) is monotone in its first argument and anti-monotone in its second argument, and \( f_2 : A \times A \to B \) is anti-monotone in its first argument and monotone in its second argument. (Such functions \( f \) are called approximations in (Vennekens, Gilis, and De necker 2006).) Composition is ordinary function composition and for each complete lattice \( A \), the identity morphism \( \text{id}_A : A \to A \) is the identity function \( \text{id}_A = \text{id}\times\text{id} : A \times A \to A \times A \).

The category CL has finite products. (Actually it has all products). Indeed, a terminal object \( T \) of CL is any 1-element lattice. Suppose that \( A_1, \ldots, A_n \) are complete lattices. Then consider the direct product \( A_1 \times \cdots \times A_n \) as an object of CL together with the following morphisms
\[
\pi_1 A_1 \times \cdots \times A_n = \langle x_1, \ldots, x_n \rangle \to A_i, i \in [n].
\]
For each \( i \), \( \pi_1 A_1 \times \cdots \times A_n \) is the function \( A_1 \times \cdots \times A_n \to A_i \times A_i \) defined by \( \pi_1 A_1 \times \cdots \times A_n (x_1, \ldots, x_n) = (x_i, x_i) \). Thus, in SET, \( \pi_1 A_1 \times \cdots \times A_n \) can be written as
\[
\langle \pi_1 A_1 \times \cdots \times A_n, \pi_1 A_1 \times \cdots \times A_n \rangle = \pi_1 A_1 \times \cdots \times A_n.
\]

It is easy to see that the morphisms \( \pi_i A_1 \times \cdots \times A_n, i \in [n] \), determine a product diagram in CL. To this end, let \( f^i = \langle f_1^i, f_2^i \rangle : C \to A_i \) in CL, for all \( i \in [n] \), so that each \( f^i \) is a \( \leq \)-monotone function \( C \times C \to A_i \times A_i \). Then \( h = \langle h_1, h_2 \rangle \) be the function \( C \times C \to A_1 \times \cdots \times A_n \times A_1 \times \cdots \times A_n \), where \( h_1 = \langle f_1^1, \ldots, f_1^n \rangle \) and \( h_2 = \langle f_2^1, \ldots, f_2^n \rangle \). Thus, \( h_1 \) and \( h_2 \) are functions \( C \times C \to A_1 \times \cdots \times A_n \). The tupling of any sequence of morphisms \( f^i = \langle f_1^i, f_2^i \rangle : C \to A_i \) in CL is \( h = \langle h_1, h_2 \rangle \). We will denote it by \( \langle f^1, \ldots, f^n \rangle : C \to A_1 \times \cdots \times A_n \).

**Proposition 1** CL is a cartesian category in which the product of any objects \( A_1, \ldots, A_n \) agrees with their product in CL.

For further use, we note the following. Suppose that \( \rho : [m] \to [n] \) and \( A_1, \ldots, A_n \) are complete lattices. Then the associated morphism \( \rho A_1 \times \cdots \times A_n : A_1 \times \cdots \times A_n \to A_{\rho(1)} \times \cdots \times A_{\rho(m)} \) in the category CL is the function
\[
A_1 \times \cdots \times A_n \times A_1 \times \cdots \times A_n \to A_{\rho(1)} \times \cdots \times A_{\rho(m)} \times A_{\rho(1)} \times \cdots \times A_{\rho(m)}
\]
given by
\[
(x_1, \ldots, x_n, x'_1, \ldots, x'_n) \mapsto (x_{\rho(1)}, \ldots, x_{\rho(m)}, x'_{\rho(1)}, \ldots, x'_{\rho(m)}).
\]

Thus, \( \rho A_1 \times \cdots \times A_n = \rho A_1 \times \cdots \times \rho A_{\rho(m)} \), where \( \rho A_1 \times \cdots \times A_n \) is the morphism associated with \( \rho \) and \( A_1, \ldots, A_n \) in SET (or CL). This is in accordance with \( \text{id}_A = \text{id}_A \times \text{id}_A \).

