Interpreting Topological Logics Over Euclidean Spaces

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Abstract

Topological logics are a family of languages for representing and reasoning about topological data. In this paper, we consider propositional topological logics able to express the property of *connectedness*. The satisfiability problem for such logics is shown to depend not only on the spaces they are interpreted in, but also on the subsets of those spaces over which their variables are allowed to range. We identify the crucial notion of *tameness*, and chart the surprising patterns of sensitivity to the presence of non-tame regions exhibited by a range of topological logics in low-dimensional Euclidean spaces.

Introduction

Topological logics are a family of languages for representing and reasoning about topological data. The non-logical primitives of these languages stand for various topological relations and operations, and their valid formulas encode our knowledge about those relations and operations. Consider, for example, the six relations illustrated in Fig. 1. By employing the binary predicates NTPP (non-tangential proper parthood), TPP (tangential proper parthood), EQ (equality), PO (partial overlap), EC (external contact) and DC (disconnection) to stand for these relations, the formula

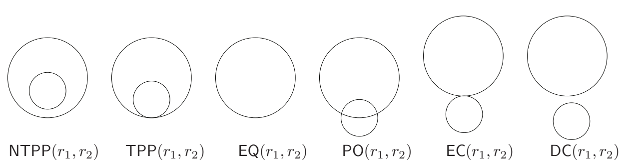


Figure 1: $\mathcal{RCC}8$-relations over discs in $\mathbb{R}^2$: except in the case of EQ, the smaller disc depicts r_1, and the larger, r_2.

$$\mathsf{EC}(r_1, r_2) \wedge \mathsf{NTPP}(r_1, r_3) \rightarrow \mathsf{PO}(r_2, r_3) \vee \mathsf{TPP}(r_2, r_3) \vee \mathsf{NTPP}(r_2, r_3) \quad (1)$$

makes the intuitively reasonable assertion that, if region r_1 externally contacts region r_2 and is a non-tangential proper part of region r_3, then r_2 either partially overlaps, or else is a proper part (tangential or non-tangential) of r_3. This particular topological logic, known as $\mathcal{RCC}8$, has been intensively analysed in the literature on qualitative spatial reasoning; see e.g., (Egenhofer and Franzosa 1991; Randell, Cui, and Cohn 1992; Renz and Nebel 1999).

We referred to r_1, r_2 and r_3 above as 'regions,' and depicted them as discs in the plane. But a moment's thought shows that the set of valid formulas of any topological logic depends on the precise collection of regions we have in mind. For instance, there are $\mathcal{RCC}8$-formulas that are valid as long as regions are taken to be discs in the plane, but invalid when regions are allowed to be *disconnected* (i.e. to consist of more than one 'piece'). Thus, an important semantic issue for a topological logic like $\mathcal{RCC}8$ is to identify the intended models. In this paper, we consider *propositional* topological logics with *connectedness*, i.e. topological logics in which the only logical connectives are the usual Boolean operators, but where there is a non-logical primitive expressing the property of topological connectedness (or a variant thereof). We show that such topological logics are typically sensitive both to the spaces they are interpreted over and—more particularly—to the subsets of those spaces over which their variables are allowed to range. We identify the crucial notion of *tameness*, and chart the surprising patterns of sensitivity to the presence of non-tame regions exhibited by a range of topological logics in low-dimensional Euclidean spaces.

Historically, $\mathcal{RCC}8$ has typically been interpreted over the *regular closed* sets of *arbitrary* topological spaces. (A set is *regular closed* if it is the topological closure of an open set.) This very general interpretation is *prima facie* surprising: after all, in qualitative spatial reasoning, it is the low-dimensional Euclidean spaces $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$ that interest us—not the arbitrary topological spaces found in mathematics textbooks! The answer to this objection is that if an $\mathcal{RCC}8$-formula is valid over regular closed subsets of $\mathbb{R}^n$ for some $n \geq 1$, then it is valid in all topological spaces whatsoever (Renz 1998). Put another way: the language $\mathcal{RCC}8$ is almost totally *in*sensitive to the underlying space.

However, this insensitivity disappears when we increase the expressive resources at our command. To illustrate, consider the effect of adding a unary predicate c, where $c(r)$ means 'r is connected.' We call the resulting language

$\mathcal{RCC}8c$. Thus, the $\mathcal{RCC}8c$-formula

$$\bigwedge_{1\leq i\leq 3} c(r_i) \;\to\; \bigvee_{1\leq i<j\leq 3} \neg\mathsf{EC}(r_i, r_j) \tag{2}$$

states that no three connected regions r_1, r_2 and r_3 can externally contact each other. In $\mathbb{R}^2$ (or in $\mathbb{R}^3$), this formula is clearly *in*valid. However, (2) is *valid* when interpreted over $\mathbb{R}$, since the non-empty connected regular closed subsets of $\mathbb{R}$ are simply the (non-empty) closed intervals.

