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Bayesian Deduction with Subjective Opinions

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Abstract

Bayesian deduction is the process of deriving the probability distribution of a random variable given the distribution of another variable when the relevant conditional distributions are available. We present an extension of Bayesian deduction to the framework of subjective logic. The latter represents uncertain probabilistic information in the form of subjective opinions, which allow for explicitly modelling and operating with degrees of uncertainty about the probability distributions. We discuss how the defined deduction operation can be used for predictive reasoning in subjective Bayesian networks.

A Bayesian network (BN) is a compact representation of a joint probability distribution in the form of a directed acyclic graph (DAG) with random variables as nodes, and a set of conditional probability distributions associated with each node representing the probabilistic connection of the node and its parents in the graph. Bayesian networks reasoning algorithms provide a way to propagate probabilistic information through the graph from an evidence to a target set of variables, in that way updating the probability of the target upon observation of the evidence. One serious limitation of the BNs reasoning methods is that all the input probabilities must be assigned a precise value in order for the inference algorithms to work and the model to be analysed. This is problematic in situations of uncertain or incomplete information, where probabilities can not be reliably estimated and the inference needs to be based on what is available while providing the most accurate conclusions possible.

In the literature, many different approaches have been proposed for dealing with incomplete Bayesian networks and uncertain probabilistic information in general, like for example, Bayesian logic (Andersen and Hooker 1994), credal networks (Cozman 2000), the probabilistic logics and networks discussed in (Haenni et al. 2011), the logics of likelihood in (Fagin, Halpern, and Megiddo 1990) and later in (Ivanovska and Giese 2011), imprecise probabilities (Walley 1991), (Walley 1996), interval probabilities (Tessem 1992), etc. None of the mentioned approaches models the *degree* of uncertainty about a probability distribution as a particular numerical value that can be propagated through the network in the process of inference.

Subjective logic (Jøsang 2001) is a formalism that offers explicit treatment of the uncertainty about probabilities in both representation and inference. The basic entities in subjective logic are *subjective opinions* on random variables. A subjective opinion includes a *belief mass distribution* over the states of the variable, complemented with an *uncertainty mass*, which together reflect a current analysis of the probability distribution of the variable by an expert, based on a test, etc; and a *base rate* probability distribution of the variable, reflecting a prior domain knowledge that is relevant to the current analysis. Subjective opinions can represent uncertain probabilistic information of any kind, minor or major imprecision and even total ignorance about the probability distribution, by varying the uncertainty mass between 0 and 1. By simply substituting every input conditional probability distribution in a BN with a subjective opinion, we obtain what we call a *subjective Bayesian network*. An introduction to subjective networks along with a brief discussion about the perspectives and challenges of the reasoning in them can be found in (Ivanovska et al. 2015).

This paper focuses on extending *Bayesian deduction* from random variables to subjective opinions, as a first step in solving the problem of inference in subjective networks. We call *Bayesian deduction* the process of deriving the probability distribution of a variable Y based on available conditional probabilities $p(y|x)$, for the states x of another variable X, and the distribution of the variable X itself. In the context of Bayesian networks, this amounts to deriving the marginal probability distribution of Y in a two-node Bayesian network where X is the parent and Y is the child node. In this sense, Bayesian deduction is the reasoning from the cause to the effect (under assumption that the arrow from X to Y denotes a causal relation between the variables).

Every subjective opinion can be "projected" to a single probability distribution, called *projected probability distribution* which is an important characteristic of the opinion since it unifies all of its defining parameters. The starting point in defining the operation of deduction with subjective opinions is deriving the projected probability distribution of the deduced opinion applying standard Bayesian deduction. Then the deduced opinion is fully determined by applying additional constrains on its beliefs and uncertainty mass. The whole process is motivated and illustrated by the geometrical representation of subjective opinions in barycentric coordinate systems.

The operation of deduction in subjective logic was first introduced for binomial opinions in (Jøsang, Pope, and Daniel 2005), and then for multinomial opinions in (Jøsang 2008). The deduction procedure for multinomial opinions proposed here is a generalised version of the one given in (Jøsang 2008). In this paper we also provide a way to apply the deduction operation to *hyper opinions*, which are subjective opinions that assign beliefs to *sets* of values of the variable.

We explore the use of the introduced deduction operation for predictive reasoning in subjective Bayesian networks. Predictive reasoning in Bayesian networks is the reasoning from causes to effects, which propagates "downwards" the new information to the target variables from their ancestors, using only the available input conditionals. We propose a method for predictive reasoning in subjective Bayesian networks that considers evidence in the form of a subjective opinion. The method combines the introduced operation of deduction and the operation for multiplication of subjective opinions on independent variables introduced in (Jøsang and McAnally 2004).

This paper is structured as follows: In Section 1 we give the necessary preliminaries from probability theory and Bayesian networks. In Section 2 we provide the definition of subjective opinion and its representation in a barycentric coordinate system. In Section 3 we introduce Bayesian deduction for subjective opinions and its geometrical interpretation and we demonstrate its application in an intelligence analysis example. In Section 4 we discuss the use of the introduced deduction operation for predictive reasoning in subjective BNs. In Section 5 we summarize the results of the paper and topics for future work.

1 Bayesian Deduction

This section introduces Bayesian deduction for random variables, providing the necessary definitions from probability theory.

We assume a simplified definition of a *random variable* as a variable that takes its values with certain probabilities. For example, the *weather today* is a random variable, that can be *cloudy* with probability 0.4, or *sunny* with probability 0.6. More formally, let X be a variable with a *domain* $\mathbb{X} = \{x_1, \ldots, x_k\}$. A *probability distribution* p of X is a function $p : \mathbb{X} \to [0, 1]$, such that:

$$\sum_{x \in \mathbb{X}} p(x) = 1 \ . \tag{1}$$

The domain $\mathbb{X}$ is the set of *values*, or *states* of the variable, which are assumed to be mutually exclusive and exhaustive. $k = |\mathbb{X}|$ is the cardinality of the variable. If $k = 2$, X is a binary random variable and $\mathbb{X} = \{x, \bar{x}\}$, where $\bar{x}$ denotes the state "not x". $p(x)$ is the probability that the variable X takes the value x.

A *joint probability distribution* of a set of variables $X_1, \ldots, X_n$ is a probability distribution defined on the Cartesian product of their domains. Given two random variables X and Y, a *conditional probability distribution* of Y given that X takes the value x, $p(Y|x)$, is a function from $\mathbb{Y}$ to $[0, 1]$ defined by the following equation:

$$p(y|x) = \frac{p(y, x)}{p(x)} \ , \tag{2}$$

where $p(x, y)$ is a joint probability distribution value. $p(y|x)$ is the probability that Y takes the value y, given that the value of X is x. The expressions $y|x$ and $p(y|x)$ are called *conditionals*, where x is the *antecedent*, and y is the *consequent*.

