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Quantifying Conflicts for Spatial and Temporal Information

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Abstract

This paper tackles the problem of evaluating the degree of inconsistency in spatial and temporal qualitative reasoning. We first introduce postulates to propose a formal framework for measuring inconsistency in this context. Then, we provide two inconsistency measures that can be useful in various AI applications. The first one is based on the number of constraints that we need to relax to get a consistent qualitative constraint network. The second inconsistency measure is based on variable restrictions to restore consistency. It is defined from the minimum number of variables that we need to ignore to recover consistency. We show that our proposed measures satisfy required postulates and other appropriate properties. Finally, we discuss the impact of our inconsistency measures on belief merging in qualitative reasoning.

1 Introduction

Spatial and temporal reasoning is a central and well-studied topic in Artificial Intelligence, particularly in Knowledge Representation and Reasoning. This field (Renz and Nebel 2007; Hazarika 2012) has gained a lot of attention during the last few years as it extends to a plethora of domains, including natural language processing (Song and Cohen 1988), planning (Feiner et al. 1991), geographic information systems, computer vision, robot navigation, database theory, archaeology and genetics, and other autonomous systems that act in the real-world and need to reason about time and space. In this context, various qualitative approaches have been proposed to represent the spatial and temporal entities and their relations.

Qualitative formalisms have different advantages compared to quantitative spatial and temporal representations such as coordinate systems. They are closer to everyday human cognition, deal well with incomplete knowledge, and can be computationally more efficient than, say, the full machinery of metric spaces. In the context of these formalisms, *Qualitative Constraints Networks* (QCNs) can be used to represent spatial/temporal information about a system. In a QCN, a constraint represents a set of acceptable qualitative configurations between temporal/spatial entities and is defined by a set of base relations of the considered formalism.

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Conflicts in QCNs may be present for several reasons. For instance, when a QCN is used to represent beliefs of several agents about spatial and/or temporal entities (Condotta et al. 2010). Indeed, some agents in this context may have divergent and conflictual beliefs. Using standard reasoning, inconsistent QCNs are not informative. To deal with QCN inconsistency as an informative concept, we need to handle it with approaches similar to those used in the case of classical theories, such as argumentation theory, paraconsistent logics, belief revision and measuring inconsistency. In this paper, we focus in studying the latter approach.

Inconsistency measuring is a promising approach for handling inconsistent knowledge bases. In a sense, this approach is based on the identification of syntactic and/or semantic internal sub-parts of that base involved in the conflicts. In the literature, an inconsistency measure is defined as a function that associates a non negative value to each knowledge base (Hunter and Konieczny 2010). In the same way as the notion of utility function, it provides in particular a rank ordering of inconsistent knowledge bases. Several inconsistency measures have been proposed and studied in the literature (e.g. (Grant 1978; Knight 2002; Qi, Liu, and Bell 2005; Hunter and Konieczny 2010; Mu, Liu, and Jin 2011; Jabbour and Raddaoui 2013; Grant and Hunter 2013; Hunter, Parsons, and Wooldridge 2014; Jabbour, Ma, and Raddaoui 2014; Jabbour et al. 2014; Ammoura et al. 2015; Jabbour et al. 2015; Thimm 2016)). It has been shown that inconsistency measures are useful and attractive in diverse scenarios, including software specifications (Martinez, Arias, and Vilas 2004), e-commerce protocols (Chen, Zhang, and Zhang 2004), belief merging (Qi, Liu, and Bell 2005), news reports (Hunter 2006), integrity constraints (Grant and Hunter 2006), requirements engineering (Martinez, Arias, and Vilas 2004), databases (Martinez et al. 2007), semantic web (Zhou et al. 2009), and network intrusion detection (McAreavey et al. 2011). In this work, we aim at extending the approach of measuring inconsistency to qualitative reasoning. To the best of our knowledge, this paper is the first work on measuring inconsistency in qualitative reasoning.

We first propose postulates for measuring QCN inconsistency that allow us to formalize the intuition behind our inconsistency measures. Our two first postulates are requirements for every QCN inconsistency measure. They are similar to the postulates of *Consistency* and *Monotonicity* introduced in (Hunter and Konieczny 2010). It is worth noting that Hunter and Konieczny have proposed in their basic system two additional postulates that every inconsistency measure should satisfy, namely *Free Formula Independence* and *Dominance*. We here introduce a postulate similar to Dominance and three variants of Free Formula independence. We do not require these postulates for any QCN inconsistency measure because we have objections against them similar to Besnard's objections against Free Formula Independence and Dominance (Besnard 2014). However, by our objections, we are not arguing that our postulates related to Dominance and Free Formula independence are not needed for any QCN inconsistency measure. Indeed, we think that they are suitable in various cases where we need to reason with QCN inconsistency. We also propose other additional properties coming from qualitative reasoning.

In addition, we introduce two QCN inconsistency measures. The first measure is defined from relaxing constraints to get consistency. Indeed, it is based on the minimum number of constraints that we have to relax in order to obtain a consistent QCN. The second QCN inconsistency measure is defined from variable restrictions that allow to restore consistency. It is based on the minimum number of variables that we need to ignore to get a consistent QCN. We show that our two QCN inconsistency measures satisfy required postulates and other interesting properties.

It is important to note that QCN inconsistency measures can be a useful contribution in applications for which information is represented by temporal or spatial qualitative constraints. For instance, in the context of an intelligent system for human temporal annotations of texts using the standard TimeML relations (Pustejovsky et al. 2003), inconsistency measures for QCNs of the Interval Algebra can be used to control the addition of a new piece of information. In the same vein, QCN inconsistency measures can provide a guide to repair inconsistent corpus. Moreover, one of the motivations for introducing and studying inconsistency measures for qualitative formalisms is their application to information merging (see Section 7). Indeed, the inconsistency measures can be used in defining various operators for belief revision. For instance, we can decide that an agent accepts a new piece of information (that expresses his preference/belief on relative positions of spatial and/or temporal entities) only if it brings few contradictions.

2 Temporal and Spatial Qualitative Calculi

This section reviews basic notions in qualitative reasoning.

Relations A (binary) temporal or spatial qualitative calculus uses a domain D to represent temporal or spatial entities and a finite set B of binary relations defined on this domain D. The elements of B are referred to as *base relations* and represent the set of possible configurations over two temporal or spatial entities. The base relations of B are *jointly exhaustive and pairwise disjoint* and closed for the inverse. Moreover, the identity relation on D, denoted by Id, belongs to B. Given two elements x and y of D and a base relation $b \in \mathsf{B}$, $x\ b\ y$ denotes that x and y satisfies b, *i.e.* $(x, y) \in b$. For a given qualitative calculus, each (complex) relation is the union of base relations. A relation is represented as the set of base relations included in the corresponding union. Hence, we have $2^{|\mathsf{B}|}$ possible relations represented by the set 2^{B}. Given $x, y \in \mathsf{D}$ and $r \in 2^{\mathsf{B}}$, $x\ r\ y$ will denote that x and y satisfies a base relation $b \in r$. The set 2^{B} is equipped with the usual set-theoretic operations: union ($\cup$), intersection ($\cap$) and, converse ($^{-1}$). Notice that the converse of a relation is the union of the converses of its base relations.

To illustrate these concepts, let us consider the well-known Allen's temporal calculus, called the *Interval Algebra* (IA) (Allen 1981). The set D_{IA} representing the temporal entities is the set of the intervals on the line. Moreover, IA uses thirteen base relations denoted as $\mathsf{B}_{\mathsf{IA}} = \{eq, p, pi, m, mi, o, oi, s, si, d, di, f, fi\}$. Each base relation of B_{IA} represents a particular ordering of the four endpoints of two intervals on the rational line (see Figure 1). In the past, numerous qualitative calculus have been proposed and studied; see e.g. (Randell, Cohn, and Cui 1992; Vilain, Kautz, and Beek 1990; Pujari, Kumari, and Sattar 1999; Balbiani, Condotta, and Fariñas del Cerro 1998; Balbiani and Osmani 2000).

Relation	Symbol	Inverse	Meaning
precedes	p	pi	
meets	m	mi	
overlaps	o	oi	
starts	s	si	
during	d	di	
finishes	f	fi	
equals	eq	eq	

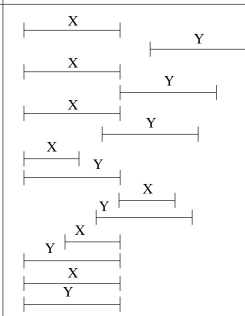


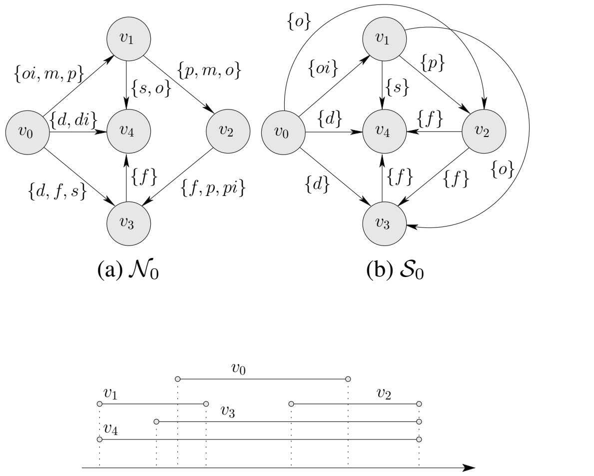
Figure 1: The base relations of IA

In what follows, we shall assume a temporal or spatial qualitative calculus based on a finite set B of base relations on a domain D.

Qualitative Constraint Networks Temporal or spatial information about the relative positions of a set of entities can be represented by a Qualitative Constraint Network. A QCN is a pair composed of a set of variables and a set of constraints. Intuitively, the variables represent the set of temporal/spatial entities of the system, and each constraint consists of a set of acceptable qualitative configurations between two entities. More formally, a QCN is defined as follows:

Definition 1 *A* QCN *is a pair* $\mathcal{N} = (V, C)$ *where* V *is a non-empty finite set of variables, and* C *is a mapping that associates a relation* $C(v, v') \in 2^{\mathsf{B}}$ *with each pair* (v, v') *of* $V \times V$. C *is such that* $C(v, v) = \{\mathsf{Id}\}$ *and* $C(v, v') = (C(v', v))^{-1}$.

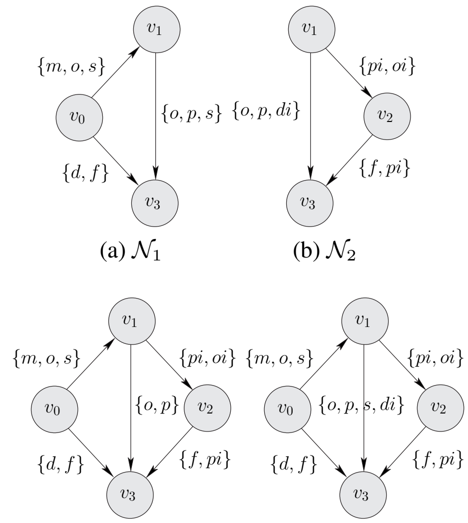
Let $\mathcal{N} = (V, C)$ be a QCN. For all $v, v' \in V$, the relation $C(v, v')$ will be also denoted by $\mathcal{N}[v, v']$. $\mathcal{N}_{[v,v']/r}$



(a) $\mathcal{N}_0$ (b) $\mathcal{S}_0$

(c) σ

Figure 2: (a) A QCN $\mathcal{N}_0$ of IA, (b) a consistent scenario $\mathcal{S}_0$ of $\mathcal{N}_0$ and (c) a solution σ of $\mathcal{N}_0$ for which the corresponding scenario is $\mathcal{S}_0$.