Suppose that \( f : C \to A \) and \( g : D \to B \) in CL, so that \( f \) is a function \( C \times C \to A \times A \) and \( g \) is a function \( D \times D \to B \times B \). Then \( f \circ g : C \times D \to A \times B \) in the category CL is the function \( (\text{id}_A \times \pi_2, \pi_1 \times \text{id}_A) \circ h \circ (\pi_2 \times C, \pi_1 \times D) \) from \( C \times D \to A \times B \times A \times B \), where \( h \) is a function \( C \times C \times D \to A \times B \times A \times B \), where \( h \) is a function \( C \times C \times D \to A \times B \times A \times B \).

Some subcategories

Motivated by (Denecker, Marek, and Truszczyński 2000; Vennekens, Gilis, and Denecker 2006), we define several subcategories of CL. Suppose that \( A, B \) are complete lattices. Following (Denecker, Marek, and Truszczyński 2000), we call an ordered pair \( (x, x') \in A \times A \) consistent if \( x \leq x' \). Moreover, we call \( f : A \to B \) in CL consistent if it maps consistent pairs to consistent pairs. The consistent morphisms in CL determine a cartesian subcategory of CL with the same product diagrams. Let CCL denote this subcategory. We define two subcategories of CCL. The first one, ACCL is the subcategory determined by those morphisms \( f = \langle f_1, f_2 \rangle : A \to B \) in CL such that \( f_1 \langle x, x \rangle \leq f_2 \langle x, x \rangle \) for all \( x \in A \). The second, AECCL is the subcategory determined by those \( f : A \to B \) with \( f_1 \langle x, x \rangle = f_2 \langle x, x \rangle \). These are again cartesian subcategories with the same product diagrams.

As noted in (Denecker, Marek, and Truszczyński 2000), most applications of approximation fixed point theory use symmetric functions. We introduce the subcategory of CL having complete lattices as object but only symmetric \( \leq \)-preserving functions as morphisms. Suppose that \( f : A \to B \) in CL, say \( f = \langle f_1, f_2 \rangle \). We call \( f \) symmetric if \( f_2 \langle x, x' \rangle = f_1 \langle x, x' \rangle \), i.e., when
\[
f_2 = f_1 \circ \langle \pi_2, \pi_1 \rangle : A \times A \to B.
\]
The symmetric morphisms determine a subcategory of CL, denoted SCL. In fact, SCL is a subcategory of ACCL. Moreover, it is again a cartesian subcategory with the same products.
Fixed points

Suppose that \( A \) and \( B \) are complete lattices, ordered by \( \leq \), and let \( f : A \times B \to A \) be a monotonic function. The least fixed point operation on \( C \) maps \( f \) to the monotonic function \( f^\dagger : B \to A \) such that for all \( y \in B \), \( f^\dagger(y) \) is the least solution of the fixed point equation \( x = f(x,y) \). The existence of \( f^\dagger(y) \) is guaranteed by the Knaster-Tarski fixed point theorem. It is also known that \( f^\dagger(y) \) is the least \( z \in A \) such that \( f(z,y) \leq z \).

In this section, we recall from (Denecke, Marek, and Truszczyński 2000) the construction of stable and well-founded fixed points. Suppose that \( f = \langle f_1, f_2 \rangle : A \to A \) in \( CL \), so that \( f \) is a \( \leq_p \)-monotonic function \( A \times A \to A \times A \). Then \( f_1 : A \times A \to A \) is monotonic in its first argument and anti-monotonic in its second argument, and \( f_2 : A \times A \to A \) is monotonic in its second argument and anti-monotonic in its first argument. Define the functions \( s_1, s_2 : A \to A \) by

\[ s_1(x') = \mu x. f_1(x, x') \quad \text{and} \quad s_2(x) = \mu x'. f_2(x, x'), \]