Let X and Y be two random variables. Let us assume that the analyst knows the probabilities $p(y|x)$ for every $y \in \mathbb{Y}$ and $x \in \mathbb{X}$, i.e. the set of conditional distributions $p(Y|X) = \{p(Y|x) \mid x \in \mathbb{X}\}$ is available. Assume further that a probability distribution of the variable X, $p(X)$, is also available. The above information determines a two-node Bayesian network where X is the parent and Y is the child node. Given this information, we can determine the probability distribution of Y, $p(Y)$, as follows:

$$p(y) = \sum_{x \in \mathbb{X}} p(y|x)p(x) \ . \tag{3}$$

The process of deriving $p(Y)$ from $p(X)$ and $p(Y|X)$ we call a *Bayesian deduction*. This type of reasoning follows the direction of the available conditionals in the sense that the direction of reasoning is from the antecedent to the consequent. Assuming that the connection $X \to Y$ is causal, we can say that Bayesian deduction is reasoning from the cause to the effect.

An important probability theory concept, especially in the context of Bayesian networks, is that of probabilistic *independence*. Let $V = \{X_1, \ldots, X_n\}$ be the set of all random variables that are of interest in a given context and let X, Y, and Z be disjoint subsets of V. Then X is *conditionally independent* of Y given Z, denoted $I(X, Y|Z)$, if the following holds:

$$p(x|y, z) = p(x|z) \text{ whenever } p(y, z) > 0 \ , \tag{4}$$

for every choice of assignments x, y, and z to the variables in the corresponding sets.[1]

A *Bayesian network* (Pearl 1988) with n variables is a directed acyclic graph (DAG) with random variables $X_1, \ldots, X_n$ as nodes, and a set of conditional probability distributions $p(X_i|Pa(X_i))$ associated with each node X_i containing one conditional probability distribution $p(X_i|pa(X_i))$ of X_i for every assignment of values $pa(X_i)$ to its parent nodes $Pa(X_i)$. If we assume that the Markov property holds: Every node is conditionally independent on its non-descendant nodes given its parent nodes in the graph,

$$I(X_i, ND(X_i)|Pa(X_i)) \ , \tag{5}$$

for the given DAG and the joint distribution p of the variables $X_1, \ldots, X_n$, then p is determined by:

$$p(x_1, \ldots, x_n) = \prod_{i=1}^{n} p(x_i|pa(X_i)) \ , \tag{6}$$

[1] A set of variables $Y = \{Y_1, \ldots, Y_k\}$ can also be considered a variable with a domain $\mathbb{Y} = \mathbb{Y}_1 \times \cdots \times \mathbb{Y}_k$. As standard in Bayesian networks literature, we use the notation of a variable also for a set of variables, making the obvious identifications (see (Pearl 1988)).

where $pa(X_i)$ is the assignment of the parents of X_i that corresponds to the tuple $(x_1, \dots, x_n)$.

Having the joint probability distribution of the variables determined by Eq.(6), we can condition it upon new information - observed values of some of the variables (*evidence variables*) - to obtain an updated probability distribution of a set of *target variables*. This process is called *probabilistic inference* or *belief update* in BNs. *Predictive reasoning* in Bayesian networks (Korb and Nicholson 2010) is the probabilistic inference that generalizes Bayesian deduction in the sense that it goes from new information about causes to new beliefs about effects: If we imagine we have a causal BN (all the arrows are causal), this type of reasoning follows the direction of the arrows, i.e. the target variables are descendant of the evidence variables.

2 Subjective Opinions

Let X be a random variable. A *(multinomial) subjective opinion* on X (Jøsang 2008) is a tuple:

$$\omega_X = (b_X, u_X, a_X), \tag{7}$$

where $b_X : \mathbb{X} \to [0,1]$ is a *belief mass distribution*, $u_X \in [0,1]$ is an *uncertainty mass*, and $a_X : \mathbb{X} \to [0,1]$ is a *base rate distribution* of X, satisfying the following additivity constraints:

$$u_X + \sum_{x \in \mathbb{X}} b_X(x) = 1 \,, \tag{8}$$

$$\sum_{x \in \mathbb{X}} a_X(x) = 1 \,. \tag{9}$$

The beliefs and the uncertainty mass reflect the results of a current analysis of the random variable obtained by applying expert knowledge, experiments, or a combination of the two. $b_X(x)$ is the belief that X takes the value x expressed as a degree in $[0,1]$. It represents the amount of experimental or analytical evidence in favour of x. u_X is a single value, representing the degree of uncertainty about the probability distribution of X. It represents lack of evidence in the analysis that can be due to lack of knowledge or expertise, or insufficient experimental analysis. The base rate a_X is a prior probability distribution of X that reflects domain knowledge relevant to the current analysis, most usually relevant statistical information. Hence, subjective opinion is a composite representation of our uncertain knowledge about the probability distribution of a variable that combines (subjective) beliefs, uncertainty, and statistical information.

For example, a GP wants to determine whether a patient suffers from depression through a series of different tests. Based on the test results, the GP concludes that the collected evidence is 10% inconclusive, but is still two times more in support of the diagnosis that the patient suffers from depression than of the opposite one. As a result, the GP assigns 0.6 belief mass to the diagnosis that the patient suffers from depression and 0.3 belief mass to the opposite diagnosis, complemented by 0.1 uncertainty mass. The probability that a random person in the population suffers from depression is 5% and this fact determines the base rate distribution in the GPs subjective opinion on the condition of the patient.

In some cases of modelling it is useful to be able to distribute belief mass to set of values, subsets of $\mathbb{X}$. This leads to generalization of multinomial subjective opinions to *hyper opinions*, in which the belief mass distribution b_X is a function defined on a restricted power set of the domain $\mathcal{R}(\mathbb{X}) = \mathcal{P}(\mathbb{X}) \setminus \{\mathbb{X}, \emptyset\}$. We call $\mathcal{R}(\mathbb{X})$ a *hyperdomain* of X. The sets $\mathbb{X}$ and $\emptyset$ are excluded from the hyperdomain since they do not represent specific observations which discriminate among the values of the domain and which therefore can be assigned a belief mass.