(a) $\mathcal{N}_1$ (b) $\mathcal{N}_2$

(c) $\mathcal{N}_3$ (d) $\mathcal{N}_4$

Figure 3: Four QCNs $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$ and $\mathcal{N}_4$ of IA such that $\mathcal{N}_3 = \mathcal{N}_1 \sqcup \mathcal{N}_2$ and $\mathcal{N}_4 = \mathcal{N}_1 \cup \mathcal{N}_2$.

with $v, v' \in V$ and $r \in 2^{\mathsf{B}}$, is the QCN (V, C') defined by $C'(v, v') = r$, $C'(v', v) = r^{-1}$, and $C'(v'', v^3) = C(v'', v^3)$ for all $(v'', v^3) \in V \times V \setminus \{(v, v'), (v', v)\}$. An *instantiation* of V is a mapping σ defined from V to D. A *solution* σ of $\mathcal{N}$ is an instantiation of V such that for every pair (v, v') of variables in V, $(\sigma(v), \sigma(v'))$ satisfies $C(v, v')$, i.e., there exists a base relation $b \in C(v, v')$ such that $(\sigma(v), \sigma(v')) \in b$. $\mathcal{N}$ is *consistent* iff it admits a solution. Let $\mathcal{N} = (V, C)$ and $\mathcal{N}' = (V', C')$ be two QCNs. Then, $\mathcal{N}'$ is a subQCN of $\mathcal{N}$ iff $V' \subseteq V$ and for all $v, v' \in V'$, $C'(v, v') \subseteq C(v, v')$. We write $\mathcal{N}' \subseteq \mathcal{N}$ in such cases. In particular cases where $\mathcal{N}' \neq \mathcal{N}$, we use the notation $\mathcal{N}' \subset \mathcal{N}$. A *scenario* $\mathcal{S}$ is a QCN whose constraints are defined by one, and exactly one, base relation for all pairs of variables of the QCN. A scenario $\mathcal{S} = (V', C')$ of $\mathcal{N} = (V, C)$ is a scenario which is a sub-QCN of $\mathcal{N}$ and such that $V' = V$.

For illustration, consider the two QCNs $\mathcal{N}_0$ and $\mathcal{S}_0$ depicted in Figure 2. On each of these QCNs, a variable is represented by a node, and a constraint by an edge labeled with the associated relation. For the sake of simplicity, an edge labeled by the universal relation (B), or linking a same node or two nodes for which there is already another edge, is omitted. The QCN $\mathcal{N}_0$ is a consistent QCN of IA. The QCN $\mathcal{S}_0$ is a consistent scenario of $\mathcal{N}$. Figure 2c describes a solution of $\mathcal{N}_0$. The corresponding consistent scenario of σ is $\mathcal{S}_0$.

3 Operations on QCNs

In this section, we introduce some basic operations on QCNs that are useful in the sequel for defining our framework for measuring QCN inconsistency.

Amalgamation of QCNs Let us begin by introducing the intersection, optimistic union and pessimistic union operations.

Definition 2 *Let $\mathcal{N} = (V, C)$ and $\mathcal{N}' = (V', C')$ be two* QCN*s. The* intersection *of $\mathcal{N}$ and $\mathcal{N}'$, denoted by $\mathcal{N} \cap \mathcal{N}'$, is the* QCN *$\mathcal{N}'' = (V \cap V', C'')$ s.t. for all $v, v' \in V \cap V'$, $C''(v, v') = C(v, v') \cap C'(v, v')$.*

That is, given two QCNs $\mathcal{N}$ and $\mathcal{N}'$, it is clear that $\mathcal{N} \cap \mathcal{N}' \subseteq \mathcal{N}$ and $\mathcal{N} \cap \mathcal{N}' \subseteq \mathcal{N}'$. We now define operations for combining two QCNs to obtain a QCN in which the set of variables corresponds to the union of the variables of these two QCNs. Notice that the two operations that we define differ in the way to handle the common constraints.

Definition 3 *Let $\mathcal{N} = (V, C)$ and $\mathcal{N}' = (V', C')$ be two* QCN*s. The* optimistic union *and the* pessimistic union *of $\mathcal{N}$ and $\mathcal{N}'$, denoted respectively by $\mathcal{N} \sqcup \mathcal{N}'$ and $\mathcal{N} \cup \mathcal{N}'$, are respectively the* QCN*s $\mathcal{N}^1 = (V^1, C^1)$ and $\mathcal{N}^2 = (V^2, C^2)$ defined as follows:*

- $V^1 = V^2 = V \cup V'$,
- $\forall\, v, v' \in V \setminus V'$, $C^1(v, v') = C^2(v, v') = C(v, v')$,
- $\forall\, v, v' \in V' \setminus V$, $C^1(v, v') = C^2(v, v') = C'(v, v')$,
- $\forall\, v, v' \in V$ *s.t.* $v \in V \setminus V'$ *and* $v' \in V' \setminus V$, *or* $v' \in V \setminus V'$ *and* $v \in V' \setminus V$, $C^1(v, v') = C^2(v, v') = \mathsf{B}$,
- $\forall v, v' \in V' \cap V$, $C^1(v, v') = C(v, v') \cap C'(v, v')$ *and* $C^2(v, v') = C(v, v') \cup C'(v, v')$.

In order to illustrate the above notions, let us consider the QCNs of IA $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ and $\mathcal{N}_4$ described in Figure 3. Then, we have $\mathcal{N}_3 = \mathcal{N}_1 \sqcup \mathcal{N}_2$, and $\mathcal{N}_4 = \mathcal{N}_1 \cup \mathcal{N}_2$.

Clearly, for two QCNs $\mathcal{N}$ and $\mathcal{N}'$ we have $\mathcal{N} \cap \mathcal{N}' \subseteq \mathcal{N} \sqcup \mathcal{N}' \subseteq \mathcal{N} \cup \mathcal{N}'$. Moreover, $\mathcal{N} \subseteq \mathcal{N} \cup \mathcal{N}'$ and $\mathcal{N}' \subseteq \mathcal{N} \cup \mathcal{N}'$.

The next definition describes a property that concerns particular amalgamations of QCNs:

Definition 4 *Given a set of relations $\mathcal{R} \subseteq 2^{\mathsf{B}}$, three* QCN*s $\mathcal{N} = (V, C)$, $\mathcal{N}' = (V', C')$ and $\mathcal{N}'' = (V'', C'')$. $\mathcal{N}''$ is said to be an* amalgamation *of $\mathcal{N}$ and $\mathcal{N}'$ through $\mathcal{R}$, denoted by $\mathcal{N}'' = \mathcal{N} \uplus^{\mathcal{R}} \mathcal{N}'$, iff $V \cap V' = \emptyset$, $V'' = V \cup V'$, and for all $v, v' \in V$, $C''(v, v') = C(v, v')$, for all $v, v' \in V'$, $C''(v, v') = C'(v, v')$ and for all $v \in V$, $v' \in V'$, $C''(v, v') \in \mathcal{R}$.*

In other words, $\mathcal{N}'' = \mathcal{N} \uplus^{\mathcal{R}} \mathcal{N}'$ means that $\mathcal{N}''$ is composed of two disjoint QCNs $\mathcal{N}$ and $\mathcal{N}'$ related by relations belonging to $\mathcal{R}$. Intuitively, $\uplus$ can be seen as a variant of the standard disjoint union.

Relaxations and Restrictions of QCNs We now turn to the notion of constraint relaxations in QCN which is a transformation paradigm to recover consistency for inconsistent QCN. As in the context of general CSPs (Constraint Satisfaction Problems), we can regard a constraint relaxation (C-relaxation, for short) of QCN as a QCN obtained by replacing some constraints with weaker ones. Several different relaxations have been proposed in the literature. For example, we can relax a disjunctive constraint to its conceptual neighbourhood as suggested in (Li and Li 2013). Now, we proceed by describing three forms of C-relaxations: a constraint can be enlarged by adding at most one relation on its relation, a constraint can be enlarged by adding an arbitrary number of relations on its relation, or, a constraint can be enlarged by adding all the missing base relations of B.

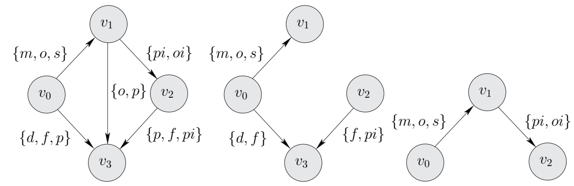
More formally, these different ways of relaxing QCN are defined in the following manner:

Definition 5 (C-Relaxations) *Let $\mathcal{N} = (V, C)$ be a* QCN*. Then:*

- *a* C-relaxation *of $\mathcal{N}$ is a* QCN *$\mathcal{N}' = (V', C')$ s.t. $V = V'$ and $\mathcal{N} \subseteq \mathcal{N}'$.*
- *a* unit C-relaxation *of $\mathcal{N}$ is a C-relaxation of $\mathcal{N}$ s.t. for all $v, v' \in V$, $|C'(v, v') \setminus C(v, v')| \leqslant 1$.*
- *a* trivial C-relaxation *of $\mathcal{N}$ is a* QCN *$\mathcal{N}' = (V, C')$ s.t. $\mathcal{N}'$ is C-relaxation of $\mathcal{N}$ and for all $v, v' \in V$, if $C'(v, v') \neq C(v, v')$ then $C'(v, v') = \mathsf{B}$.*

We use $\mathsf{CCR}(\mathcal{N})$, $\mathsf{CCR}^{\mathsf{U}}(\mathcal{N})$ and $\mathsf{CCR}^{*}(\mathcal{N})$ to denote the set of consistent C-relaxations, the set of unit consistent C-relaxations, and the set of trivial consistent C-relaxations of $\mathcal{N}$, respectively. In addition, $\mathcal{N}'$ is said to be a *minimal consistent C-relaxation* (resp. *minimal consistent trivial C-relaxation*) of $\mathcal{N}$ if $\mathcal{N}'$ is a consistent C-relaxation (resp. trivial C-relaxation) of $\mathcal{N}$ and there exists no $\mathcal{N}''$ such that $\mathcal{N}'' \subset \mathcal{N}$ and $\mathcal{N}'' \in \mathsf{CCR}(\mathcal{N})$ (resp. $\mathcal{N}'' \in \mathsf{CCR}^{*}(\mathcal{N})$). $\mathcal{N}'$ is said to be a *maximal inconsistent trivial C-relaxation* of $\mathcal{N}$ if it is an inconsistent trivial C-relaxation of $\mathcal{N}$ and, for all trivial C-relaxation $\mathcal{N}''$ of $\mathcal{N}$ with $\mathcal{N}' \subset \mathcal{N}''$, $\mathcal{N}''$ is a consistent QCN. Intuitively, the notion of maximal inconsistent trivial C-relaxation is similar to that of minimal inconsistent subset in the case of propositional setting.

Given a QCN $\mathcal{N} = (V, C)$ and a subset of variables $V' \subseteq V$, the *projection* of $\mathcal{N}$ on V', called also variable restriction (V-restriction in short), is defined as a QCN on V' having the same constraints that $\mathcal{N}$ has on the variables in V'.