and let \( S(f) : A \times A \to A \times A \) be the function \( S(f)(x,x') = (s_1(x'), s_2(x)) \). Since \( s_1 \) and \( s_2 \) are anti-monotonic, \( S(f) \) is a morphism \( A \to A \) in \( CL \). We call \( S(f) \) the stable function for \( f \). It is known that every fixed point of \( S(f) \) is a fixed point of \( f \), called a stable fixed point of \( f \). Since \( S(f) \) is \( \leq_p \)-monotonic, there is a \( \leq_p \)-least stable fixed point \( f^\dagger \), called the well-founded fixed point of \( f \).

The above construction can slightly be extended. When \( f : A \times B \to A \) is in \( CL \) and \( (y', y') \in B \times B \), then let \( g : A \to A \) be given by \( g(y, x') = f(x, y, x') \). Then we define \( \rho(f, y, y') = g^\dagger \).

Proposition 2 Suppose that \( f : A \times B \to A \) is in \( CL \). Then \( f^\dagger : B \to A \) is also in \( CL \). However, neither of the subcategories \( CCL, ACL, ECCL, and SCL \) is closed under \( \dagger \).

Some valid identities

Iteration categories capture the equational properties of several fixed point operations including the least fixed point operation over \( C \). Axiomatizations of iteration categories can be conveniently divided into two parts, axioms for Conway categories and the commutative (Bloom and Ésik 1993) or group identities (Ésik 1999b), or the generalized power identities of (Ésik 1999a). Known axiomatizations of Conway categories include the group consisting of the parameter (1), composition (2) and double dagger (7) identities, and the group consisting of the fixed point (2), parameter (1), pairing (6) and permutation (3) identities. In this section we establish several of the above mentioned identities for the parametrized well-founded fixed point operation over \( C \). In the next section we will show that several others fail.

Proposition 3 The following parameter (1), fixed point (2) and permutation (3) identities hold:

\[ (f \circ (\text{id}_A \times g))^\dagger = f^\dagger \circ g, \quad (f \circ \text{id}_B)^\dagger = f^\dagger, \]

for all \( f : A \times B \to A \) and \( g : C \to B \).

\[ f \circ (f^\dagger \circ \text{id}_B) = f^\dagger, \]

\[ (p \circ f \circ (p^{-1} \circ \text{id}_B))^\dagger = p \circ f^\dagger, \]

for all \( f : A_1 \times \cdots \times A_n \times B \to A_1 \times \cdots \times A_n \) and permutation \( p : [n] \to [n] \).

Also, a special case of the pairing identity (6) holds:

Proposition 4 The following identity holds:

\[ \langle f, g \circ (\pi_2^{A \times C} \circ \text{id}_C) \rangle^\dagger = \langle f^\dagger \circ (h^\dagger \circ \text{id}_C), g^\dagger \rangle, \]

where \( f : A \times B \times C \to A \) and \( g : B \times C \to B \).

The identity (4) has already been established in Theorem 3.11 of (Vennekens, Gilis, and Denecke 2006), see also the Splitting Set Theorem of (Lisitzin and Turner 1994).

Some identities that fail

Proposition 6 The composition identity

\[ (f \circ g, \pi_2^{A \times C})^\dagger = (f \circ (g \circ (f, \pi_2^{B \times C})))^\dagger \circ \text{id}_C, \]

fails in \( CL \), even in the following simple case: \( f \circ (f \circ f)^\dagger = (f \circ f)^\dagger \), where \( f : A \to A \).

Proposition 7 The squaring identity \( f \circ f \) fails, where \( f : A \to A \).

Since the fixed point, parameter and permutation identities hold but the composition identity fails, the pairing identity (6) also must fail, see (Bloom and Ésik 1993).