Actually, the standard notion of connectedness may be inappropriate for many applications of qualitative spatial reasoning, particularly in the context of geographical information systems (GIS). Consider, for example, the (closed) region formed by two triangles touching externally at a common vertex. Mathematically speaking, this set is connected; yet we are loath to take it to represent, say, a connected plot of land on a map. Accordingly, we introduce the unary predicate c°, where $c^\circ(r)$ means 'the topological interior of r is connected,' and denote by $\mathcal{RCC}8c^\circ$ the result of adding c° to $\mathcal{RCC}8$. Again, one can show that $\mathcal{RCC}8c^\circ$, like $\mathcal{RCC}8c$, is sensitive to the topological space in which it is interpreted.

Another way to increase the expressive power of $\mathcal{RCC}8$ is to provide the means to talk about *combinations* of regions. Thus, in the language known as *Boolean* $\mathcal{RCC}8$ (Wolter and Zakharyaschev 2000), we use $r_1 + r_2$, $r_1 \cdot r_2$ and $-r$ for the regular closures of $r_1 \cup r_2$, $r_1 \cap r_2$ and the complement of r, respectively. We denote this extension of $\mathcal{RCC}8$ by $\mathcal{C}$ (this nomenclature will be justified in the sequel).

By extending $\mathcal{C}$ with either of the predicates c or c°, we obtain the languages $\mathcal{C}c$ and $\mathcal{C}c^\circ$. For example, the $\mathcal{C}c^\circ$-formula

$$c^\circ(-r_1) \wedge c^\circ(-r_2) \wedge \mathsf{DC}(r_1, r_2) \to c^\circ(-(r_1 + r_2)) \tag{3}$$

can be shown to be valid for regular closed sets in Euclidean space of any dimension (see Theorem 12); yet it is invalid in other topological spaces—for example, the torus (Fig. 3).

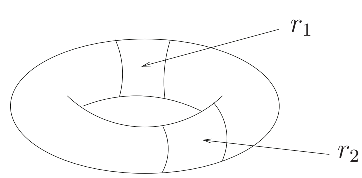


Figure 3: Invalidating (3) in the torus.

Once we have connectedness predicates at our disposal, two further topological languages suggest themselves. We denote by $\mathcal{B}c$ the language featuring the function symbols $+$, $\cdot$ and $-$, together with the equality predicate $=$ and the connectedness predicate c; the language $\mathcal{B}c^\circ$ is defined similarly, but with c replaced by c°. Note that the languages $\mathcal{B}c$ and $\mathcal{B}c^\circ$ do not contain the $\mathcal{RCC}8$-primitives.

The language $\mathcal{B}c^\circ$ nicely illustrates a subtle but important semantic issue which is often neglected in discussions of topological logics. Consider the $\mathcal{B}c^\circ$-formulas (for $m \geq 3$):

$$\bigwedge_{1\leq i\leq m} c^\circ(r_i) \;\wedge\; c^\circ\Big(\sum_{1\leq i\leq m} r_i\Big) \;\to\; \bigvee_{2\leq i\leq m} c^\circ(r_1 + r_i). \tag{4}$$

Interpreted over the regular closed subsets of $\mathbb{R}^2$, these formulas are *in*valid: Fig. 4 shows a counterexample (with $m = 3$), in which the boundary between r_2 and r_3 is formed by the curve $\sin(1/x)$ over the interval $(0, 1]$. Yet r_2 and

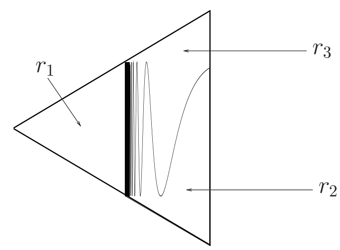


Figure 4: Three regular closed sets in $\mathbb{R}^2$ satisfying (4).

r_3 are hardly plausible models of, for instance, regions occupied by physical objects resting on a surface, or plots of land in a cadastre. Crucially, it can be shown (Lemma 1) that (4) becomes valid as soon as we restrict attention to 'well-behaved' (as we shall say: *tame*) subsets of $\mathbb{R}^n$—in particular, to polyhedra (or polygons). Therefore, in deciding how to interpret topological logics for spatial representation and reasoning, it is not sufficient merely to fix the topological space in question: we must also specify which subsets of that space we wish to count as *bona fide* regions.