(a) $\mathcal{N}_5$ (b) $\mathcal{N}_6$ (c) $\mathcal{N}_7$

Figure 4: Three QCNs $\mathcal{N}_5$, $\mathcal{N}_6$, $\mathcal{N}_7$ with $\mathcal{N}_5$ a consistent unit C-relaxation of $\mathcal{N}_3$, $\mathcal{N}_6$ a consistent trivial C-relaxation of $\mathcal{N}_3$ and $\mathcal{N}_7$ a consistent V-restriction of $\mathcal{N}_3$.

Definition 6 (V-Restrictions) *Let $\mathcal{N} = (V, C)$ and $\mathcal{N}' = (V', C')$ be two* QCN*s. We say that $\mathcal{N}'$ is a* V-restriction *of $\mathcal{N}$ if $V' \subseteq V$, and $C(v, v') = C'(v, v')$ for every $v, v' \in V'$. We use $\mathcal{N}_{\downarrow V'}$ to denote the V-restriction of $\mathcal{N}$ on $V' \subseteq V$.*

The set of consistent V-restrictions of $\mathcal{N}$ is denoted by $\mathsf{CVR}(\mathcal{N})$. Moreover, $\mathcal{N}'$ is said to be a *minimal inconsistent V-restriction* of $\mathcal{N}$ if $\mathcal{N}'$ is a V-restriction of $\mathcal{N}$ and, for all V-restriction $\mathcal{N}''$ of $\mathcal{N}$ with $\mathcal{N}'' \subset \mathcal{N}'$, $\mathcal{N}''$ is a consistent QCN.

For instance, we consider the three QCNs $\mathcal{N}_5$, $\mathcal{N}_6$ and $\mathcal{N}_7$ described in Figure 4 and again the QCN $\mathcal{N}_3$ depicted in Figure 3. As one can see, $\mathcal{N}_5$, $\mathcal{N}_6$ and $\mathcal{N}_7$ are consistent, whereas $\mathcal{N}_3$ is inconsistent. Indeed, the sub-QCN of $\mathcal{N}_3$ corresponding to the variables (v_1, v_2, v_3) is inconsistent. Moreover, it holds that $\mathcal{N}_5$, $\mathcal{N}_6$ and $\mathcal{N}_7$ are respectively a unit C-relaxation, a trivial C-relaxation and a V-restriction of $\mathcal{N}_3$.

4 Rational Postulates for Inconsistency Measures

An inconsistency measure I is a function that maps a (possibly inconsistent) QCN onto a non-negative real value, i.e., an inconsistency measure I is a function $I : \mathsf{QCN} \longrightarrow [0, \infty]$.

In order to formalize the intuition behind inconsistency measures, we propose a list of postulates that should be satisfied by any reasonable/desired inconsistency measure. These postulates are also useful for comparing inconsistency measures.

In the sequel, we make the following requirements for any inconsistency measure.

Definition 7 (Basic Postulates) *Let $\mathcal{N} = (V, C)$ be a* QCN*.*

P1. *$I(\mathcal{N}) = 0$ iff $\mathcal{N}$ is a consistent* QCN *(Consistency Null - CN);*

P2. *For all $V' \subseteq V$, $I(\mathcal{N}_{\downarrow V'}) \leqslant I(\mathcal{N})$ (Variables Monotonicity - VM).*

The postulates introduced in Definition 7 are related to postulates in the basic system introduced in (Hunter and Konieczny 2010). The postulate Consistency Null ensures that all and only consistent QCNs get 0 as amount of inconsistency. In the same way as Hunter and Konieczny's Consistency postulate in the basic system for measuring inconsistency in propositional logic, this property expresses that every inconsistency measure should be able to discriminate between consistency and inconsistency. Regarding the postulate **P2**, it is similar to Hunter and Konieczny's Monotonicity postulate, since Variables Monotonicity expresses the fact that adding constraints does not decrease the amount of inconsistency.

We now introduce notions that are used in the definition of additional postulates.

Definition 8 (Free Constraint) *Let $\mathcal{N} = (V, C)$ be a* QCN *and $v, v' \in V$. The pair of variables $\{v, v'\}$ is said to be a* free constraint *in $\mathcal{N}$ if for all minimal consistent C-relaxation $\mathcal{N}' = (V, C')$, $C'(v, v') = C(v, v')$ holds.*

Definition 9 (T-Free Constraint) *Let $\mathcal{N} = (V, C)$ be a* QCN *and $v, v' \in V$. The pair of variables $\{v, v'\}$ is said to be a* T-free constraint *in $\mathcal{N}$ if for all minimal consistent trivial C-relaxation $\mathcal{N}' = (V, C')$, $C'(v, v') = C(v, v')$ holds.*

We use $\mathsf{FC}(\mathcal{N})$ (resp. $\mathsf{FC}^*(\mathcal{N})$) to denote the set of free constraints (resp. T-free constraints) of $\mathcal{N}$.

Proposition 1 *Given a* QCN *$\mathcal{N}$, we have $\mathsf{FC}(\mathcal{N}) \subseteq \mathsf{FC}^*(\mathcal{N})$.*

Proof: Let $\mathcal{N} = (V, C)$ be a QCN and $\{v, v'\} \in \mathsf{FC}(\mathcal{N})$. Assume that $\{v, v'\} \notin \mathsf{FC}^*(\mathcal{N})$. Then, there exists a minimal consistent trivial C-relaxation $\mathcal{N}' = (V, C')$ of $\mathcal{N}$ s. t. $C'(v, v') \neq C(v, v')$. Clearly, there exists a minimal consistent C-relaxation $\mathcal{N}'' = (V, C'')$ of $\mathcal{N}$ s.t. $\mathcal{N}'' \subseteq \mathcal{N}'$. Thus, we get $C''(v, v') = C(v, v')$, since $\{v, v'\} \in \mathsf{FC}(\mathcal{N})$. Let $\mathcal{N}^3 = (V, C^3)$ be a trivial C-relaxation of $\mathcal{N}$ defined from $\mathcal{N}''$ by: for all $v_1, v_2 \in V$, $C^3(v_1, v_2) = \mathsf{B}$ if $C''(v_1, v_2) \neq C(v_1, v_2)$; $C^3(v_1, v_2) = C(v_1, v_2)$ otherwise. One can see that $\mathcal{N}^3$ is consistent, since $\mathcal{N}''$ is consistent. Moreover, we get $\mathcal{N}^3 \subset \mathcal{N}'$, since $\mathcal{N}'' \subseteq \mathcal{N}'$ and $C^3(v, v') = C''(v, v') = C(v, v')$. Thus, we get a contradiction, since $\mathcal{N}'$ is a minimal consistent trivial C-relaxation. $\square$

Let us now show that $\mathsf{FC}(\mathcal{N})$ is not always equal to $\mathsf{FC}^*(\mathcal{N})$. To this end, we provide an example in IA.

Example 1 *Let $\mathcal{N} = (\{v_1, v_2, v_3\}, C)$ be a* QCN *in* IA *s.t. $C(v_1, v_2) = \emptyset$, $C(v_2, v_3) = \{pi\}$ and $C(v_3, v_1) = \{pi\}$. Clearly, the* QCN *$\mathcal{N}' = (\{v_1, v_2, v_3\}, C')$, with $C'(v_1, v_2) = \{eq\}$, $C'(v_2, v_3) = \{pi, eq\}$ and $C'(v_3, v_1) = \{pi, eq\}$, is a minimal consistent C-relaxation of $\mathcal{N}$. Thus, we get $\mathsf{FC}(\mathcal{N}) = \emptyset$, since all the constraints are relaxed. However, there is a unique minimal consistent trivial C-relaxation: $\mathcal{N}'' = (\{v_1, v_2, v_3\}, C'')$ with $C(v_1, v_2) = \mathsf{B}_{\mathsf{IA}}$, $C(v_2, v_3) = \{pi\}$ and $C(v_3, v_1) = \{pi\}$. It follows that $\mathsf{FC}^*(\mathcal{N}) = \{\{v_2, v_3\}, \{v_1, v_3\}\}$.*

The notion of T-free constraint is similar to that of free formula in the case of propositional logic (Hunter and Konieczny 2010). Let us recall that a free formula in a knowledge base is a formula that does not belong to any minimal inconsistent subsets of this knowledge base. In the case of QCNs, the notion of minimal inconsistent subset of constraints can be associated to that of maximal inconsistent trivial C-relaxation. Indeed, the constraints having relations different from B by a maximal inconsistent trivial C-relaxation corresponds to minimal inconsistent subset of constraints.

Proposition 2 *Let $\mathcal{N} = (V, C)$ be a* QCN *and $\{v, v'\} \in \mathsf{FC}^*(\mathcal{N})$. Then, for all maximal inconsistent trivial C-relaxation $\mathcal{N}' = (V, C')$ of $\mathcal{N}$, $C'(v, v') = \mathsf{B}$ holds.*

Proof: Assume that there exists a maximal inconsistent trivial C-relaxation $\mathcal{N}' = (V, C')$ of $\mathcal{N}$ s.t. $C'(v, v') \neq \mathsf{B}$. We thus know that $C(v, v') \neq B$. Moreover, we know that the trivial C-relaxation $\mathcal{N}'' = (V, C'')$ of $\mathcal{N}$ is consistent, where $C''(v, v') = \mathsf{B}$, and $C''(v_1, v_2) = C'(v_1, v_2)$ for every $\{v_1, v_2\} \neq \{v, v'\}$. Then, there exists a minimal consistent trivial C-relaxation $\mathcal{N}^3 = (V, C^3)$ of $\mathcal{N}$ s.t. $\mathcal{N}^3 \subseteq \mathcal{N}''$ and $C^3(v, v') = \mathsf{B}$. Thus, we get a contradiction, since $\{v, v'\} \in \mathsf{FC}^*(\mathcal{N})$. $\square$

After introducing the notions of free constraint, we now define a similar notion for variables.

Definition 10 (Free Variable) *Let $\mathcal{N} = (V, C)$ be a* QCN *and $v \in V$. The variable v is said to be a* free variable *in $\mathcal{N}$ if, for all minimal inconsistent V-restriction $\mathcal{N}' = (V', C')$, $v \notin V'$ holds.*

We use $\mathsf{FV}(\mathcal{N})$ to denote the set of free variables of $\mathcal{N}$.

In this work, we also consider the following postulates for every two QCNs $\mathcal{N} = (V, C)$ and $\mathcal{N}' = (V', C')$:

- **P3.** If $V = V'$ and $\mathcal{N} \subseteq \mathcal{N}'$ then $I(\mathcal{N}) \geqslant I(\mathcal{N}')$ (Relation Monotonicity - CM).
- **P4.** For all $\{v, v'\} \in \mathsf{FC}(\mathcal{N})$, $I(\mathcal{N}) = I(\mathcal{N}_{[v,v']/\mathsf{B}})$ (Free Constraint Independence - FCI).
- **P5.** For all $\{v, v'\} \in \mathsf{FC}^*(\mathcal{N})$, $I(\mathcal{N}) = I(\mathcal{N}_{[v,v']/\mathsf{B}})$ (T-Free Constraint Independence - TFCI).
- **P6.** For all $v \in \mathsf{FV}(\mathcal{N})$, $I(\mathcal{N}) = I(\mathcal{N}_{\downarrow(V \setminus \{v\})})$ (Free Variable Independence - FVI).