Technically, if we ignore the reduction of the hyperdomain, the beliefs and the uncertainty mass of a subjective opinion correspond to a *basic belief assignment* in belief theory (Shafer 1976). This correspondence gives the same interpretation of the beliefs, but a rather different interpretation of u_X, namely as $b_X(\mathbb{X})$, a belief mass in support of the whole $\mathbb{X}$, while we interpret u_X as a measure for the lack of evidence. The last can be more clearly seen from the correspondence between multinomial opinions and multinomial Dirichlet models, for details see (Jøsang and Elouedi 2007).

A subjective opinion in which $u_X = 0$, i.e. an opinion without any uncertainty, is called a *dogmatic opinion*. Dogmatic multinomial opinions correspond to probability distributions. A dogmatic opinion for which $b_X(x) = 1$, for some $x \in \mathbb{X}$, is called an *absolute opinion*. Absolute multinomial opinions correspond to assigning values to variables, i.e. to observations. In contrast, an opinion for which $u_X = 1$, and consequently $b_X(x) = 0$, for every $x \in \mathcal{R}(\mathbb{X})$, i.e. an opinion with complete uncertainty, is called a *vacuous opinion*. Vacuous opinions correspond to complete ignorance about the probability distribution, where the only relevant information is the base rate.

A multinomial opinion ω_X is "projected" to a probability distribution $\mathrm{P}_X : \mathbb{X} \to [0,1]$, defined in the following way:

$$\mathrm{P}_X(x) = b_X(x) \; + \; a_X(x) \, u_X \,. \tag{10}$$

We call the function P_X a *projected probability distribution* of ω_X. According to Eq.(10), $\mathrm{P}_X(x)$ is the belief mass in support of x increased by a portion of the base rate of x determined by u_X. It provides an estimate of the probability of x which varies from the base rate value, in the case of complete ignorance, to the actual belief in the case of zero uncertainty. The projected probability is an important characteristic of the opinion since it unifies all of the opinion parameters in a single probability distribution and thus enables reasoning about subjective opinions in the classical probability theory.

We call *focal elements* the elements of $\mathcal{R}(\mathbb{X})$ that are assigned a non-zero belief mass. In the case of a hyper opinion, there can be focal elements that have a non-empty intersection. For that reason, for hyper opinions the definition of projected probability distribution is generalized as follows:

$$\mathrm{P}_X(x) = \sum_{x' \in \mathcal{R}(\mathbb{X})} a_X(x|x') \, b_X(x') + a_X(x) \, u_X \,, \tag{11}$$

for $x \in \mathbb{X}$, where $a_X(x|x')$[2] is a conditional probability if

[2]Note that we make an abuse of the notation using the same type of letters for the elements of $\mathbb{X}$ and $\mathcal{R}(\mathbb{X})$, identifying the elements of $\mathbb{X}$ with the singletons of $\mathcal{R}(\mathbb{X})$.

a_X is extended to $\mathcal{P}(X)$ additively (and $a_X(x') > 0$). If we denote the sum in Eq.(11) by b'_X:

$$b'_X(x) = \sum_{x' \in \mathcal{R}(\mathbb{X})} a_X(x|x')\ b_X(x')\ , \qquad (12)$$

then it is easy to check that $b'_X : \mathbb{X} \to [0,1]$, and that b'_X together with u_X satisfies the additivity property in Eq.(8), i.e. $\omega'_X = (b'_X, u_X, a_X)$ is a multinomial opinion. From Eq.(11) and Eq.(12) we obtain $\mathrm{P}_X = \mathrm{P'}_X$. This means that every hyper opinion can be approximated with a multinomial opinion which has the same projected probability distribution as the initial hyper one.

A special type of a multinomial opinion is the subjective opinion on a binary random variable called a *binomial opinion*. A binomial opinion $\omega_X = (b_X, u_X, a_X)$ on a variable X with a domain $\mathbb{X} = \{x, \bar{x}\}$ can be represented as a tuple:

$$\omega_X = (b_x, d_x, u, a_x)\ , \qquad (13)$$

where $b_x = b_X(x)$ is the *belief* in $X = x$, $d_x = b_X(\bar{x})$ is the *disbelief* in it (the belief in the opposite), $u = u_X$ is the *uncertainty*, and the base rate of x, $a_x = a_X(x)$, determines the base rate probability distribution of X. According to Eq.(8), the parameters of a binomial opinion satisfy:

$$b_x + d_x + u = 1, \qquad (14)$$

and the projected probability distribution is fully determined by the value $\mathrm{P}_x = \mathrm{P}_X(x)$ obtained according to Eq.(10):

$$\mathrm{P}_x = b_x + a_x u. \qquad (15)$$

For example, if a reviewer of a paper scores a paper with 70% of acceptance, but is just partially relevant to judge the paper (is 50% uncertain in her expertise), then this reviewer's score can be described by the binary opinion $\omega_X = (0.35, 0.15, 0.50, 0.20)$, where the base rate 0.20 represents the 20% acceptance rate of the conference.

2.1 Geometrical Representation of Subjective Opinions

Multinomial opinions on variables with cardinality k can be represented in a *barycentric coordinate system* (Ungar 2010) of dimension k, which is a regular k-simplex where the coordinates of each point specify the distances of the point to the sides of the simplex. In that way binomial opinions are represented in an equilateral triangle and trinomial in a tetrahedron as shown in Fig.1 and Fig.2. The belief and uncertainty masses of the opinion are represented as a point ω_X inside of the simplex, and the base rate distribution is represented with a point a_X on one designated side of the simplex called a *base*. The distance of the point ω_X to the base equals the uncertainty mass, while its distances to the other sides of the simplex equal the belief masses.[3]

For a given variable X of cardinality k, we denote by Ω_X the k-simplex with a designated base side where we can represent all the possible opinions on X, and we call it an *opinion simplex* or *opinion space* of X. An *apex* of the opinion simplex is the vertex opposite the base side. It corresponds to a vacuous opinion on X. The other vertices correspond to absolute opinions. The points on the base represent dogmatic opinions. A strong positive opinion, for example, would be represented by a point towards the bottom right belief vertex in Fig.1, etc.

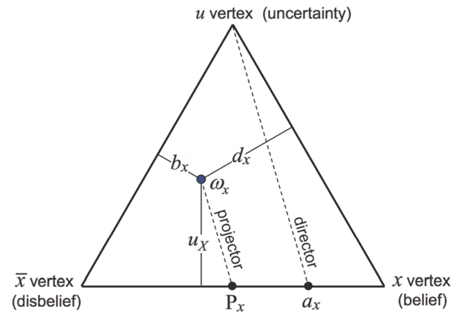

Figure 1: Graphical representation of binomial opinions

The line that joins the apex of the simplex and the base rate point a_X is called a *director line* or a *director*. The line parallel to the director that passes through the opinion point ω_X is called a *projector*. It can be checked that the intersection of the projector with the base side is a point with coordinates that correspond to the projected probability distribution of the opinion ω_X.