This paper undertakes the kind of semantic analysis of topological logics we have just argued for. We consider the six languages introduced above—namely, $\mathcal{RCC}8c$, $\mathcal{RCC}8c^\circ$, $\mathcal{B}c$, $\mathcal{B}c^\circ$, $\mathcal{C}c$ and $\mathcal{C}c^\circ$—and their various interpretations in the spaces $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$. We show that the dimensionality of the space is important for all of our languages, but that, in addition, these languages exhibit varying patterns of sensitivity to tameness in different dimensions. It turns out that only $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$ are insensitive to tameness in $\mathbb{R}^2$ and $\mathbb{R}^3$, while—surprisingly—only $\mathcal{B}c$ and $\mathcal{B}c^\circ$ do not feel it in $\mathbb{R}$; in all these cases reasoning (satisfiability) proves to be NP-complete. Reasoning with $\mathcal{B}c$, $\mathcal{B}c^\circ$, $\mathcal{C}c$ and $\mathcal{C}c^\circ$ in $\mathbb{R}^n$, for $n \geq 2$, is shown to be generally EXPTIME-hard (apart from two cases that are still open); and a matching upper bound is obtained for $\mathcal{B}c^\circ$ over polyhedra in $\mathbb{R}^n$, $n \geq 3$. The obtained results are collected in Fig. 2.

Preliminaries

Let T be a topological space. We denote the closure of any $X \subseteq T$ by X^-, the interior of X by X° and the boundary of X by $\delta X = X^- \setminus X^\circ$. We call X *regular closed* if $X = X^{\circ -}$, and denote by $\mathsf{RC}(T)$ the set of all regular closed subsets of T. It is known that $\mathsf{RC}(T)$ forms a Boolean algebra under the operations $r_1 + r_2 = r_1 \cup r_2$, $r_1 \cdot r_2 = (r_1 \cap r_2)^{\circ -}$ and $-r = (T \setminus r)^-$. As usual, we write $r_1 \leq r_2$ for $r_1 \cdot (-r_2) = \emptyset$. A subset $X \subseteq T$ is *connected* if it cannot be covered by the union of two non-empty and disjoint subsets which are open in the subspace topology on X. We say that X is *interior-connected* if X° is connected. If W is a set and R a reflexive and transitive relation (i.e., a quasi-order) on W, we obtain a topological space over W by taking the open sets to be those of the form

lang.	$\mathbb{R}$			$\mathbb{R}^2$			$\mathbb{R}^3$				RC
	$\mathsf{RCP}(\mathbb{R})$		$\mathsf{RC}(\mathbb{R})$	$\mathsf{RCP}(\mathbb{R}^2)$		$\mathsf{RC}(\mathbb{R}^2)$	$\mathsf{RCP}(\mathbb{R}^3)$		$\mathsf{RC}(\mathbb{R}^3)$		
$\mathcal{RCC}8c^\circ$	NP	$\neq$	NP	NP Th. 6, 9			NP				
$\mathcal{RCC}8c$	Th. 4	Th. 2	Th. 4	NP Th. 6, 9			Th. 11				
$\mathcal{B}c^\circ$	NP			$\geq$ ExpTime Th. 14	$\neq$ Th. 7	?	ExpTime Th. 14	$\neq$ Th. 13	?	?	NP Th. 15
$\mathcal{B}c$	Th. 3			$\geq$ ExpTime Th. 10	$\neq$ Th. 8	$\geq$ ExpTime Th. 10	$\geq$ ExpTime	?	$\geq$ ExpTime	?	ExpTime
$\mathcal{C}c^\circ$	PSpace	$\neq$	PSpace	$\geq$ ExpTime Th. 10	$\neq$ Th. 7	$\geq$ ExpTime Th. 10	$\geq$ ExpTime	$\neq$ Th. 13	$\geq$ ExpTime	$\neq$ Th. 12	ExpTime
$\mathcal{C}c$	Th. 4	Th. 2	Th. 4	$\geq$ ExpTime Th. 10	$\neq$ Th. 8	$\geq$ ExpTime Th. 10	$\geq$ ExpTime	?	$\geq$ ExpTime	?	ExpTime

Figure 2: Summary of the expressiveness and complexity results.

$\{x \in X \mid \forall y \in W(xRy \to y \in X)\}$, for $X \subseteq W$. We call such a space an *Aleksandrov space* and denote it by (W, R).

By a *topological language* we mean a language featuring an infinite set of variables, a fixed non-logical signature of function symbols and predicates (with standard meanings as topological operations and relations), and the usual connectives of propositional logic. Note that the topological languages we deal with in this paper do not feature quantifiers. If $\mathcal{L}$ is a topological language, a *frame* for $\mathcal{L}$ is a collection $\mathfrak{F}$ of subsets of some topological space T; and a *model* for $\mathcal{L}$ over $\mathfrak{F}$ is a pair $(\mathfrak{F}, \sigma)$, where σ is a function from variables to elements of $\mathfrak{F}$. Thus, for any topological space T, $\mathsf{RC}(T)$ is a frame for any of our topological languages; we denote the class of all such frames by RC. In this paper, we shall be concerned exclusively with frames which form subsets of $\mathsf{RC}(T)$ for some T. Restricting attention to regular closed sets is regarded as a convenient means of ignoring the boundaries of regions.