The postulate **P3** can be seen as an adaptation of the postulate of Dominance in the case of propositional logic (Hunter and Konieczny 2010). Indeed, we have $C(v, v') \subseteq C'(v, v')$ for every two variables v and v', and that means that the truth of $C(v, v')$ implies the truth of $C'(v, v')$. Thus, following Dominance, replacing $C(v, v')$ with $C'(v, v')$ does not increase the amount of inconsistency, which means that the amount of inconsistency in $\mathcal{N}$ is greater than or equal to that in $\mathcal{N}'$. Moreover, the postulate **P4** expresses that the amount of inconsistency does not change by ignoring the constraints that are not involved in any minimal consistent C-relaxation. This postulate can be seen as a restriction of the postulate in the basic system called Free Formula Independence. Indeed, as we said before, the free constraints of a QCN are included in the set of its T-free constraints (see Proposition 2), and the notion of T-free constraint in QCNs corresponds to that of free formula in the case of propositional logic (see Proposition 2). Thus, the postulate **P5** is similar to that of Free Formula independence. Moreover, using Proposition 1, we know that **P5** is stronger than **P4**, i.e., TFCI implies FCI. Regarding the postulate **P6**, it is an adaptation of Free Formula Independence postulate to the variables of QCNs.

We do not consider **P3**, **P4**, **P5** and **P6** as basic postulates because we think that they can not be required for every inconsistency measure for QCNs. In a sense, we follow Besnard's objections against Dominance and Free Formula Independence in the case of propositional logic (Besnard

2014). As an example, let us provide an objection against Relation Monotonicity. To this end, we consider the natural inconsistency measure which is similar to that defined as the number of minimal inconsistent subsets (Hunter and Konieczny 2010). We expect that the equivalent inconsistency measure for QCNs has to satisfy all the basic postulates. In what follows, however, we will show that this measure does not satisfy Relation Monotonicity. As we said before, the notion of maximal inconsistent trivial C-relaxation is the equivalent notion of minimal inconsistent subset in QCNs. We use I_{MTR} to denote the natural inconsistency measure defined as the number of maximal inconsistent trivial C-relaxations. Let $\mathcal{N} = (\{v_1, v_2, v_3, v_4\}, C)$ be a QCN of IA such that $C(v_1, v_2) = \emptyset$, $C(v_2, v_3) = \{pi\}$, $C(v_3, v_1) = \{pi\}$, $C(v_2, v_4) = \{pi\}$, $C(v_4, v_1) = \{pi\}$ and $C(v_3, v_4) = \mathsf{B}_{\mathsf{IA}}$. This QCN has a unique maximal inconsistent trivial C-relaxation, which is $\mathcal{N}_0 = (V, C_0)$ where $C_0(v_1, v_2) = \emptyset$ and $C_0(v, v') = \mathsf{B}_{\mathsf{IA}}$ for all the other pairs of variables. As a consequence, we get $I_{\mathsf{MTR}}(\mathcal{N}) = 1$. Let us consider now the QCN $\mathcal{N}' = (V, C')$ where $C'(v_1, v_2) = \{pi\}$, $C'(v_2, v_3) = \{pi\}$, $C'(v_3, v_1) = \{pi\}$, $C'(v_2, v_4) = \{pi\}$, $C'(v_4, v_1) = \{pi\}$ and $C'(v_3, v_4) = \mathsf{B}_{\mathsf{IA}}$. Clearly, we have $\mathcal{N} \subset \mathcal{N}'$. Moreover, $\mathcal{N}'$ admits two maximal inconsistent trivial C-relaxation:

- $\mathcal{N}_1 = (V, C_1)$ where $C_1(v_1, v_2) = \{pi\}$, $C_1(v_2, v_3) = \{pi\}$, $C_1(v_3, v_1) = \{pi\}$, $C_1(v_2, v_4) = \mathsf{B}_{\mathsf{IA}}$, $C_1(v_4, v_1) = \mathsf{B}_{\mathsf{IA}}$ and $C_1(v_3, v_4) = \mathsf{B}_{\mathsf{IA}}$,
- $\mathcal{N}_2 = (V, C_2)$ where $C_2(v_1, v_2) = \{pi\}$, $C_1(v_2, v_3) = \mathsf{B}_{\mathsf{IA}}$, $C_1(v_3, v_1) = \mathsf{B}_{\mathsf{IA}}$, $C_1(v_2, v_4) = \{pi\}$, $C_1(v_4, v_1) = \{pi\}$ and $C_1(v_3, v_4) = \mathsf{B}_{\mathsf{IA}}$.

Thus, we get $I_{\mathsf{MTR}}(\mathcal{N}') = 2$ and, consequently, $I_{\mathsf{MTR}}(\mathcal{N}) < I_{\mathsf{MTR}}(\mathcal{N}')$. Therefore, I_{MTR} does not satisfy Relation Monotonicity.

However, we are not arguing that **P3**, **P4**, **P5** and **P6** are not needed for any inconsistency measure for QCNs. Indeed, we think that they are suitable in several contexts, especially, when the primitive conflicts are identified through consistent C-relaxations. For instance, Relation Monotonicity expresses the fact that the amount of inconsistency does not increase when we replace some constraints with weaker constraints, and this makes sense in numerous cases.

Let two consistent QCNs $\mathcal{N}$, $\mathcal{N}'$ and, a QCN $\mathcal{N}''$ such that $\mathcal{N}'' = \mathcal{N} \uplus^{\mathcal{R}} \mathcal{N}'$ with $\mathcal{R} \subseteq 2^{\mathsf{B}}$. For particular sets $\mathcal{R}$ we can show that two consistent scenarios of $\mathcal{N}$ and $\mathcal{N}'$ can be extended in order to obtain a consistent scenario of $\mathcal{N}''$. This property which concerns the set $\mathcal{R}$ is formally defined in the following manner:

Definition 11 *Let $\mathcal{R} \subseteq 2^{\mathsf{B}}$ be a subset of relations. $\mathcal{R}$ has the* consistent scenario separation *property, in short the $\mathcal{CSS}$ property, iff for all* QCN*s $\mathcal{N}$, $\mathcal{N}'$, $\mathcal{N}''$ with $\mathcal{N}'' = \mathcal{N} \uplus^{\mathcal{R}} \mathcal{N}'$, we have for all consistent scenarios $\mathcal{S}$, $\mathcal{S}'$ of $\mathcal{N}$ and $\mathcal{N}'$ respectively, $(\mathcal{S} \sqcup \mathcal{S}') \sqcup \mathcal{N}''$ is consistent.*

For instance, consider the set of relations of IA $\mathcal{R}_p$ defined by $\mathcal{R}_p = \{r \in 2^{\mathsf{B}_{\mathsf{IA}}} : p \in r\}$. We can easily show that this set has the $\mathcal{CSS}$ property, since a consistent scenario can be obtained by selecting p for each relation in $\mathcal{R}_p$.

From this property, we define the following postulate concerning a QCN inconsistency measure:

Definition 12

P7. *Let a set of relations $\mathcal{R} \subseteq 2^{\mathsf{B}}$ having the $\mathcal{CSS}$ property and* QCN*s $\mathcal{N}, \mathcal{N}', \mathcal{N}''$ such that $\mathcal{N}'' = \mathcal{N} \uplus^{\mathcal{R}} \mathcal{N}'$. We say that an inconsistency measure I satisfies the property of $\mathcal{CSS}$*-additivity *if we have $I(\mathcal{N}'') = I(\mathcal{N}) + I(\mathcal{N}')$ ($\mathcal{CSS}$-Additivity).*

In a sense, the $\mathcal{CSS}$-Additivity property can be seen as a variant of Thimm's postulate, called Super-Additivity, introduced in (Thimm 2013).

5 A C-relaxation Based Inconsistency Measure

We here introduce a first inconsistency measure defined from the notion of trivial C-relaxation. More precisely, it is defined as the number of constraints that we need to alter in order to get a consistent qualitative constraint network.

Given two QCNs $\mathcal{N} = (V, C)$ and $\mathcal{N}' = (V, C')$, we use $\#\mathsf{diffC}(\mathcal{N}, \mathcal{N}')$ to denote the number of constraints that differ between $\mathcal{N}$ and $\mathcal{N}'$, i.e., $\mathcal{N}' = (V, C') = \frac{1}{2}.|\{(v, v') \in V \times V : C(v, v') \neq C'(v, v')\}|$.

Definition 13 (I_{CCR} Inconsistency Measure) *The inconsistency measure I_{CCR} is defined by:*

$$I_{\mathsf{CCR}}(\mathcal{N}) = \min\{\#\mathsf{diffC}(\mathcal{N}, \mathcal{N}') : \mathcal{N}' \in \mathsf{CCR}(\mathcal{N})\}$$

The inconsistency measure I_{CCR} captures the minimum effort needed in terms of constraint relaxation to recover consistency.

Note that we can show that for a QCN $\mathcal{N}$, the inconsistency measure I_{CCR} can be easily calculated from a solution of the problem MAX-QCN on $\mathcal{N}$ which consists in finding a consistent scenario on the variables of $\mathcal{N}$ that maximizes the number of constraints in $\mathcal{N}$(Condotta et al. 2015).

The previous definition considers the consistent C-relaxations of the considered QCN, nevertheless, in an equivalent manner one could use the more restrictive sets corresponding to the consistent unit C-relaxations or the consistent trivial C-relaxations. Indeed, we have the following results:

Proposition 3 *Let $\mathcal{N} = (V, C)$ be a* QCN*. Then, we have:*

(a) $I_{\mathsf{CCR}}(\mathcal{N}) = \min\{\#\mathsf{diffC}(\mathcal{N}, \mathcal{N}') : \mathcal{N}' \in \mathsf{CCR}^{\mathsf{U}}(\mathcal{N})\}$.

(b) $I_{\mathsf{CCR}}(\mathcal{N}) = \min\{\#\mathsf{diffC}(\mathcal{N}, \mathcal{N}') : \mathcal{N}' \in \mathsf{CCR}^{*}(\mathcal{N})\}$.

Proof:(Sketch) Let $\mathcal{N}' = (V, C')$ be a consistent C-relaxation of $\mathcal{N}$ such that $\#\mathsf{diffC}(\mathcal{N}, \mathcal{N}') = I_{\mathsf{CCR}}(\mathcal{N})$. Let $\mathcal{S}$ be a consistent scenario of $\mathcal{N}'$. We can show that for all $v, v' \in V$, $\mathcal{S}[v, v'] \subseteq \mathcal{N}$ iff $\mathcal{N}'[v, v'] = \mathcal{N}[v, v']$. In the contrary case, it would exist a consistent C-relaxation of $\mathcal{N}''$ such that $\#\mathsf{diffC}(\mathcal{N}, \mathcal{N}'') < I_{\mathsf{CCR}}(\mathcal{N})$. From the consistent scenario $\mathcal{S}$ we define the two QCNs $\mathcal{N}^1 = (V, C^1)$ and $\mathcal{N}^2 = (V, C^2)$, by respectively :

- for all $v, v' \in V$, $\mathcal{N}^1[v, v'] = \mathcal{N}[v, v'] \cup \mathcal{S}[v, v']$;
- for all $v, v' \in V$, $\mathcal{N}^2[v, v'] = \mathcal{N}[v, v']$ if $\mathcal{S}[v, v'] \subseteq \mathcal{N}[v, v']$, $\mathcal{N}^2[v, v'] = \mathsf{B}$ otherwise.

Note that $\mathcal{N}^1$ is a consistent unit C-relaxation of $\mathcal{N}$ whereas $\mathcal{N}^2$ is a consistent trivial C-relaxation of $\mathcal{N}$. We also have $\#\mathsf{diffC}(\mathcal{N}, \mathcal{N}') = \#\mathsf{diffC}(\mathcal{N}, \mathcal{N}^1) = \#\mathsf{diffC}(\mathcal{N}, \mathcal{N}^2)$.