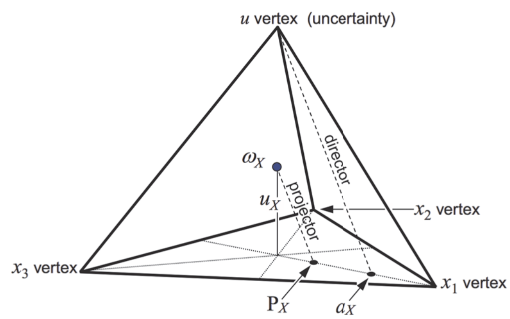

Figure 2: Graphical representation of trinomial opinions

3 Deduction Operation for Subjective Opinions

In this section we extend the Bayesian deduction for random variables from Section 1 to a deduction operation for subjective opinions on random variables.

3.1 Deduction Problem

Let X and Y be random variables with domains $\mathbb{X} = \{x_i | i = 1, \ldots, k\}$ and $\mathbb{Y} = \{y_j | j = 1, \ldots, l\}$ respectively. Assume we are given an opinion $\omega_X = (b_X, u_X, a_X)$ on X and conditional opinions $\omega_{Y|x_i}$ on Y, one for each $x_i \in \mathbb{X}$:

$$\omega_{Y|x_i} = (b_{Y|x_i}, u_{Y|x_i}, a_Y)\ . \qquad (16)$$

$b_{Y|x_i} : \mathbb{Y} \to [0,1]$ is a belief mass distribution and $u_{Y|x_i} \in [0,1]$ is an uncertainty mass, such that Eq.(8) holds, and the

[3]Note that the opinion notation ω_X in the figures refers to the belief and uncertainty masses of the represented opinion only, while the base rate distribution is represented separately by the point a_X on the base.

base rate distribution $a_Y : \mathbb{Y} \to [0, 1]$ is a probability distribution of Y.

The conditional opinion $\omega_{Y|x_i}$ is a subjective opinion on the variable Y given that X takes the value x_i. It represents our uncertain knowledge about the probability distribution $p(Y|x_i)$. In general it has the form $\omega_{Y|x_i} = (b_{Y|x_i}, u_{Y|x_i}, a_{Y|x_i})$, but here we assume $a_{Y|x_i} = a_Y$, for every $i = 1, \ldots, k$. As we will see later, this assumption comes from a mathematical necessity in the deduction procedure. It implies independence of the variables X and Y in the knowledge domain that we justify with an assumption that the base rate distribution of the variable Y is obtained independently of X, i.e. the arrow from X to Y represents just a potential dependence, the content of which is expressed by the conditional belief and uncertainty masses.

We denote by $\omega_{Y|X}$ the set of all conditional opinions on Y given the values of X:

$$\omega_{Y|X} = \{\omega_{Y|x_i} \mid i = 1, \ldots, k\} . \tag{17}$$

Given ω_X and $\omega_{Y|X}$, the goal of the deduction is to derive a subjective opinion on Y:

$$\omega_{Y\|X} = \left(b_{Y\|X}, u_{Y\|X}, a_Y\right) , \tag{18}$$

where $b_{Y\|X} : \mathbb{Y} \to [0, 1]$ is a belief mass distribution and $u_{Y\|X} \in [0, 1]$ is an uncertainty mass, such that Eq.(8) holds, and a_Y is as given in the input conditional opinions $\omega_{Y|X}$. Although $a_{Y\|X} = a_Y$ may as well be considered a consequence of the assumption $a_{Y|x_i} = a_Y$, for every $i = 1, \ldots, k$ (if the base rates form a BN with the given DAG), assuming $a_{Y\|X} = a_Y$ here means that the proposed deduction procedure does not include reasoning about the base rates, they are just given or obtained separately.

3.2 Deduction Method

In this section we give the idea behind the deduction method for obtaining $\omega_{Y\|X}$ from ω_X and $\omega_{Y|X}$. Each of the following sections focuses on one part of the method.

The deduction method determines the projected probability distribution and the uncertainty mass of the deduced opinion first, and then obtains the belief mass distribution as a consequence applying Eq.(10). While the projected probability is determined by Bayesian deduction described in Section 1, the procedure for determining an appropriate value for the uncertainty mass of the deduced opinion, $u_{Y\|X}$, is inspired by the geometrical analysis of the input opinions and their interrelations. The idea is that the input conditional opinions $\omega_{Y|X}$ define a *deduction operator* from Ω_X to Ω_Y, which maps ω_X into $\omega_{Y\|X}$. This is denoted by the following expression for multinomial deduction in subjective logic (Jøsang, Pope, and Daniel 2005):

$$\omega_{Y\|X} = \omega_X \circledcirc \omega_{Y|X} . \tag{19}$$

The following intuitive constrains are taken into consideration in providing the definition of the deduction operator:

1. The *absolute opinions* that correspond to the base vertices of the opinion space Ω_X map into the respective opinions from the set $\omega_{Y|X}$:

$$\omega_{Y|x_i} = \omega_X^i \circledcirc \omega_{Y|X} , \tag{20}$$

where $\omega_X^i = (b_X^i, u_X^i, a_X)$ is the absolute opinion on X such that $b_{x_i}^i = 1$ (consequently $b_{x_j}^i = 0$, for $j \neq i$, and $u_X^i = 0$).

2. The apex of Ω_X corresponds to the following *vacuous opinion* on X:

$$\widehat{\omega}_X = (\widehat{b}_X, \widehat{u}_X, a_X) , \tag{21}$$

where $\widehat{u}_X = 1$, and $\widehat{b}_{x_i} = 0$, for every $i = 1, \ldots, k$. Let us denote the deduction image of $\widehat{\omega}_X$ by $\omega_{Y\|\widehat{X}}$:

$$\omega_{Y\|\widehat{X}} = \widehat{\omega}_X \circledcirc \omega_{Y|X} . \tag{22}$$

Now, the idea is that $\omega_{Y\|\widehat{X}}$ is determined as the opinion with the maximum possible uncertainty mass satisfying certain constrains imposed on the projected probability and the beliefs of the opinion $\omega_{Y\|\widehat{X}}$.

3. The deduction operator maps the whole opinion space of X, Ω_X, into a sub-space of Ω_Y, which we call a *deduction sub-space*. The deduction sub-space is determined as the convex closure of the opinion points $\omega_{Y|x_i}$, $i = 1, \ldots, k$, and $\omega_{Y\|\widehat{X}}$.