Since the meanings of the non-logical primitives of $\mathcal{L}$ are fixed, any model defines a notion of truth for $\mathcal{L}$-formulas in the obvious way. If $\mathfrak{F}$ is a frame and φ an $\mathcal{L}$-formula, we say that φ is *satisfiable over* $\mathfrak{F}$ if φ is true in some model over $\mathfrak{F}$. If $\mathcal{K}$ is a class of frames, we say that φ is *satisfiable over* $\mathcal{K}$ if φ is satisfiable over some frame in $\mathcal{K}$; and we say that φ is *valid over* $\mathcal{K}$ if $\neg\varphi$ is not satisfiable over $\mathcal{K}$. Thus, in the context of discussing the satisfiability of formulas, variables are regarded as (implicitly) *existentially* quantified, while in the context of discussing the validity of formulas, they are regarded as universally quantified. A *topological logic* is a pair $(\mathcal{L}, \mathcal{K})$ where $\mathcal{L}$ is a topological language and $\mathcal{K}$ a class of frames for $\mathcal{L}$. The *satisfiability problem* for $(\mathcal{L}, \mathcal{K})$ is denoted by *Sat*$(\mathcal{L}, \mathcal{K})$.

For regular closed sets, the $\mathcal{RCC}8$-predicates are standardly interpreted as follows:

$$\begin{array}{rll} \mathsf{DC}(r_1, r_2) & \text{iff} & r_1 \cap r_2 = \emptyset, \\ \mathsf{EC}(r_1, r_2) & \text{iff} & r_1 \cap r_2 \neq \emptyset \text{ but } r_1^\circ \cap r_2^\circ = \emptyset, \\ \mathsf{PO}(r_1, r_2) & \text{iff} & r_1^\circ \cap r_2^\circ,\ r_1^\circ \setminus r_2,\ r_2^\circ \setminus r_1 \neq \emptyset, \\ \mathsf{EQ}(r_1, r_2) & \text{iff} & r_1 = r_2, \\ \mathsf{TPP}(r_1, r_2) & \text{iff} & r_1 \subseteq r_2 \text{ but } r_1 \not\subseteq r_2^\circ \text{ and } r_2 \not\subseteq r_1, \\ \mathsf{NTPP}(r_1, r_2) & \text{iff} & r_1 \subseteq r_2^\circ \text{ but } r_2 \not\subseteq r_1. \end{array}$$

Note that, in this paper, we allow variables to take the empty set as value. (Some discussions of $\mathcal{RCC}8$ in the literature adopt the opposite convention.) No loss of expressive power is entailed by this decision: r is empty if and only if it is disconnected from itself, i.e., $\mathsf{DC}(r, r)$. The unary predicates c and c° are interpreted as the properties of connectedness and interior-connectedness, respectively. The function symbols $+$, $\cdot$, $-$ and constants 0 and 1 are interpreted as the corresponding operations and elements in $\mathsf{RC}(T)$. The *contact* predicate C holds between r_1 and r_2 if and only if $r_1 \cap r_2 \neq \emptyset$. Thus, $C(r_1, r_2)$ is equivalent to $\neg\mathsf{DC}(r_1, r_2)$. In the presence of the functions $\cdot$ (or $+$) and $-$, all the $\mathcal{RCC}8$-predicates can be expressed in terms of C and $=$ (i.e., EQ), and *vice versa* (Kontchakov et al. 2009). We denote by $\mathcal{C}$ the language with predicates C and $=$ and function symbols $+$, $\cdot$ and $-$. Thus, we may regard $\mathcal{C}$ as, in effect, the extension of $\mathcal{RCC}8$ with $+$, $\cdot$ and $-$.

Fixing $n \geq 1$, any $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$ bounds two elements of $\mathsf{RC}(\mathbb{R}^n)$; let us call these sets *half-spaces*. We denote by $\mathsf{RCP}(\mathbb{R}^n)$ the Boolean subalgebra of $\mathsf{RC}(\mathbb{R}^n)$ generated by the half-spaces. We call the elements of $\mathsf{RCP}(\mathbb{R}^n)$ *polyhedra* in $\mathbb{R}^n$, and the elements of $\mathsf{RCP}(\mathbb{R}^2)$ *polygons*. Let $\mathfrak{F}$ be a set of subsets of $\mathsf{RC}(T)$ for some topological space T. We say that $\mathfrak{F}$ is *finitely decomposable* if every element of $\mathfrak{F}$ is the sum of finitely many connected elements of $\mathfrak{F}$. It is evident that, for all $n \geq 1$, $\mathsf{RCP}(\mathbb{R}^n)$ is finitely decomposable, whereas $\mathsf{RC}(\mathbb{R}^n)$ itself is not. Further, we say that a subset $X \subseteq T$ has *curve selection* if, for any point p in the closure X^- of X, there exists a Jordan arc α lying entirely within $X \cup \{p\}$, one endpoint of which is p. Again, it is readily checked that, for all $n \geq 1$, every element of $\mathsf{RCP}(\mathbb{R}^n)$ has curve selection, whereas there are elements of $\mathsf{RC}(\mathbb{R}^n)$ which do not. For example, the sets r_2 and r_3 illustrated in Fig. 4 lack curve selection, because none of the boundary points lying on the y-axis can be approached from within either r_2 or r_3 by a Jordan arc. If $\mathfrak{F}$ is a finitely decomposable subset of $\mathsf{RC}(\mathbb{R}^n)$ all of whose elements have curve selection, we say that $\mathfrak{F}$ is *tame*. Tame collections of sets are regarded as mathematically well-behaved.