Moreover, we can show that $\#\mathsf{diffC}(\mathcal{N},\mathcal{N}^1) = \min\{\#\mathsf{diffC}(\mathcal{N},\mathcal{N}'') : \mathcal{N}'' \in \mathsf{CCR}^{\mathsf{U}}(\mathcal{N})\}$ and $\#\mathsf{diffC}(\mathcal{N},\mathcal{N}^2) = \min\{\#\mathsf{diffC}(\mathcal{N},\mathcal{N}'') : \mathcal{N}'' \in \mathsf{CCR}^{\mathsf{U}}(\mathcal{N})\}$. The non-satisfaction of one of these two equalities would lead to a contradiction. More precisely, they would lead to the existence of a consistent C-relaxation of $\mathcal{N}''$ such that $\#\mathsf{diffC}(\mathcal{N},\mathcal{N}'') < I_{\mathsf{CCR}}(\mathcal{N})$. From all this, $I_{\mathsf{CCR}}(\mathcal{N}) = \min\{\#\mathsf{diffC}(\mathcal{N},\mathcal{N}') : \mathcal{N}' \in \mathsf{CCR}^{\mathsf{U}}(\mathcal{N})\} = \min\{\#\mathsf{diffC}(\mathcal{N},\mathcal{N}') : \mathcal{N}' \in \mathsf{CCR}^*(\mathcal{N})\}$ holds. □

Now, we study the inconsistency measure I_{CCR} with respect to the different postulates defined previously. First, we establish a result that will be use in the sequel.

Proposition 4 *Let $\mathcal{N},\mathcal{N}',\mathcal{N}''$ be three* QCN*s defined on a same set of variables V. If $\mathcal{N} \subseteq \mathcal{N}'$ then $\#\mathsf{diffC}(\mathcal{N},\mathcal{N}'') \geqslant \#\mathsf{diffC}(\mathcal{N}',\mathcal{N}'\cup\mathcal{N}'')$.*

Proof: Suppose that $\mathcal{N} \subseteq \mathcal{N}'$. Let $v, v' \in V$. As $\mathcal{N} \subseteq \mathcal{N}' \subseteq \mathcal{N}'\cup\mathcal{N}''$, we can assert that $\mathcal{N}[v,v'] \subseteq \mathcal{N}'[v,v']$ and $\mathcal{N}''[v,v'] \subseteq (\mathcal{N}'\cup\mathcal{N}'')[v,v']$. Now, suppose that $\mathcal{N}'[v,v'] \neq (\mathcal{N}'\cup\mathcal{N}'')[v,v']$. Then, using $\mathcal{N}'[v,v'] \subseteq (\mathcal{N}'\cup\mathcal{N}'')[v,v']$, we have that $(\mathcal{N}'[v,v'] \setminus \mathcal{N}''[v,v']) \neq \emptyset$. Since $\mathcal{N}[v,v'] \subseteq \mathcal{N}'[v,v']$, we have $(\mathcal{N}[v,v'] \setminus \mathcal{N}''[v,v']) \neq \emptyset$. Consequently, $\mathcal{N}[v,v'] \neq \mathcal{N}''[v,v']$. Thus, we can assert that, for all $v, v' \in V$, if $\mathcal{N}'[v,v'] \neq (\mathcal{N}'\cup\mathcal{N}'')[v,v']$ then $\mathcal{N}[v,v'] \neq \mathcal{N}''[v,v']$. Therefore, $\#\mathsf{diffC}(\mathcal{N},\mathcal{N}'') \geqslant \#\mathsf{diffC}(\mathcal{N}',\mathcal{N}'\cup\mathcal{N}'')$ holds. □

In the following proposition, we show that I_{CCR} is a basic inconsistency measure.

Proposition 5 *I_{CCR} satisfies the Consistency Null (**P1**) and Variable Monotonicity (**P2**) postulates.*

Proof:
- **Consistency Null.** Let $\mathcal{N}$ be a consistent QCN. Then we have $\mathcal{N} \in \mathsf{CCR}(\mathcal{N})$ and $\#\mathsf{diffC}(\mathcal{N},\mathcal{N}) = 0$. Since, for all QCN $\mathcal{M} \in \mathsf{CCR}(\mathcal{N})$, we have $\#\mathsf{diffC}(\mathcal{M},\mathcal{N}) \geqslant 0$, we can assert that, for all QCN $\mathcal{M} \in \mathsf{CCR}(\mathcal{N})$, $\#\mathsf{diffC}(\mathcal{M},\mathcal{N}) \geqslant \#\mathsf{diffC}(\mathcal{N},\mathcal{N})$ holds. We can conclude that $I_{\mathsf{CCR}}(\mathcal{N}) = 0$. Now, assume that $I_{\mathsf{CCR}}(\mathcal{N}) = 0$. Then, there exists $\mathcal{M} \in \mathsf{CCR}(\mathcal{N})$ s.t. $\#\mathsf{diffC}(\mathcal{M},\mathcal{N}) = 0$. Since $\#\mathsf{diffC}(\mathcal{M},\mathcal{N}) = 0$ we have $\mathcal{M} = \mathcal{N}$. Hence, $\mathcal{N} \in \mathsf{CCR}(\mathcal{N})$ holds and, consequently, $\mathcal{N}$ is consistent.
- **Variable Monotonicity.** Let $\mathcal{N} = (V, C)$ be a QCN and $V' \subseteq V$. Let $\mathcal{M} \in \mathsf{CCR}(\mathcal{N})$ s.t. $I_{\mathsf{CCR}}(\mathcal{N}) = \#\mathsf{diffC}(\mathcal{N},\mathcal{M})$. One can see that $\mathcal{M}_{\downarrow V'}$ is a consistent C-relaxation of $\mathcal{N}_{\downarrow V'}$. Hence, $\#\mathsf{diffC}(\mathcal{N}_{\downarrow V'},\mathcal{M}_{\downarrow V'}) \geqslant I_{\mathsf{CCR}}(\mathcal{N}_{\downarrow V'})$ holds. Moreover, we have that $\#\mathsf{diffC}(\mathcal{N},\mathcal{M}) \geqslant \#\mathsf{diffC}(\mathcal{N}_{\downarrow V'},\mathcal{M}_{\downarrow V'})$. Consequently, we have $\#\mathsf{diffC}(\mathcal{N},\mathcal{M}) \geqslant I_{\mathsf{CCR}}(\mathcal{N}_{\downarrow V'})$. Thus, $I_{\mathsf{CCR}}(\mathcal{N}) \geqslant I_{\mathsf{CCR}}(\mathcal{N}_{\downarrow V'})$ holds. □

Proposition 6 *I_{CCR} satisfies the Relation Monotonicity (**P3**) and T-Free Constraint Independence (**P5**) postulates.*

Proof:
- **Relation Monotonicity.** Let $\mathcal{N} = (V, C)$ and $\mathcal{N}' = (V, C')$ be two QCNs s.t. $\mathcal{N} \subseteq \mathcal{N}'$. Let $\mathcal{M} \in \mathsf{CCR}(\mathcal{N})$ s.t. $I_{\mathsf{CCR}}(\mathcal{N}) = \#\mathsf{diffC}(\mathcal{M},\mathcal{N})$. Using Proposition 4, we have $\#\mathsf{diffC}(\mathcal{N},\mathcal{M}) \geqslant \#\mathsf{diffC}(\mathcal{N}',\mathcal{N}'\cup\mathcal{M})$. Since $\mathcal{M}$ is a consistent QCN, $\mathcal{N}'\cup\mathcal{M}$ is consistent. Moreover, as $\mathcal{N}' \subseteq \mathcal{N}'\cup\mathcal{M}$, $\mathcal{N}'\cup\mathcal{M}$ is a consistent C-relaxation of $\mathcal{N}'$. Hence, $\#\mathsf{diffC}(\mathcal{N}',\mathcal{N}'\cup\mathcal{M}) \geqslant I_{\mathsf{CCR}}(\mathcal{N}')$ holds. Consequently, we have $I_{\mathsf{CCR}}(\mathcal{N}) = \#\mathsf{diffC}(\mathcal{N},\mathcal{M}) \geqslant \#\mathsf{diffC}(\mathcal{N}',\mathcal{N}'\cup\mathcal{M}) \geqslant I_{\mathsf{CCR}}(\mathcal{N}')$. Thus, $I_{\mathsf{CCR}}(\mathcal{N}) \geqslant I_{\mathsf{CCR}}(\mathcal{N}')$ holds.
- **T-Free Constraint Independence.** Let $\mathcal{N} = (V, C)$ be a QCN and $\{v, v'\} \in \mathsf{FC}^*(\mathcal{N})$. We use $\mathcal{N}'$ to denote the QCN $\mathcal{N}_{[v,v']/\mathsf{B}}$. The case $\mathcal{N} = \mathcal{N}'$ is trivial. We assume in the sequel that $\mathcal{N} \neq \mathcal{N}'$. Using Proposition 6, we know that $I_{\mathsf{CCR}}(\mathcal{N}) \geqslant I_{\mathsf{CCR}}(\mathcal{N}')$. Let us show that $I_{\mathsf{CCR}}(\mathcal{N}) \leqslant I_{\mathsf{CCR}}(\mathcal{N}_{[v,v']/\mathsf{B}})$. Let $\mathcal{M}' \in \mathsf{CCR}^*(\mathcal{N}')$ s.t. $I_{\mathsf{CCR}^*}(\mathcal{N}') = \#\mathsf{diffC}(\mathcal{N}',\mathcal{M}')$. Note that $\mathcal{M}'$ is also a trivial consistent C-relaxation of $\mathcal{N}$. By hypothesis we have $\mathcal{N}[v,v'] \neq \mathcal{N}'[v,v']$. Consequently, $\mathcal{N}[v,v'] \neq \mathcal{M}'[v,v']$. It follows that $\mathcal{M}'$ cannot be a minimal trivial consistent C-relaxation of $\mathcal{N}$ since $\{v, v'\}$ is a free-constraint of $\mathcal{N}$. From all this, we can assert that there exists a QCN $\mathcal{M}$ which is a minimal trivial consistent C-relaxation of $\mathcal{N}$ s.t. $\mathcal{M} \subset \mathcal{M}'$. We have $\#\mathsf{diffC}(\mathcal{N},\mathcal{M}) < \#\mathsf{diffC}(\mathcal{N},\mathcal{M}')$. Moreover, we can see that $\#\mathsf{diffC}(\mathcal{N}',\mathcal{M}') = \#\mathsf{diffC}(\mathcal{N},\mathcal{M}') - 1$. It follows that $\#\mathsf{diffC}(\mathcal{N},\mathcal{M}) \leqslant \#\mathsf{diffC}(\mathcal{N}',\mathcal{M}')$. Thus, $I_{\mathsf{CCR}}(\mathcal{N}) \leqslant I_{\mathsf{CCR}}(\mathcal{N}_{[v,v']/\mathsf{B}})$ holds. □

Let us recall that T-Free Constraint Independence is stronger than Free Constraint Independence (**P4**) (see Proposition 1). As a consequence, we obtain that I_{CCR} satisfies also **P4**.

The following proposition is used to show that I_{CCR} satisfies also $\mathcal{CSS}$-Additivity (**P7**).