4. The deduced opinion $\omega_{Y\|X}$ from an arbitrary opinion ω_X is determined as a linear projection of ω_X inside the deduction sub-space.

A visualisation of the above in the case of trinomial opinions where the opinion spaces are tetrahedrons, is given in Fig.3. The deduction sub-space is shown as a shaded tetrahedron inside the opinion tetrahedron of Y.

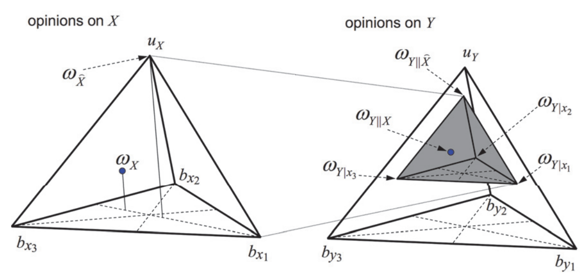

Figure 3: Projecting an opinion in a deduction subspace

3.3 Projected Probability

The definition of Bayesian deduction for subjective opinions should be compatible with the definition of Bayesian deduction for probability distributions described in Section 1. This means that the projected probability of the deduced opinion $\omega_{Y\|X}$ should satisfy the relation given in Eq.(3), i.e.:

$$\mathrm{P}(y_j\|X) = \sum_{i=1}^{k} \mathrm{P}(x_i)\mathrm{P}(y_j|x_i) , \tag{23}$$

for $j = 1, \ldots, l$, where Eq.(10) provides each factor on the right-hand side of Eq.(23).[4] On the other side, from Eq.(10),

[4]Note that we omit indices and use simplified notation for the projected probabilities: $\mathrm{P}(y_j|x_i)$ for $\mathrm{P}_{Y|x_i}(y_j)$ and $\mathrm{P}_{Y\|X}(y_j)$ for $\mathrm{P}(y_j\|X)$.

we have the following:

$$\mathrm{P}(y_j\|X) = b_{y_j\|X} + a_{y_j} u_{Y\|X} \ , \tag{24}$$

where $b_{y_j\|X} = b_{Y\|X}(y_j)$, and $a_{y_j} = a_Y(y_j)$. (We use similar short-hand notations for the belief masses and base rates appearing later in the text as well.)

Eq.(23) determines a system of l linear equations with the beliefs $b_{y_j\|X}$, $j = 1, \ldots, l$, and the uncertainty $u_{Y\|X}$ as variables. We obtain one more equation over the same variables by applying Eq.(8) to the deduced opinion:

$$u_{Y\|X} + \sum_{j=1}^{l} b_{y_j\|X} = 1 \ . \tag{25}$$

The system of equation determined by Eq.(25) and Eq.(24) does not have a unique solution, however, since Eq.(25) is the sum of Eq.(24), hence the system is dependent. This means that it has infinitely many solutions, i.e. there are infinitely many subjective opinions on Y with a base rate a_Y and projected probability determined by Eq.(23). The last observation is in correspondence with the geometrical representation of subjective opinions, namely: Once we have an opinion point ω_Y corresponding to the given a_Y and P_Y, every other point in Ω_Y lying on the projector through ω_Y and a_Y will also correspond to a solution of the system.

We apply additional constrains on the beliefs and uncertainty mass of the deduced opinion in order to completely determine the opinion.

3.4 Constraint on the Deduced Beliefs

The belief mass assigned to each value y_j of Y in any deduced opinion $\omega_{Y\|X}$ should be at least as large as the minimum of the corresponding belief masses in the given conditionals, i.e. :

$$b_{y_j\|X} \geq \min_i b_{y_j|x_i} \ , \tag{26}$$

for every $j = 1, \ldots, l$. This is intuitively clear: Our belief in y_j when we do not know the exact value of X is at least as large as the minimum of the belief we would assign to y_j knowing that a particular value of X is the case. (The assumption is, of course, that we are aware of the domain of X.) This is a natural assumption, which can be found as a *principle of plausible reasoning* in (Pearl 1990).

The condition in Eq.(26) has the following geometrical interpretation: For every ω_X, the opinion point $\omega_{Y\|X}$ must be inside the sub-space of Ω_Y bounded by the planes $b_{y_j} = \min_i b_{y_j|x_i}$ (which are parallel to the sides of Ω_Y). We call it an *auxiliary deduction sub-space*.

3.5 Deduction from a Vacuous Opinion

The deduction sub-space will be bounded by the k points $\omega_{Y|x_i}$, $i = 1, \ldots, k$, and the point that corresponds to the vacuous opinion $\omega_{Y\|\widehat{X}}$. While the former are given, the latter needs to be computed.

The projected probability of $\omega_{Y\|\widehat{X}}$ can be determined by Eq.(23). We will determine its uncertainty mass $u_{Y\|\widehat{X}}$ as the maximum possible uncertainty mass value that corresponds to the obtained projected probability and the given base rate, satisfying at the same time the belief constraint imposed by Eq.(26).

Applying Eq.(23) to the vacuous opinion on X, $\omega_{\widehat{X}}$, we obtain the following equation for the projected probability distribution of $\omega_{Y\|\widehat{X}}$:

$$\mathrm{P}(y_j\|\widehat{X}) = \sum_{i=1}^{k} a_{x_i} \mathrm{P}(y_j|x_i) \ . \tag{27}$$

On the other hand, by definition given in Eq.(10), we have the following equation:

$$\mathrm{P}(y_j\|\widehat{X}) = b_{y_j\|\widehat{X}} + a_{y_j} u_{Y\|\widehat{X}} \ . \tag{28}$$

Now, we want to find the point $\omega_{Y\|\widehat{X}} = (b_{Y\|\widehat{X}}, u_{Y\|\widehat{X}}, a_Y)$ with the greatest possible uncertainty satisfying the requirements in Eq.(28) and Eq.(26), where $\mathrm{P}(y_j\|\widehat{X})$ are determined by Eq.(27) and a_{y_j} are given.