Proposition 8 The pairing identity

\[ \langle f, g \rangle^\dagger = \langle f^\dagger \circ (h^\dagger \circ \text{id}_C), h^\dagger \rangle, \]

where \( h = g \circ (f^\dagger \circ \text{id}_B \times C) \) fails, where \( f : A \times B \times C \to A \) and \( g : A \times B \times C \to B \).

Proposition 9 The double dagger identity

\[ f^\ddagger = (f \circ (\text{id}_A \circ A, \text{id}_A \circ B)), \]

fails in \( CL \), even in the particular case when \( B = T \) (terminal object).

Some applications

The established identities can be seen as abstract versions of transformations over logic programs that preserve the well-founded fixed point operations. For one example, consider the simple propositional logic program

\[ p := q, r, q := p, r := p, q. \]

Identifying \( p, q, r \), we obtain \( p := p, q, r \). By the weak functorial implication,
the two programs are equivalent in the sense that each component of the well-founded semantics of the first program agrees with the well-founded semantics of the second.

By formulating transformations as identities, one can use standard (many-sorted) equational logic to derive other identities that in turn give rise to new transformations. For example, the following identity is an equational consequence of those established in the paper:

\[ \langle f \circ (\pi_B^{A \times B} \circ B) \rangle \triangleq \langle f \circ (\pi_A^{B \times C}) \rangle, g \circ (\pi_B^{A \times B} \circ C) \triangleq \] where \( f : A \times B \times C \rightarrow A \) and \( g : B \times C \rightarrow B. \)

This identity is an abstract version of the fold/unfold transformation (Tamaki and Sato 1984; Seki 1993). For example, it yields that the following propositional logic transformation (Tamaki and Sato 1984; Seki 1993). For example, the following identity is an equational consequence of those established in the paper:

\[ \langle f \circ (\pi_B^{A \times B} \circ B) \rangle \triangleq \langle f \circ (\pi_A^{B \times C}) \rangle, g \circ (\pi_B^{A \times B} \circ C) \triangleq \]

where \( f : A \times B \times C \rightarrow A \) and \( g : B \times C \rightarrow B. \)

This identity is an abstract version of the fold/unfold transformation (Tamaki and Sato 1984; Seki 1993). For example, it yields that the following propositional logic programs \( p := g, r; r := s, t \) and \( p := g, s, t; r := s, t \) are equivalent for the well-founded semantics. On the other hand, the following identity, which is a generalization of the above folding/unfolding identity, fails:

\[ \langle f \circ (\pi_A^{A \times B} \circ B) \rangle, g \rangle \triangleq \langle f, g \rangle \]

where \( f : A \times B \rightarrow A \) and \( g : A \times B \rightarrow B. \)

And this again follows by standard equational reasoning using our positive and negative results.

**Conclusion**

We extended the well-founded fixed point operation of (Denecker, Marek, and Truszczynski 2000; Vennekens, Gilis, and Denecker 2006) to a parametric operation and studied its equational properties. We found that several of the identities of iteration theories hold for the parametric well-founded fixed point operation, but some others fail. By showing that some identities of iteration theories do not hold, we tried to have a better understanding why logic programs with the well-founded semantics cannot be manipulated using standard fixed point methods. And by showing that some other identities hold, we tried to understand to what extent the standard techniques can be used for manipulating logic programs. Two interesting questions arise for further investigation. The first concerns the algorithmic description of the valid identities of the well-founded fixed point operation. The second concerns the axiomatic description of the valid identities. These questions are also relevant in connection with modular logic programming, cf. (Ferraris et al. 2009; Janhunen et al. 2009; Lifschitz and Turner 1994).

An alternative semantics of logic programs with negation based on an infinite domain of truth values was proposed in (Rondogiannis and Wadge 2005). The infinite valued approach has been further developed in the abstract setting of ‘stratified complete lattices’. It has been proved in (Esik 2015b) that the stratified least fixed point operation arising in this approach does satisfy all identities of iteration theories. So in this regards, the infinite valued semantics behaves just as the Kripke-Kleene semantics (Fitting 2002).

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**References**