Proposition 7 *Let a set of relations $\mathcal{R} \subseteq 2^{\mathsf{B}}$ having the $\mathcal{CSS}$ property and* QCN*s $\mathcal{N} = (V, C)$, $\mathcal{N}' = (V', C')$, $\mathcal{N}''$ such that $\mathcal{N}'' = \mathcal{N} \uplus^{\mathcal{R}} \mathcal{N}'$. For all* QCN *$\mathcal{M} \in \mathsf{CCR}(\mathcal{N}'')$ with $I(\mathcal{N}'') = \#\mathsf{diffC}(\mathcal{N}'',\mathcal{M})$, we have:*

(a) for all $v \in V$ and $v' \in V'$, $\mathcal{M}[v,v'] = \mathcal{N}''[v,v']$;

(b) $\#\mathsf{diffC}(\mathcal{N}'',\mathcal{M}) = \#\mathsf{diffC}(\mathcal{N},\mathcal{M}_{\downarrow V}) + \#\mathsf{diffC}(\mathcal{N}',\mathcal{M}_{\downarrow V'})$.

Proof: Let $\mathcal{M} \in \mathsf{CCR}(\mathcal{N}'')$ s.t. $I(\mathcal{N}'') = \#\mathsf{diffC}(\mathcal{N}'',\mathcal{M})$. Let $\mathcal{M}'$ be the QCN on $V \cup V'$ defined by $\mathcal{M}'_{\downarrow V} = \mathcal{M}_{\downarrow V}$, $\mathcal{M}'_{\downarrow V'} = \mathcal{M}_{\downarrow V'}$ and for all $(v, v') \in V \times V'$, $\mathcal{M}'[v,v'] = \mathcal{N}''][v,v']$. Note that $\mathcal{M}' = \mathcal{M}_{\downarrow V} \uplus^{\mathcal{R}} \mathcal{M}_{\downarrow V'}$ and $\mathcal{N}'' \subseteq \mathcal{M}' \subseteq \mathcal{M}$. Since $\mathcal{M}$ is a consistent QCN, $\mathcal{M}_{\downarrow V}$ and $\mathcal{M}_{\downarrow V'}$ are consistent. Then, using the fact that $\mathcal{R}$ has the $\mathcal{CSS}$ property, we can assert that $\mathcal{M}$ is consistent. Moreover, as $\mathcal{N}'' \subseteq \mathcal{M}' \subseteq \mathcal{M}$, we have $\#\mathsf{diffC}(\mathcal{N}'',\mathcal{M}') \leqslant \#\mathsf{diffC}(\mathcal{N}'',\mathcal{M})$. Consequently, by definition of $\mathcal{M}$, $\#\mathsf{diffC}(\mathcal{N}'',\mathcal{M}') = \#\mathsf{diffC}(\mathcal{N}'',\mathcal{M})$ holds. In other hand, we have $\#\mathsf{diffC}(\mathcal{N}'',\mathcal{M}) = \#\mathsf{diffC}(\mathcal{N}''_{\downarrow V},\mathcal{M}_{\downarrow V}) + \#\mathsf{diffC}(\mathcal{N}''_{\downarrow V'},\mathcal{M}_{\downarrow V'}) + |\{(v, v') \in V \times V' : \mathcal{N}''[v,v'] \neq \mathcal{M}[v,v']\}|$ and $\#\mathsf{diffC}(\mathcal{N}'',\mathcal{M}') = \#\mathsf{diffC}(\mathcal{N}''_{\downarrow V},\mathcal{M}'_{\downarrow V}) + \#\mathsf{diffC}(\mathcal{N}''_{\downarrow V'},\mathcal{M}'_{\downarrow V'}) + |\{(v, v') \in V \times V' : \mathcal{N}''[v,v'] \neq \mathcal{M}'[v,v']\}|$. We know that $\mathcal{M}'_{\downarrow V} = \mathcal{M}_{\downarrow V}$, $\mathcal{M}'_{\downarrow V'} = \mathcal{M}_{\downarrow V'}$ and $\{(v, v') \in V \times V' : \mathcal{N}''[v,v'] \neq \mathcal{M}'[v,v']\} = \emptyset$. Consequently, $\#\mathsf{diffC}(\mathcal{N}'',\mathcal{M}') = \#\mathsf{diffC}(\mathcal{N}''_{\downarrow V},\mathcal{M}_{\downarrow V}) + \#\mathsf{diffC}(\mathcal{N}''_{\downarrow V'},\mathcal{M}_{\downarrow V'})$. We also know that $\#\mathsf{diffC}(\mathcal{N}'',\mathcal{M}') = \#\mathsf{diffC}(\mathcal{N}'',\mathcal{M})$. Hence, we have $\#\mathsf{diffC}(\mathcal{N}'',\mathcal{M}) = \#\mathsf{diffC}(\mathcal{N}''_{\downarrow V},\mathcal{M}_{\downarrow V}) + \#\mathsf{diffC}(\mathcal{N}''_{\downarrow V'},\mathcal{M}_{\downarrow V'}) + |\{(v,v') \in V \times V' : \mathcal{N}''[v,v'] \neq \mathcal{M}[v,v']\}| = \#\mathsf{diffC}(\mathcal{N}''_{\downarrow V},\mathcal{M}_{\downarrow V}) + \#\mathsf{diffC}(\mathcal{N}''_{\downarrow V'},\mathcal{M}_{\downarrow V'})$. From this, we can conclude, on the one hand, that $\{(v,v') \in V \times V' : \mathcal{N}''[v,v'] \neq \mathcal{M}[v,v']\} = \emptyset$ and, on the other hand, that $\#\mathsf{diffC}(\mathcal{N}'',\mathcal{M}) = \#\mathsf{diffC}(\mathcal{N}''_{\downarrow V},\mathcal{M}_{\downarrow V}) + \#\mathsf{diffC}(\mathcal{N}''_{\downarrow V'},\mathcal{M}_{\downarrow V'})$. □

Proposition 8 *I_{CCR} satisfies $\mathcal{CSS}$-Additivity (***P7***).*

Proof: Let a set of relations $\mathcal{R} \subseteq 2^{\mathsf{B}}$ having the $\mathcal{CSS}$ property and QCNs $\mathcal{N} = (V,C)$, $\mathcal{N}' = (V',C')$, $\mathcal{N}''$ s.t. $\mathcal{N}'' = \mathcal{N} \uplus^{\mathcal{R}} \mathcal{N}'$.
- **Case of** $I_{\mathsf{CCR}}(\mathcal{N}'') \geqslant I_{\mathsf{CCR}}(\mathcal{N}) + I_{\mathsf{CCR}}(\mathcal{N}')$. Let $\mathcal{M} \in \mathsf{CCR}(\mathcal{N}'')$ s.t. $I(\mathcal{N}'') = \#\mathsf{diffC}(\mathcal{N}'',\mathcal{M})$. Using Proposition 7, we have $\#\mathsf{diffC}(\mathcal{N}'',\mathcal{M}) = \#\mathsf{diffC}(\mathcal{N},\mathcal{M}_{\downarrow V}) + \#\mathsf{diffC}(\mathcal{N}',\mathcal{M}_{\downarrow V'})$. We can see that $\mathcal{M}_{\downarrow V}$ and $\mathcal{M}_{\downarrow V'}$ are consistent QCNs, $\mathcal{N} \subseteq \mathcal{M}_{\downarrow V}$ and $\mathcal{N}' \subseteq \mathcal{M}_{\downarrow V'}$. Consequently, $\mathcal{M}_{\downarrow V} \in \mathsf{CCR}(\mathcal{N})$ and $\mathcal{M}_{\downarrow V'} \in \mathsf{CCR}(\mathcal{N}')$ hold. Hence, we have $\#\mathsf{diffC}(\mathcal{N},\mathcal{M}_{\downarrow V}) \geqslant I_{\mathsf{CCR}}(\mathcal{N})$ and $\#\mathsf{diffC}(\mathcal{N},\mathcal{M}_{\downarrow V'}) \geqslant I_{\mathsf{CCR}}(\mathcal{N}')$. Therefore, we have $I_{\mathsf{CCR}}(\mathcal{N}'') = \#\mathsf{diffC}(\mathcal{N}'',\mathcal{M}) \geqslant I_{\mathsf{CCR}}(\mathcal{N}) + I_{\mathsf{CCR}}(\mathcal{N}')$.
- **Case of** $I_{\mathsf{CCR}}(\mathcal{N}'') \leqslant I_{\mathsf{CCR}}(\mathcal{N}) + I_{\mathsf{CCR}}(\mathcal{N}')$**.** Let $\mathcal{M} \in \mathsf{CCR}(\mathcal{N})$ s.t. $I_{\mathsf{CCR}}(\mathcal{N}) = \#\mathsf{diffC}(\mathcal{N},\mathcal{M})$ and $\mathcal{M}' \in \mathsf{CCR}(\mathcal{N}')$ s.t. $I(\mathcal{N}') = \#\mathsf{diffC}(\mathcal{N}',\mathcal{M}')$. Consider the QCN $\mathcal{M}''$ defined on $V \cup V'$ by $\mathcal{M}''_{\downarrow V} = \mathcal{M}$, $\mathcal{M}''_{\downarrow V'} = \mathcal{M}'$ and, for all $(v,v') \in V \times V'$, $\mathcal{M}''[v,v'] = \mathcal{N}''[v,v']$. We can see that $\#\mathsf{diffC}(\mathcal{N}'',\mathcal{M}'') = \#\mathsf{diffC}(\mathcal{N},\mathcal{M}) + \#\mathsf{diffC}(\mathcal{N}',\mathcal{M}') = I_{\mathsf{CCR}}(\mathcal{N}) + I_{\mathsf{CCR}}(\mathcal{N}')$. Moreover, we have $\mathcal{M}'' = \mathcal{M} \uplus^{\mathcal{R}} \mathcal{M}'$ and, $\mathcal{M}$ and $\mathcal{M}'$ are consistent. Thus, using the fact that $\mathcal{R}$ has the $\mathcal{CSS}$ property, we can assert that $\mathcal{M}''$ is consistent. Consequently, from the fact that $\mathcal{N}'' \subseteq \mathcal{M}''$, $\mathcal{M}'' \in \mathsf{CCR}(\mathcal{N}'')$ holds. Therefore, we have $I_{\mathsf{CCR}}(\mathcal{N}'') \leqslant \#\mathsf{diffC}(\mathcal{N}'',\mathcal{M}'') = I_{\mathsf{CCR}}(\mathcal{N}) + I_{\mathsf{CCR}}(\mathcal{N}')$. □

6 A V-restrictions Based Inconsistency Measure

In this section, we introduce an inconsistency measure based on the notion of V-restriction. More precisely, this measure is defined as the minimum number of variables that we need to ignore in order to obtain a consistent QCN.

Definition 14 (I_{CVR} Inconsistency Measure) *The inconsistency measure I_{CVR} is defined by:*

$$I_{\mathsf{CVR}}(\mathcal{N}) = \min\{|V \setminus V'| : \mathcal{N}' = (V',C') \in \mathsf{CVR}(\mathcal{N})\}$$

Let us first show that I_{CVR} is a basic inconsistency measure.