From Eq.(28) and Eq.(26) we obtain the following inequality:

$$u_{Y\|\widehat{X}} \leq \frac{\mathrm{P}(y_j\|\widehat{X}) - \min_i b_{y_j|x_i}}{a_{y_j}} \ , \tag{29}$$

for every $j = 1, \ldots, l$. For simplicity, let us denote the right-hand side of Eq.(29) by u_j. Hence we have:

$$u_{Y\|\widehat{X}} \leq u_j \ , \tag{30}$$

for every $j = 1, \ldots, l$. Now, the greatest $u_{Y\|\widehat{X}}$ for which Eq.(30) holds is determined as:

$$u_{Y\|\widehat{X}} = \min_j u_j \ . \tag{31}$$

Namely, from Eq.(28) and Eq.(26), it follows that $u_j \geq 0$, for every j (proof in the Appendix), hence the value determined by Eq.(31) is non-negative. Also $u_{Y\|\widehat{X}} \leq 1$, since, if we assume the opposite, it will follow that $u_j > 1$, for every $j = 1, \ldots, l$, which leads to $\mathrm{P}(y_j\|\widehat{X}) > \min_i b_{y_j|x_i} + a_{y_j}$, for every $j = 1, \ldots, l$; summing up by j in the last inequality we obtain a contradiction, since both the projected probabilities and the base rates of Y sum up to 1. Hence, $u_{Y\|\widehat{X}}$ determined by Eq.(31) is a well-defined uncertainty mass value. It is obviously the greatest value satisfying Eq.(30), hence also the initial requirements.

Having determined $u_{Y\|\widehat{X}}$, we determine the corresponding belief masses $b_{y_j\|\widehat{X}}$, $j = 1, \ldots, l$, from Eq.(28) by which the opinion $\omega_{Y\|\widehat{X}}$ is fully determined.

In the geometrical representation of subjective opinions in tetrahedrons, the described procedure for determining the opinion $\omega_{Y\|\widehat{X}}$ corresponds to determining the intersection between the surface of the auxiliary deduction sub-space and the *projector* passing through the base point that represents the projected probability determined by Eq.(27), when the corresponding *director* is determined by a_Y.

3.6 Deduction from an Arbitrary Opinion

The given opinion ω_X is then linearly projected in the deduction sub-space determined by the points $\omega_{Y|x_i}$, $i = 1,\dots,k$, and $\omega_{Y\|\widehat{X}}$. This means that its uncertainty mass $u_{Y\|X}$ is determined by a linear transformation of the parameters of ω_X, and the belief masses are determined accordingly. This linear transformation is obtained based on the constraints 1. and 2. in Section 3.2. The vertices of the opinion simplex of X map into the vertices of the deduction sub-space. The latter leads to the following linear expression for the uncertainty $u_{Y\|X}$:

$$u_{Y\|X} = u_X u_{Y\|\widehat{X}} + \sum_{i=1}^{k} u_{Y|x_i} b_{x_i} \ . \tag{32}$$

We obtain the last expression as the unique transformation on the beliefs and uncertainty of an opinion on X that maps the beliefs and uncertainty mass of the opinions ω_X^i, $i = 1,\dots,k$, and $\widehat{\omega}_X$, into the uncertainty masses of $\omega_{Y|x_i}$, $i = 1,\dots,k$, and $\omega_{Y\|\widehat{X}}$ respectively.

Having deduced the uncertainty $u_{Y\|X}$, and the projected probability distribution by Eq.(23), the beliefs of the deduced opinion are determined by Eq.(24). We prove in the Appendix that the beliefs $b_{y_j\|X}$ determined in this way satisfy the requirement in Eq.(26). We remark that the initial assumption for a fixed (unconditional) base rate of Y is essential in this proof.

The above described deduction procedure can also be applied if some of the input opinions are hyper opinions. In that case, we first determine the corresponding projections of the hyper opinions into multinomial opinions, in the way described in Section 2, and then deduce an opinion from the projections. The deduced opinion will be a multinomial one. Allowing hyper input opinions while deducing only multinomial ones is an advantage in some sense, since one usually has the input information in a more vague, hyper opinion form, but prefers to have the derived conclusions in a sharper, multinomial opinion form, i.e. to have a distribution of beliefs over the values rather than set of values.

3.7 Example: Intelligence Analysis with Subjective Logic Deduction

Two neighbouring countries A and B are in conflict, and intelligence analysts of country B want to find out whether country A intends to use *military aggression* (random variable Y). The analysts of country B consider the following possible alternatives regarding country A's plans:

$$\begin{array}{ll} y_1: & \text{No military aggression from country } A \\ y_2: & \text{Minor military operations by country } A \\ y_3: & \text{Full invasion of country } B \text{ by country } A \end{array} \tag{33}$$

The way the analysts of country B determine the most likely plan of country A is by observing *mobilization of troops* along the border in country A (random variable X) using satellite photos. In that way, the following possibilities for mobilization of troops are considered:

$$\begin{array}{ll} x_1: & \text{No mobilization of country } A\text{'s troops} \\ x_2: & \text{Minor mobilization of country } A\text{'s troops} \\ x_3: & \text{Major mobilization of country } A\text{'s troops} \end{array} \tag{34}$$

The available evidence from the satellite photos gives the following distribution of beliefs over the possible options:

$$b_X = (0.0, 0.5, 0.2) \ , \tag{35}$$

and the uncertainty mass $u_X = 0.3$. This means that the experts' beliefs are significantly in favour of a minor troops mobilization, with some space for a major mobilization too, but they are uncertain to some degree (in their material evidence, in their expertise, etc.).

Based on statistical evidence about mobilization of troops along the border of country A, the following probability distribution of the values of X is available and is taken as a base rate a_X in the current analysis:

$$a_X = (0.7, 0.2, 0.1) \ . \tag{36}$$

In that way a subjective opinion $\omega_X = (b_X, u_X, a_X)$ on the mobilization of troops is formed.

The values in Eq.(36) and Eq.(35) give the following projected probability distribution of the troop mobilization in country A:

$$\mathrm{P}(X) = (0.21, 0.56, 0.23) \ . \tag{37}$$

The projected probabilities are experts' beliefs adjusted by a portion of the corresponding base rates, according to the uncertainty value u_X.

Now, the target variable is Y and the evidence variable is X. Based on statistical data and professional expertise, the analysts form an opinion on how the military plans (Y) probabilistically depend on the troop movements (X). They express this opinion through the belief mass distributions and uncertainty masses given in Table 1.

	y_1	y_2	y_3	Y
$\omega_{Y\|x_1}$	$b_{y_1} = 0.9$	$b_{y_2} = 0.0$	$b_{y_3} = 0.0$	$u_Y = 0.1$
$\omega_{Y\|x_2}$	$b_{y_1} = 0.2$	$b_{y_2} = 0.3$	$b_{y_3} = 0.1$	$u_Y = 0.4$
$\omega_{Y\|x_3}$	$b_{y_1} = 0.0$	$b_{y_2} = 0.3$	$b_{y_3} = 0.5$	$u_Y = 0.2$

Table 1: Conditional opinions on Y given X

The values in Table 1 together with the base rate of Y, a_Y defined by:

$$a_Y = (0.90, 0.09, 0.01) \ , \tag{38}$$

form a set of conditional opinions $\omega_{Y|X}$. These opinions have projected probability distributions as given in Table 2.

	y_1	y_2	y_3
$\mathrm{P}(Y\|x_1)$	0.990	0.009	0.001
$\mathrm{P}(Y\|x_2)$	0.560	0.336	0.104
$\mathrm{P}(Y\|x_3)$	0.180	0.318	0.502

Table 2: Projected probabilities of Y conditional on X

Next we apply a deduction on the given opinion $\omega_X = (b_X, u_X, a_X)$ and the set of conditional opinions $\omega_{Y|X}$ to deduce an opinion on Y.