The polyhedra are by no means the only tame subsets of $\mathbb{R}^n$. A subset of $\mathbb{R}^n$ is *semi-algebraic* if it is definable by a formula with n free variables in the first-order language of fields (with parameters); we denote the collection of regular closed semi-algebraic subsets of $\mathbb{R}^n$ by $\mathsf{RCS}(\mathbb{R}^n)$. It is routine to show that $\mathsf{RCP}(\mathbb{R}^n)$ is a Boolean subalgebra of $\mathsf{RCS}(\mathbb{R}^n)$, which is in turn a Boolean subalgebra of $\mathsf{RC}(\mathbb{R}^n)$. Arguably, $\mathsf{RCS}(\mathbb{R}^n)$ constitutes a more natural and flexible model of spatial regions than $\mathsf{RCP}(\mathbb{R}^n)$; yet $\mathsf{RCS}(\mathbb{R}^n)$ is still tame in the above sense. In particular, the Cell Decomposition Theorem (Bochnak et al. 1998, p. 33) easily entails that $\mathsf{RCS}(\mathbb{R}^n)$ is finitely decomposable; further, the Curve Selection Theorem (*op. cit.* p. 38) states that all semi-algebraic sets have a strong form of the curve selection property. For simplicity, however, we focus on the polyhedra as an example of a tame collection of regions.

One-dimensional Euclidean Space

First we consider the logics $(\mathcal{L}, \mathsf{RC}(\mathbb{R}))$ and $(\mathcal{L}, \mathsf{RCP}(\mathbb{R}))$, where $\mathcal{L}$ is any of the topological languages introduced above. The one-dimensional case is simple to analyse, yet illustrates well the kinds of phenomena that will occupy us at greater length when we come to the 2D and 3D cases.

Over $\mathbb{R}$, the notions of connectedness and interior-connectedness coincide; hence, we have only the languages $\mathcal{RCC}8c$, $\mathcal{B}c$ and $\mathcal{C}c$ to consider. We begin by observing that the dimensionality of the space is significant.

Theorem 1. *For any* $\mathcal{L} \in \{\mathcal{RCC}8c, \mathcal{B}c, \mathcal{C}c\}$ *and* $n \geq 2$, $Sat(\mathcal{L}, \mathsf{RC}(\mathbb{R})) \subsetneqq Sat(\mathcal{L}, \mathsf{RC}(\mathbb{R}^n))$.

Proof. The inclusion holds because any tuple in $\mathsf{RC}(\mathbb{R})$ can easily be cylindrified to form a tuple in $\mathsf{RC}(\mathbb{R}^n)$, $n \geq 2$, satisfying the same $\mathcal{L}$-formulas. To see that it is proper, we observed above that (2) is invalid over $\mathsf{RC}(\mathbb{R}^n)$ for $n \geq 2$; but it is valid over $\mathsf{RC}(\mathbb{R})$. For $\mathcal{B}c$, consider $\bigwedge_{1 \leq i \leq 3}\big(c(r_i) \wedge (r_i \neq 0)\big) \rightarrow \bigvee_{1 \leq i < j \leq 3} \neg\big((r_i \cdot r_j = 0) \wedge c(r_i + r_j)\big)$. ☐

Next, we consider the issue of tameness. Over $\mathbb{R}$, the languages $\mathcal{RCC}8c$ and $\mathcal{C}c$ are sensitive to the presence of non-tame sets:

Theorem 2. $Sat(\mathcal{RCC}8c, \mathsf{RCP}(\mathbb{R})) \subsetneqq Sat(\mathcal{RCC}8c, \mathsf{RC}(\mathbb{R}))$ *and* $Sat(\mathcal{C}c, \mathsf{RCP}(\mathbb{R})) \subsetneqq Sat(\mathcal{C}c, \mathsf{RC}(\mathbb{R}))$.