Proposition 9 *I_{CVR} satisfies the Consistency Null (***P1***) and Variable Monotonicity (***P2***) postulates.*

Proof:
- **Consistency Null.** Let $\mathcal{N} = (V,C)$ be a QCN. Suppose that $\mathcal{N}$ is consistent. We have $\mathcal{N} \in \mathsf{CVR}(\mathcal{N})$. Hence, $\min\{|V \setminus V'| : \mathcal{N}' = (V',C') \in \mathsf{CVR}(\mathcal{N})\} = 0$ holds. Consequently, we obtain $I_{\mathsf{CVR}}(\mathcal{N}) = 0$. Now, suppose that $I_{\mathsf{CVR}}(\mathcal{N}) = 0$. Then, there exists $\mathcal{N}' = (V',C') \in \mathsf{CVR}(\mathcal{N})\}$ s.t. $|V \setminus V'| = 0$. Hence, $V = V'$ and $\mathcal{N}' = \mathcal{N}$ hold. Thus, we have $\mathcal{N} \in \mathsf{CVR}(\mathcal{N})$, and, as a consequence, $\mathcal{N}$ is consistent.
- **Variable Monotonicity.** Let $\mathcal{N} = (V,C)$ be a QCN and $V' \subseteq V$. Let $\mathcal{M} = (V'',C'') \in \mathsf{CVR}(\mathcal{N})$ s.t. $I_{\mathsf{CVR}}(\mathcal{N}) = |V \setminus V''|$. Then, we have $\mathcal{M}_{\downarrow(V' \cap V'')} \in \mathsf{CVR}(\mathcal{N}_{\downarrow V'})$. Hence, $|V' \setminus (V' \cap V'')| \geqslant I_{\mathsf{CVR}}(\mathcal{N}_{\downarrow V'})$ holds. We have $V' \setminus (V' \cap V'') = V' \setminus V''$. Moreover, as $V' \subseteq V$, we can assert that $V' \setminus V'' \subseteq V \setminus V''$. Consequently, $|V' \setminus (V' \cap V'')| = |V' \setminus V''| \leqslant |V \setminus V''|$. Hence, we have $I_{\mathsf{CVR}}(\mathcal{N}) = |V \setminus V''| \geqslant |V' \setminus (V' \cap V'')| \geqslant I_{\mathsf{CVR}}(\mathcal{N}_{\downarrow V'})$. Therefore, $I_{\mathsf{CVR}}(\mathcal{N}) \geqslant I_{\mathsf{CVR}}(\mathcal{N}_{\downarrow V'})$ holds. □

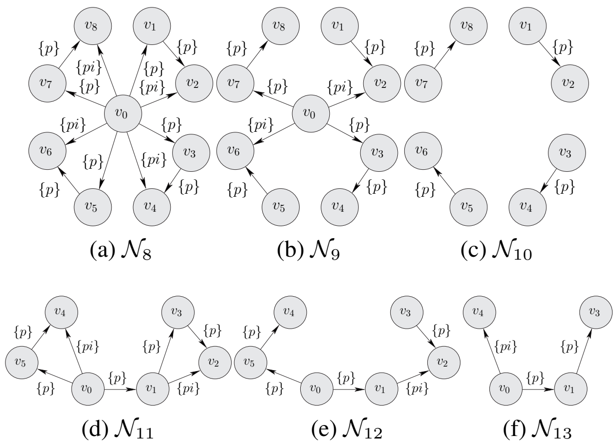
We now show that I_{CVR} satisfies the additional postulates **P3**, **P4**, **P5** and **P6**. To this end, it suffices to show that I_{CVR} satisfies the postulates **P3**, **P5** and **P6**, since **P5** is stronger than **P4**.

Proposition 10 *I_{CVR} satisfies the Relation Monotonicity (***P3***), T-Free Formula Independence (***P5***) and Free Variable Independence (***P6***) postulates.*

Proof:
- **Relation Monotonicity.** Let $\mathcal{N} = (V,C)$ and $\mathcal{N}' = (V,C')$ be two QCNs s.t. $\mathcal{N} \subseteq \mathcal{N}'$. Let $\mathcal{M} = (V',C'') \in \mathsf{CVR}(\mathcal{N})$ s.t. $I_{\mathsf{CVR}}(\mathcal{N}) = |V \setminus V'|$. Since $\mathcal{M} \subseteq \mathcal{N}'_{\downarrow V'}$ and $\mathcal{M}$ is consistent, $\mathcal{N}'_{\downarrow V'}$ is consistent. Hence, $\mathcal{N}'_{\downarrow V'} \in \mathsf{CVR}(\mathcal{N}')$ holds. Consequently, we have $I_{\mathsf{CVR}}(\mathcal{N}') \leqslant |V \setminus V'|$. Therefore, we have $I_{\mathsf{CVR}}(\mathcal{N}') \leqslant I_{\mathsf{CVR}}(\mathcal{N})$.
- **T-Free Formula Independence.** Let $\mathcal{N} = (V,C)$ be a QCN and $\{v,v'\} \in \mathsf{FC}^*(\mathcal{N})$. We use $\mathcal{N}'$ to denote the QCN $\mathcal{N}_{[v,v']/\mathsf{B}}$. Using Relation Monotonicity, $I_{\mathsf{CVR}}(\mathcal{N}') \leqslant I_{\mathsf{CVR}}(\mathcal{N})$ holds, since $\mathcal{N} \subseteq \mathcal{N}'$. We now show that $I_{\mathsf{CVR}}(\mathcal{N}') \geqslant I_{\mathsf{CVR}}(\mathcal{N})$. Let $V' \subseteq V$ and $\mathcal{N}'_{\downarrow V'} \in \mathsf{CVR}(\mathcal{N}')$ s.t. $I_{\mathsf{CVR}}(\mathcal{N}') = |V \setminus V'|$. If $\{v,v'\} \cap V' = \emptyset$, we get $\mathcal{N}'_{\downarrow V'} \in \mathsf{CVR}(\mathcal{N})$ and, as a consequence, $I_{\mathsf{CVR}}(\mathcal{N}') \geqslant I_{\mathsf{CVR}}(\mathcal{N})$ holds. Otherwise, we have $\{v,v'\} \subseteq V'$. Assume that $\mathcal{N}_{\downarrow V'}$ is an inconsistent QCN. Then, $\mathcal{N}_{\downarrow V'}$ includes an inconsistent subset of constraints that does not contain the constraint associated to $\{v,v'\}$ since $\{v,v'\} \in \mathsf{FC}^*(\mathcal{N})$. Thus, $\mathcal{N}'_{\downarrow V'}$ is an inconsistent QCN and we get a contradiction. Therefore, $\mathcal{N}_{\downarrow V'}$ is a consistent QCN and, as a consequence, $I_{\mathsf{CVR}}(\mathcal{N}') \geqslant I_{\mathsf{CVR}}(\mathcal{N})$ holds.
- **Free Variable Independence.** Let $\mathcal{N} = (V,C)$ be a QCN and $v \in \mathsf{FV}(\mathcal{N})$. It is worth noticing that (i) $I_{\mathsf{CVR}}(\mathcal{N}_{\downarrow V\setminus\{v\}}) \leqslant I_{\mathsf{CVR}}(\mathcal{N})$, since $\mathcal{N}$ includes $\mathcal{N}_{\downarrow V\setminus\{v\}}$. Let $V' \subset V \setminus \{v\}$ s.t. $\mathcal{N}_{\downarrow V\setminus(V'\cup\{v\})}$ is consistent and $I_{\mathsf{CVR}}(\mathcal{N}_{\downarrow V\setminus\{v\}}) = |V'|$. Thus, we know that $\mathcal{N}_{\downarrow V\setminus V'}$ is consistent, since v is a free variable in $\mathcal{N}$. As a consequence, we get (ii) $I_{\mathsf{CVR}}(\mathcal{N}_{\downarrow V\setminus\{v\}}) \geqslant I_{\mathsf{CVR}}(\mathcal{N})$. Therefore, using (i) and (ii), $I_{\mathsf{CVR}}(\mathcal{N}_{\downarrow V\setminus\{v\}}) = I_{\mathsf{CVR}}(\mathcal{N})$ holds. □

Proposition 11 *I_{CVR} satisfies $\mathcal{CSS}$-Additivity (***P7***).*

Proof: Let $\mathcal{R} \subseteq 2^{\mathsf{B}}$ s.t. $\mathcal{R}$ satisfies the $\mathcal{CSS}$ property and $\mathcal{N} = (V,C)$, $\mathcal{N}' = (V',C')$, $\mathcal{N}'' = (V'',C'')$ three QCNs s.t. $\mathcal{N}'' = \mathcal{N} \uplus^{\mathcal{R}} \mathcal{N}'$.
- **Case of** $I_{\mathsf{CVR}}(\mathcal{N}'') \leqslant I_{\mathsf{CVR}}(\mathcal{N}) + I_{\mathsf{CVR}}(\mathcal{N}')$. Let $\mathcal{N}^0 = (V^0,C^0) \in \mathsf{CVR}(\mathcal{N})$ s.t. $I_{\mathsf{CVR}}(\mathcal{N}) = |V \setminus V^0|$ and



(a) $\mathcal{N}_8$ (b) $\mathcal{N}_9$ (c) $\mathcal{N}_{10}$

(d) $\mathcal{N}_{11}$ (e) $\mathcal{N}_{12}$ (f) $\mathcal{N}_{13}$

Figure 5: Six QCNs $\mathcal{N}_8, \mathcal{N}_9, \mathcal{N}_{10}, \mathcal{N}_{11}, \mathcal{N}_{12}, \mathcal{N}_{13}$.

$\mathcal{N}^1 = (V^1, C^1) \in \mathsf{CVR}(\mathcal{N}')$ s.t. $I_{\mathsf{CVR}}(\mathcal{N}') = |V' \setminus V^1|$. Let $\mathcal{N}^2 = (V^0 \cup V^1, C^2)$ a QCN defined by $\mathcal{N}^2_{\downarrow V^0} = \mathcal{N}^0$, $\mathcal{N}^2_{\downarrow V^1} = \mathcal{N}^1$ and, for all $v \in V^0$ and $v' \in V^1$, $C^2(v, v') = \mathcal{N}''[v, v']$. Note that $\mathcal{N}^2$ is a V-restriction of $\mathcal{N}''$. Moreover, we have $\mathcal{N}^2 = \mathcal{N}^0 \uplus^{\mathcal{R}} \mathcal{N}^1$. As $\mathcal{N}^0$ and $\mathcal{N}^1$ are consistent, from the $\mathcal{CSS}$ property, $\mathcal{N}^2$ is consistent. Consequently, $\mathcal{N}^2$ is a consistent V-restriction of $\mathcal{N}''$. It follows that $I_{\mathsf{CVR}}(\mathcal{N}'') \leqslant |V'' \setminus (V^0 \cup V^1)|$. As $|V'' \setminus (V^0 \cup V^1)| = |V \setminus V^0| + |V' \setminus V^1|$, we have $I_{\mathsf{CVR}}(\mathcal{N}'') \leqslant |V \setminus V^0| + |V' \setminus V^1|$. We can conclude that $I_{\mathsf{CVR}}(\mathcal{N}'') \leqslant I_{\mathsf{CVR}}(\mathcal{N}) + I_{\mathsf{CVR}}(\mathcal{N}')$.
- **Case of** $I_{\mathsf{CVR}}(\mathcal{N}'') \geqslant I_{\mathsf{CVR}}(\mathcal{N}) + I_{\mathsf{CVR}}(\mathcal{N}')$. Let $\mathcal{N}^2 = (V^2, C^2) \in \mathsf{CVR}(\mathcal{N}'')$ s.t. $I_{\mathsf{CVR}}(\mathcal{N}'') = |V'' \setminus V^2|$. Let $V^0 = V \setminus V^2$ and $V^1 = V' \setminus V^2$. We have, $I_{\mathsf{CVR}}(\mathcal{N}'') = |V'' \setminus V^2| = |V'' \setminus V^0| + |V'' \setminus V^1|$. We also have $\mathcal{N}^2_{\downarrow V_0} \in \mathsf{CVR}(\mathcal{N})$ and $\mathcal{N}^2_{\downarrow V_1} \in \mathsf{CVR}(\mathcal{N}')$. Consequently, $|V'' \setminus V^0| \geqslant I_{\mathsf{CVR}}(\mathcal{N})$ and $|V'' \setminus V^1| \geqslant I_{\mathsf{CVR}}(\mathcal{N}')$ hold. Thus, we obtain $I_{\mathsf{CVR}}(\mathcal{N}'') \geqslant I_{\mathsf{CVR}}(\mathcal{N}) + I_{\mathsf{CVR}}(\mathcal{N}')$. $\square$

In summary, our V-restriction based inconsistency measure I_{CVR} satisfies all the postulates described in Section 4.