The values from Eq.(36) and Table 2, applied in Eq.(27), give the following projected probabilities of Y given the vacuous opinion on X:

$$\mathrm{P}(Y\|\widehat{X}) = (0.8072, 0.1084, 0.0844) \ . \tag{39}$$

Then according to Eq.(31), we determine the uncertainty of the opinion on Y that corresponds to the vacuous opinion on X:

$$u_{Y\|\widehat{X}} = 1 \ . \tag{40}$$

And, finally from Eq.(32), we find the actual uncertainty of the deduced opinion on Y that corresponds to the given opinion ω_X on X:

$$u_{Y\|X} = 0.52 \ . \tag{41}$$

According to Eq.(23), we determine the projected probability distribution of $\omega_{Y\|X}$:

$$\mathrm{P}(Y\|X) = (0.5629, 0.2632, 0.1739) \ . \tag{42}$$

Using the uncertainty value in Eq.(41) and the projected probabilities from Eq.(42) in Eq.(24), we obtain the following deduced beliefs:

$$b_{Y\|X} = (0.09, 0.22, 0.17) \ . \tag{43}$$

Eq.(42) and Eq.(43) show the difference between classical and subjective logic Bayesian deduction: although y_1 (no aggression) seems to be country A's most likely plan in probabilistic terms, this likelihood is based mostly on prior probabilities and uncertainty. Indeed, beliefs on y_2 (minor aggression) or y_3 (full invasion) are much more stronger than the one on y_1, with y_2 having the strongest support.

A likelihood expressed as a simple probability value can thus hide important aspects of the analysis, which will only come to light when uncertainty is explicitly expressed, as done in the example above.

4 Deduction in Subjective Networks

A *subjective network* (SN) of n random variables is a directed acyclic graph and subjective opinions associated with it. We focus on applying the deduction operation in the type of subjective networks that we call Bayesian. A *subjective Bayesian network* is a generalization of a classical BN where the probability distributions associated with the nodes are substituted with subjective opinions on the corresponding random variables (Fig.4).[5]

We define the inference goal in subjective networks as follows: Given the input conditional subjective opinions in the network and a subjective opinion on a variable X (*evidence variable*), derive a subjective opinion on a *target variable* Y different than X. Note that the evidence subjective opinion can be the one already given in the subjective BN (if X is a root node) or provided as a new information in addition to the network's input. We limit ourselves to the case when X is an ancestor of Y, i.e. to the inference problems of *predictive reasoning*, reasoning that follows the directions of the network arrows propagating new information about causes to new beliefs about the effects.

We discuss how the introduced deduction operation, in combination with the multiplication operation for opinions on independent variables described in (Jøsang and McAnally 2004), can be used for predictive reasoning in subjective Bayesian networks.

[5]In (Ivanovska et al. 2015), we also introduce another type of subjective networks, that we call *fused*, where the sets of subjective opinions are associated with the arrows rather than the nodes of the graph.

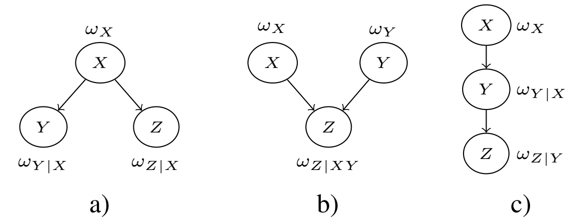

Figure 4: Three-node subjective networks

4.1 Deduction in Three-Node Structures

In the Naïve Bayes SN given in Fig.4(a), deduction operation is used in deriving the opinions $\omega_{Y\|X}$ and $\omega_{Z\|X}$ from the opinion ω_X and the corresponding sets of conditionals.

In the V-network in Fig.4(b), we apply deduction to obtain the opinion $\omega_{Z\|XY}$ from the opinion ω_{XY}. Since the variables X and Y in this network are probabilistically independent, the opinion ω_{XY} can be obtained by applying the multiplication operation on the subjective opinions ω_X and ω_Y.

In the chain network in Fig.4(c), deduction is used to derive $\omega_{Y\|X}$, and then another deduction operation is applied on $\omega_{Y\|X}$ and $\omega_{Z|Y}$ to derive $\omega_{Z\|X}$. If evidence in the form of a subjective opinion ω_Y is available at Y, then applying deduction on ω_Y and $\omega_{Z|Y}$ we can derive $\omega_{Z\|Y}$. In the latter inference we ignore the input opinion ω_X that is "above" the evidence variable Y, and the opinion $\omega_{Y\|X}$ that can be deduced from it, since we have a new opinion (a new information) on the variable Y.

In Fig.4(a), we could also use deduction operation to derive the opinion $\omega_{YZ\|X}$ by first obtaining the set $\omega_{YZ|X}$ by mutiplication: $\omega_{YZ|x} = \omega_{Y|x} \times \omega_{Z|x}, x \in \mathbb{X}$. Multiplication operation can be applied here because the independence relation $I(Y, Z|X)$ holds.

4.2 Predictive Reasoning in Subjective Networks

In this section we propose a solution to the inference problem of predictive reasoning in subjective Bayesian networks with a singly-connected DAG (a graph with only one path between every two nodes).

Let X be the evidence variable and Y be the target variable in the inference. This means that we are given a subjective opinion ω_X and want to derive a subjective opinion $\omega_{Y\|X}$. We distinguish between the following two cases:

1. The DAG is in the form of a *tree*. This means that every node has only one parent, so there is an ancestor chain between the evidence and the target: $X_1 \to \cdots \to X_n$ where $X_1 = X$ and $X_n = Y$. Then the reasoning from the evidence to the target is a generalization of the reasoning in the chain network in Fig.4(c), i.e. $\omega_{Y\|X}$ is obtained by $n-1$ consecutive deduction operations. (If X is not a root node, then its ancestors in the graph are ignored in the inference.)