Proof. The inclusions are trivial. To show that they are proper, the $\mathcal{RCC}8c$-formula $c(r_1) \wedge \bigwedge_{1 \leq i < j \leq 4} \mathsf{EC}(r_i, r_j)$ is satisfiable over $\mathsf{RC}(\mathbb{R})$, but not over $\mathsf{RCP}(\mathbb{R})$; see Fig. 5. ☐

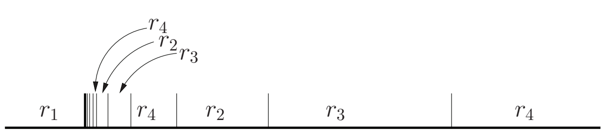


Figure 5: Subsets of $\mathbb{R}$ used in the proof of Theorem 2.

The language $\mathcal{B}c$, by contrast, is not sensitive to tameness:

Theorem 3. $Sat(\mathcal{B}c, \mathsf{RC}(\mathbb{R})) = Sat(\mathcal{B}c, \mathsf{RCP}(\mathbb{R}))$. *Further, this problem is* NP*-complete.*

Proof. Let a $\mathcal{B}c$-formula φ be given. We may assume without loss of generality that φ is of the form

$$(\rho = 0) \wedge \bigwedge_{1 \leq j \leq m} (\sigma_j \neq 0) \wedge \bigwedge_{1 \leq i \leq n} \big(c(\pi_i) \wedge (\pi_i \neq 0)\big) \wedge \bigwedge_{1 \leq k \leq p} \neg c(\tau_k).$$

We describe a non-deterministic procedure which, given a formula φ of the form above, terminates with either success or failure in time bounded by a polynomial function of $|\varphi|$. We show that, if the procedure has a successful run, then φ is satisfiable over $\mathsf{RCP}(\mathbb{R})$ and that, if φ is satisfiable over $\mathsf{RC}(\mathbb{R})$, then the procedure has a successful run.

Let $E = 2(m + n + 3p)$. Denote by Ξ the set of regular closed intervals $(-\infty, 0], [0, 1], \ldots, [E-1, E], [E, +\infty)$ and by Δ the set of integers in the interval $[0, E]$. In the sequel, we construct a function λ, which maps Ξ to the power set of the set of subterms of φ. An interval $[a, b]$ with $a, b \in \Delta$ is regular closed if $b - a \geq 1$. We start off with $\lambda(I) = \emptyset$ for every $I \in \Xi$.

1. For every j ($1 \leq j \leq m$), choose a regular closed interval $[a, b]$, $a, b \in \Delta$, and add σ_j to $\lambda(I)$, for each $I \in \Xi$ with $I \subseteq [a, b]$.
2. For every i ($1 \leq i \leq n$), choose a regular closed interval $[a, b]$, $a, b \in \Delta$, and add π_i to $\lambda(I)$, for each $I \in \Xi$ with $I \subseteq [a, b]$ and add $-\pi_i$ to $\lambda(I)$, for each $I \in \Xi$ with $I \subseteq [0, a] \cup [b, E]$; if $a > 0$ add $-\pi_i$ to $\lambda((-\infty, 0])$, otherwise, add either π_i or $-\pi_i$ to $\lambda((-\infty, 0])$; if $b < E$ add $-\pi_i$ to $\lambda([E, +\infty))$, otherwise, add either π_i or $-\pi_i$ to $\lambda([E, +\infty))$.
3. For every k ($1 \leq k \leq p$), choose a pair of regular closed intervals $[a_1, b_1]$ and $[a_2, b_2]$ with $a_1, b_1, a_2, b_2 \in \Delta$ such that $a_2 - b_1 \geq 1$ and add τ_k to $\lambda(I)$, for each $I \in \Xi$ with $I \subseteq [a_1, b_1] \cup [a_2, b_2]$, and add $-\tau_k$ to $\lambda(I)$, for each $I \in \Xi$ with $I \subseteq [b_1, a_2]$ (the latter is possible for at least one $I \in \Xi$).
4. For every $I \in \Xi$, guess a $\mathcal{B}c$-term ξ_I of the form $\prod_{1 \leq i \leq \ell} \pm r_i$, where $r_1, \ldots, r_\ell$ are all the variables of φ, and fail if $\xi_I \leq \rho$ or $\xi_I \leq -\prod \lambda(I)$. Succeed otherwise.