Given an inconsistency measure I, let the binary order relation $\leqslant^I$ on the set of QCNs defined by $\mathcal{N} \leqslant^I \mathcal{N}'$ iff $I(\mathcal{N}) \leqslant I(\mathcal{N}')$ for every QCNs $\mathcal{N}$ and $\mathcal{N}'$. Consider the order relations related to the two measures I_{CCR} and I_{CVR} previously proposed. We can show that one of these relations does not include the other and vice versa. Indeed, consider the QCNs $\mathcal{N}_8$ and $\mathcal{N}_{11}$ depicted in Figure 5. These two QCNs are inconsistent. In order to obtain a consistent QCN from $\mathcal{N}_8$, we must necessary modify at least four constraints or remove at least one variable. Thus, we get $I_{\mathsf{CCR}}(\mathcal{N}_8) = 4$ and $I_{\mathsf{CVR}}(\mathcal{N}_8) = 1$. To obtain a consistent QCN from $\mathcal{N}_{11}$, we should modify at least two constraints or remove at least two variables. Consequently, we have $I_{\mathsf{CCR}}(\mathcal{N}_{11}) = 2$ and $I_{\mathsf{CVR}}(\mathcal{N}_{11}) = 2$. Therefore, we have $\mathcal{N}_8 \not\leqslant^{I_{\mathsf{CCR}}} \mathcal{N}_{11}$, $\mathcal{N}_8 \leqslant^{I_{\mathsf{CVR}}} \mathcal{N}_{11}$, $\mathcal{N}_{11} \leqslant^{I_{\mathsf{CCR}}} \mathcal{N}_8$, and $\mathcal{N}_{11} \not\leqslant^{I_{\mathsf{CVR}}} \mathcal{N}_8$.

7 Discussion

Inconsistencies arise in many applications, e.g., when several experts share their beliefs in order to solve a problem by building a joint knowledge base. Merging multiple sources information has been widely studied in literature, and is an important issue of many AI fields (see (Bloch and Hunter 2001) for more details). In distributed knowledge systems in qualitative reasoning, spatial or temporal information about entities come from several sources, each source providing a QCN defined on the same set of entities. Due to the multiplicity of sources, we generally have to deal with conflicting QCNs which makes their merging a nontrivial issue. In this context, it is potentially useful to investigate how inconsistency measures can be used to judge the closeness between conflicting sources of information. There have been few papers discussing this issue in the context of classical theories, among them (Qi, Liu, and Bell 2005; Hunter and Konieczny 2010). In the QCN merging topic, several families of merging operators have been pointed out in (Condotta et al. 2009a; 2009b). In (Condotta et al. 2010), the authors have proposed a family of syntactical merging operators called Δ_1, which can be used where the sources are not consistent. Given a multiset of QCNs, possibly inconsistent, representing explicit preferences or beliefs of several agents about the relative positions of spatial or temporal entities, the Δ_1 operator aims at defining a non-empty set of consistent scenarios which represent a global view of the input QCNs. Notice that the operator Δ_1 is a syntactical operator, that is, it does not take the consistent scenarios of the different sources into account but rather the constraints defining them.

In this context, a QCN inconsistency measure can be served for quantifying the different input QCNs. Intuitively, each source can be labeled by a weight that can be given in terms of the amount of conflicts brought by the information in that source. By following this principle, a source with a high inconsistency value is less reliable than the one with a low value. Hence, we can define an ordering relation to compare different QCNs based on their degree of conflict, which will be also useful to refine the behaviour of Δ_1 operator. Lastly, it will be sensible to force the less reliable sources to be less considered in the result of the merging operation.

8 Conclusion

The contribution of this paper is threefold: First, setting up a list of postulates that allow us to characterize inconsistency measures in QCNs, Second, defining two inconsistency measures, and showing that they satisfy these postulates, Third, sketching the possible application of our framework of inconsistency measurement for belief merging in qualitative reasoning.

As a future work, we plan to conduct a comparative empirical evaluation of our inconsistency measures. We also intend to deeply study the impact of the use of our inconsistency measures on the merging operators Δ_1 introduced in (Condotta et al. 2010).

References

Allen, J. F. 1981. An interval-based representation of temporal knowledge. In *IJCAI*, 221–226.

Ammoura, M.; Raddaoui, B.; Salhi, Y.; and Oukacha, B. 2015. On measuring inconsistency using maximal consistent sets. In *ECSQARU*, 267–276.

Balbiani, P., and Osmani, A. 2000. A model for reasoning about topologic relations between cyclic intervals. In *KR*, 378–385.

Balbiani, P.; Condotta, J.-F.; and Fariñas del Cerro, L. 1998. A model for reasoning about bidimensional temporal relations. In *KR*, 124–130.

Besnard, P. 2014. Revisiting postulates for inconsistency measures. In *JELIA*, 383–396.

Bloch, I., and Hunter, A. 2001. Fusion: General concepts and characteristics. *Int. J. Intell. Syst.* 16(10):1107–1134.

Chen, Q.; Zhang, C.; and Zhang, S. 2004. A verification model for electronic transaction protocols. In *APWeb*, 824–833.

Condotta, J.-F.; Kaci, S.; Marquis, P.; and Schwind, N. 2009a. Merging qualitative constraint networks defined on different qualitative formalisms. In Hornsby, K. S.; Claramunt, C.; Denis, M.; and Ligozat, G., eds., *COSIT*, Lecture Notes in Computer Science, 106–123.

Condotta, J.-F.; Kaci, S.; Marquis, P.; and Schwind, N. 2009b. Merging qualitative constraints networks using propositional logic. In *ECSQARU*, 347–358.

Condotta, J.; Kaci, S.; Marquis, P.; and Schwind, N. 2010. A syntactical approach to qualitative constraint networks merging. In *LPAR*, 233–247.

Condotta, J.; Mensi, A.; Nouaouri, I.; Sioutis, M.; and Said, L. B. 2015. A practical approach for maximizing satisfiability in qualitative spatial and temporal constraint networks. In *ICTAI*, 445–452.

Feiner, S.; Litman, D. J.; McKeown, K.; and Passonneau, R. J. 1991. Towards coordinated temporal multimedia presentations. In *AAAI Workshop on Intelligent Multimedia Interfaces*, 139–147.

Grant, J., and Hunter, A. 2006. Measuring inconsistency in knowledgebases. *J. Intell. Inf. Syst.* 27(2):159–184.

Grant, J., and Hunter, A. 2013. Distance-based measures of inconsistency. In *ECSQARU*, 230–241.

Grant, J. 1978. Classifications for inconsistent theories. *Notre Dame Journal of Formal Logic* 19(3):435–444.

Hazarika, S. 2012. *Qualitative Spatio-Temporal Representation and Reasoning: Trends and Future Directions*.

Hunter, A., and Konieczny, S. 2010. On the measure of conflicts: Shapley inconsistency values. *Artif. Intell.* 174(14):1007–1026.

Hunter, A.; Parsons, S.; and Wooldridge, M. 2014. Measuring inconsistency in multi-agent systems. *Kunstliche Intelligenz* 28:169–178.

Hunter, A. 2006. How to act on inconsistent news: Ignore, resolve, or reject. *Data Knowl. Eng.* 57(3):221–239.

Jabbour, S., and Raddaoui, B. 2013. Measuring inconsistency through minimal proofs. In *ECSQARU*, 290–301.

Jabbour, S.; Ma, Y.; Raddaoui, B.; and Saïs, L. 2014. Prime implicates based inconsistency characterization. In *ECAI*, 1037–1038.

Jabbour, S.; Ma, Y.; Raddaoui, B.; Sais, L.; and Salhi, Y. 2015. On structure-based inconsistency measures and their computations via closed set packing. In *AAMAS*, 1749–1750.

Jabbour, S.; Ma, Y.; and Raddaoui, B. 2014. Inconsistency measurement thanks to MUS-decomposition. In *AAMAS*, 877–884.

Knight, K. 2002. Measuring inconsistency. *J. Philosophical Logic* 31(1):77–98.

Li, J. J., and Li, S. 2013. On finding approximate solutions of qualitative constraint networks. In *ICTAI*, 30–37.

Martinez, A. B. B.; Arias, J. J. P.; and Vilas, A. F. 2004. On measuring levels of inconsistency in multi-perspective requirements specifications. In *PRISE*, 21–30.

Martinez, M. V.; Pugliese, A.; Simari, G. I.; Subrahmanian, V. S.; and Prade, H. 2007. How dirty is your relational database? an axiomatic approach. In *ECSQARU*, 103–114.

McAreavey, K.; Liu, W.; Miller, P.; and Mu, K. 2011. Measuring inconsistency in a network intrusion detection rule set based on snort. *Int. J. Semantic Computing* 5(3).

Mu, K.; Liu, W.; and Jin, Z. 2011. A general framework for measuring inconsistency through minimal inconsistent sets. *Knowl. Inf. Syst.* 27(1):85–114.

Pujari, A.; Kumari, G.; and Sattar, A. 1999. INDU: an interval and duration network. In *AI*, 291–303.

Pustejovsky, J.; Castaño, J. M.; Ingria, R.; Sauri, R.; Gaizauskas, R. J.; Setzer, A.; Katz, G.; and Radev, D. R. 2003. Timeml: Robust specification of event and temporal expressions in text. In *AAAI Spring Symposium*, 28–34.

Qi, G.; Liu, W.; and Bell, D. A. 2005. Measuring conflict and agreement between two prioritized belief bases. In *IJCAI*, 552–557.

Randell, D.; Cohn, A.; and Cui, Z. 1992. Computing transivity tables: a challenge for automated theorem provers. *Lecture Notes in Computer Science* 607:786–790.

Renz, J., and Nebel, B. 2007. Qualitative spatial reasoning using constraint calculi. In *Handbook of Spatial Logics*. 161–215.

Song, F., and Cohen, P. R. 1988. The interpretation of temporal relations in narrative. In *AAAI*, 745–750.

Thimm, M. 2013. Inconsistency measures for probabilistic logics. *Artif. Intell.* 197:1–24.

Thimm, M. 2016. Stream-based inconsistency measurement. *Int. J. Approx. Reasoning* 68:68–87.

Vilain, M.; Kautz, H.; and Beek, P. V. 1990. Constraint propagation algorithms for temporal reasoning: a revised report. *Qualitative Reasoning about Physical Systems* 372–381.

Zhou, L.; Huang, H.; Qi, G.; Ma, Y.; Huang, Z.; and Qu, Y. 2009. Measuring inconsistency in dl-lite ontologies. In *Web Intelligence*, 349–356.