2. The DAG contains V-structures, i.e. there are nodes that have multiple parents. Suppose Z is a node on the path

between X and Y that has multiple parents. Then the parents of Z are probabilistically independent variables according to the d-separation criterion (Neapolitan 2003) since the only path between each two of them passes through Z. This means that we can first derive subjective opinion on each of the parents of Z separately, and then use the multiplication operation to find $\omega_{Pa(Z)\|X}$ which we further propagate to Z. Because the graph is singly-connected, the parents of Z have sets of ancestors that are non-intersecting, hence the deduced opinions on them are derived independently and can be multiplied.

The inference problem becomes more complicated in a multiple-connected graph where the multiplication operation can not always be applied due to absence of the necessary independencies. For example, if we add an arrow between X and Y in the V-network in Fig.4(b), the network becomes multiply-connected and the required independence is lost.

5 Conclusions and Future Work

In many practical situations of modelling probabilistic knowledge, one can only estimate the probabilities with a degree of uncertainty and would like to account for this uncertainty during the inference. Subjective logic offers an explicit treatment of uncertainty about probability distributions, representing it as particular numerical values that can be propagated through the probabilistic network in the process of inference.

We defined a procedure for Bayesian deduction in subjective logic and showed the advantage of its use for modelling. We proposed a way to use this operation in the reasoning in subjective Bayesian networks where both the network's input and the evidence is given in the form of subjective opinions on the variables. In particular, we focused on predictive reasoning in subjective Bayesian networks with a single evidence and target node.

Bayesian deduction with subjective opinions is the first step in dealing with conditional reasoning with subjective opinions. Conditional reasoning has been a great part of other theories of uncertain probabilistic information mentioned in the introduction. It has also been analysed in the context of belief theory (Shafer 1976), (Smets and Kennes 1994), (Xu and Smets 1994). Compared to these approaches, and also to the traditional Bayesian deduction, the advantage of the subjective logic deduction is that it incorporates reasoning over beliefs, uncertainty about the beliefs, and statistical information at the same time. In that way it enables control over more complex information while doing probabilistic inference, returning a more accurate portrait of the modelled situation.

In future work, we want to provide a procedure for predictive reasoning in subjective networks of any kind and graph structure. A further goal is to provide methods for inference in subjective networks in general, which would include diagnostic and combined reasoning and reasoning with multiple evidence and target nodes.

Appendix

1. We prove that $u_j \geq 0$, $j = 1, \ldots, n$, for u_j defined as:

$$u_j = \frac{\mathrm{P}(y_j\|\widehat{X}) - \min_i b_{y_j|x_i}}{a_{y_j}} \ .$$

By definition of projected probability in Eq.(23), we have:

$$\mathrm{P}(y_j|x_i) = b_{y_j|x_i} + a_{y_i} u_{Y|x_i} \ ,$$

from which we obtain $\mathrm{P}(y_j|x_i) \geq b_{y_j|x_i}$. If we multiply by a_{x_i} and sum by i in the last inequality, we obtain:

$$\sum_i a_{x_i} \mathrm{P}(y_j|x_i) \geq \sum_i a_{x_i} b_{y_j|x_i} \ .$$

Now, applying the last inequality in Eq.(27), we obtain:

$$\begin{aligned} \mathrm{P}(y_j\|\widehat{X}) &\geq \sum_i a_{x_i} b_{y_j|x_i} \geq \sum_i a_{x_i} \min_i b_{y_j|x_i} \\ &= \min_i b_{y_j|x_i} \sum_i a_{x_i} = \min_i b_{y_j|x_i} \ . \end{aligned}$$

Hence, $\mathrm{P}(y_j\|\widehat{X}) - \min_i b_{y_j|x_i} \geq 0$, and $u_j \geq 0$ as well.

2. We prove that $b_{y_j\|X} \geq \min_i b_{y_j|x_i}$.

Applying appropriate equations from Sections 3.3 - 3.6, we obtain:

$$\begin{aligned} & b_{y_j\|X} = \mathrm{P}(y_j\|X) - a_{y_j} u_{Y\|X} \\ &= \sum_i \mathrm{P}(y_j|x_i)\mathrm{P}(x_i) - a_{y_j}(u_X u_{Y\|\widehat{X}} + \sum_i b_{x_i} u_{Y|x_i}) \\ &= \sum_i \mathrm{P}(y_j|x_i) b_{x_i} + \sum_i \mathrm{P}(y_j|x_i) a_{x_i} u_X - a_{y_j} u_X u_{Y\|\widehat{X}} \\ &\qquad - \sum_i a_{y_j} b_{x_i} u_{Y|x_i} \\ &= \sum_i \mathrm{P}(y_j|x_i) b_{x_i} + u_X(\sum_i \mathrm{P}(y_j|x_i) a_{x_i} - a_{y_j} u_{Y\|\widehat{X}}) \\ &\qquad - \sum_i a_{y_j} b_{x_i} u_{Y|x_i} \\ &= u_X b_{y_j\|\widehat{X}} + \sum_i (\mathrm{P}(y_j|x_i) b_{x_i} - a_{y_j} b_{x_i} u_{Y|x_i}) \\ &= u_X b_{y_j\|\widehat{X}} + \sum_i b_{x_i} (\mathrm{P}(y_j|x_i) - a_{y_j} u_{Y|x_i}) \\ &= u_X b_{y_j\|\widehat{X}} + \sum_i b_{x_i} b_{y_j|x_i} \\ &\geq u_X b_{y_j\|\widehat{X}} + \sum_i b_{x_i} \min_i b_{y_j|x_i} \\ &\geq u_X \min_i b_{y_j|x_i} + \min_i b_{y_j|x_i} \sum_i b_{x_i} \\ &= \min_i b_{y_j|x_i} (u_X + \sum_i b_{x_i}) = \min_i b_{y_j|x_i} \ . \end{aligned}$$

Note that the proof in part 2. above contains the proof of the following equation:

$$b_{y_j\|X} = u_X b_{y_j\|\widehat{X}} + \sum_i b_{x_i} b_{y_j|x_i} \,, \tag{44}$$

which shows that the obtained belief mass $b_{y_j\|X}$ can be represented as linear combinations of the corresponding belief masses of the opinions $\omega_{Y\|\widehat{X}}$ and $\omega_{Y|x_i}$, $x_i \in \mathbb{X}$ and the coefficients of the transformation are the same as in the corresponding transformation for the uncertainty masses given in Eq.(32). This shows that the deduced opinion $\omega_{Y\|X}$ is in the convex closure of the points $\omega_{Y\|\widehat{X}}$ and $\omega_{Y|x_i}$, $x_i \in \mathbb{X}$.

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