If the procedure has a successful termination, we easily construct a model for φ over $\mathsf{RCP}(\mathbb{R})$ by setting r to be the union of all the intervals $I \in \Xi$ with $\xi_I \leq r$. Conversely, it is routine to show that, if φ has a model over $\mathsf{RC}(\mathbb{R})$, then the procedure has a terminating run. ☐

Theorem 4. *The problems* $Sat(\mathcal{RCC}8c, \mathsf{RC}(\mathbb{R}))$ *and* $Sat(\mathcal{RCC}8c, \mathsf{RCP}(\mathbb{R}))$ *are* NP*-complete*; $Sat(\mathcal{C}c, \mathsf{RC}(\mathbb{R}))$ *and* $Sat(\mathcal{C}c, \mathsf{RCP}(\mathbb{R}))$ *are* PSPACE*-complete.*

Proof. We prove here that $Sat(\mathcal{RCC}8c, \mathsf{RCP}(\mathbb{R}))$ is in NP (NP-hardness is trivial). That $Sat(\mathcal{C}c, \mathsf{RC}(\mathbb{R}))$ is in PSPACE follows from (Kontchakov et al. 2009, Theorem 6.6), and PSPACE-hardness from (Wolter and Zakharyaschev 2000, Theorem 10). The remaining statements can be proved by routine adaptation of these ideas.

Let an $\mathcal{RCC}8c$-formula φ be given. We may assume without loss of generality that φ is a conjunction of atoms of the forms $R(r, r')$, $c(r)$ and $\neg c(r)$, where R is one of the $\mathcal{RCC}8$-primitives and r, r' are variables. It will be convenient to rewrite each of these literals as an equisatisfiable conjunction of $\mathcal{C}c$-literals of the forms

$$\begin{array}{lll} r \cdot r' \neq 0, & r \cdot (-r') \neq 0, & \neg C(r, r'), \\ r \cdot r' = 0, & r \cdot (-r') = 0, & C(r, r'), \\ C(r, -r'), & \neg C(r, -r'), & c(r). \end{array} \qquad (5)$$

(Note that a literal $\neg c(r)$ can be equisatisfiably rewritten as

$$\bigwedge_{1\le i<j\le 3}(r_i\cdot r_j=0)\wedge\bigwedge_{1\le i\le 3}c(r_i)\wedge(r_1\cdot r\neq 0)\wedge$$
$$(r_2\cdot -r\neq 0)\wedge(r_3\cdot r\neq 0)\wedge C(r_1,r_2)\wedge C(r_2,r_3),$$

where the r_i are fresh variables.) Henceforth then, we assume that φ is a conjunction of literals of the forms (5). We show that φ is satisfiable over $\mathsf{RCP}(\mathbb{R})$ if and only if it has a model $\mathfrak{B}$ over an Aleksandrov space (W,R), where $W=\{x_0,\dots,x_n,z_0,\dots,z_{n-1}\}$ ($n\le|\varphi|^2$) and R is the reflexive closure of $\{(z_i,x_i),(z_i,x_{i+1})\mid 1\le i<n\}$.

Suppose $\mathfrak{M}\models\varphi$, where $\mathfrak{M}$ is a model over $\mathsf{RCP}(\mathbb{R})$. We construct our model $\mathfrak{B}$ in a two-step process.

Step 1. First we take W to contain the following points, where τ and τ' are any terms occurring in φ:

(f$_0$) if $\mathfrak{M}\models(\tau\neq 0)$, we pick a point $x\in\tau^{\mathfrak{M}}$; and

(f$_1$) if $\mathfrak{M}\models C(\tau,\tau')$, we pick a pair of points $x\in\tau^{\mathfrak{M}}$ and $x'\in(\tau')^{\mathfrak{M}}$.

Without loss of generality, we may assume that, for any pair of points picked in **(f$_1$)**, none of the other picked points lies strictly between them: this can be done by selecting that pair close enough to some point of contact of τ and τ' (we allow that the same point may be picked twice). Denote by W_0 the set of all the points picked above and let $\prec$ be the strict linear order on W_0 induced by their natural order in $\mathfrak{M}$. Let $W=W_0$ and $R=\emptyset$. Next, for each pair x,x' of points picked in **(f$_1$)**, take a fresh point z, add it to W and add (z,x), (z,x') to R. Note that (W,R) is a subgraph of the required quasi-order (i.e., each z has at most two successors and each x has at most two predecessors). We turn the space (W,R) into a model $\mathfrak{B}$ by setting $x\in r^{\mathfrak{B}}$ if and only if $x\in r^{\mathfrak{M}}$, for each $x\in W_0$ (the valuation at $z\in W\setminus W_0$ needs no definition, since, if $r^{\mathfrak{B}}$ are to be regular closed, we must have $z\in r^{\mathfrak{B}}$ if and only if $x\in r^{\mathfrak{B}}$, for some $x\in W_0$ with zRx). Note that $\mathfrak{B}\models(\tau=0)$ if and only if $\mathfrak{M}\models(\tau=0)$, and $\mathfrak{B}\models C(\tau,\tau')$ if and only if $\mathfrak{M}\models C(\tau,\tau')$. However, $\mathfrak{B}$ may require further modification to ensure that $\mathfrak{B}\models c(r)$ whenever $\mathfrak{M}\models c(r)$.

Step 2. Suppose $c(r_0)$ is true in $\mathfrak{M}$, but false in $\mathfrak{B}$. Pick any two neighbouring (with respect to $\prec$) points $x,x'\in r_0^{\mathfrak{B}}$ without a common R-predecessor, take a fresh point y, and add y to W with $y\in r^{\mathfrak{B}}$ if and only if $x,x'\in r^{\mathfrak{B}}$. Clearly, adding this point to the model does not change the truth values of subformulas of the form $(\tau\neq 0)$, $C(\tau,\tau')$, $(\tau=0)$ or $\neg C(\tau,\tau')$. So, it remains to connect y to both x and x'. We cannot, however, directly connect y to, say, x by creating a common R-predecessor because that might make one of the $\neg C(\tau,\tau')$ subformulas false. Let $r_{j_1},\dots,r_{j_k}$ be a linear order on the regions containing x such that $\mathfrak{M}\models(r_{j_i}\le r_{j_{i'}})$ whenever $j_i\le j_{i'}$. We proceed in a step-by-step way. For step 0, let $x_0=y$. For step i ($1\le i\le k$), take fresh points x_i and z_i, add them to W, add (z_i,x_{i-1}) and (z_i,x_i) to R, let

$$x_i\in r^{\mathfrak{B}}\quad\text{if and only if}\quad x_{i-1}\in r^{\mathfrak{B}}\ \text{ or }\ \mathfrak{M}\models r_{j_i}\le r$$

and $z_i\in r^{\mathfrak{B}}$ if and only if $\{x_{i-1},x_i\}\cap r^{\mathfrak{B}}\neq\emptyset$. Clearly, all subformulas of the form $(\tau\neq 0)$, $C(\tau,\tau')$ and $(\tau=0)$ that are true in $\mathfrak{M}$ are also true in $\mathfrak{B}$. It is also clear that the same holds for subformulas of the form $\neg C(r,r')$. So, it remains to show that the same holds for subformulas of the form $\neg C(r,-r')$. To this end we observe that each triple x_{i-1},z_i,x_i is either entirely in $r^{\mathfrak{B}}$ or z_i is on the boundary of $r^{\mathfrak{B}}$, in which case $x_{i-1}\notin r^{\mathfrak{B}}$ and $x_i\in r^{\mathfrak{B}}$. It also follows that in the latter case $\mathfrak{M}\models r_{j_k}\le r$ if and only if $k\ge i$, and so z_i cannot be a point of contact of r and any $-r'$. Finally, points x_k and x belong to precisely the same regions, and can be identified. In the same way we connect y to x'.

By repeating **Step 2** for each pair of neighbouring (with respect to $\prec$) points in W_0 without a common R-predecessor, we secure $\mathfrak{B}\models\varphi$, as required. □

Two-dimensional Euclidean Space

We first observe that, for the languages we consider, confining attention to the space $\mathbb{R}^2$ is significant for satisfiability.

Theorem 5. *For any of the languages $\mathcal{L}$ considered in this paper and any $n\ge 3$, Sat$(\mathcal{L},\mathsf{RC}(\mathbb{R}^2))\subsetneq$ Sat$(\mathcal{L},\mathsf{RC}(\mathbb{R}^n))$.*

Proof. Again, inclusion follows by cylindrification.

To show that it is proper, let r_i ($1\le i\le 5$) and $r_{i,j}$ ($1\le i<j\le 5$) be variables, let φ be the $\mathcal{RCC}8c$-formula

$$\bigwedge_{1\le i<j\le 5}c(r_{i,j})\ \wedge\bigwedge_{\{i,j\}\cap\{k,\ell\}=\emptyset}\mathsf{DC}(r_{i,j},r_{k,\ell})\ \wedge\bigwedge_{i\in\{j,k\}}\mathsf{TPP}(r_i,r_{j,k}),$$

and let φ° be the $\mathcal{RCC}8c^\circ$-formula obtained by replacing all occurrences of c by c°. Thus, φ° entails φ. A simple argument based on the non-planarity of the graph K_5 shows that φ (and hence φ°) is not satisfiable over $\mathsf{RC}(\mathbb{R}^2)$. On the other hand, φ° (and hence φ) is satisfiable over $\mathsf{RC}(\mathbb{R}^n)$ for $n\ge 3$. This deals with $\mathcal{RCC}8c$, $\mathcal{RCC}8c^\circ$, $\mathcal{C}$ and $\mathcal{C}c^\circ$. For $\mathcal{B}c$, replace $\mathsf{DC}(r_{i,j},r_{k,\ell})$ by $\neg c(r_{i,j}+r_{k,\ell})$ and $\mathsf{TPP}(r_i,r_{j,k})$ by $(r_i\cdot r_{j,k}=r_i)\wedge(r_i\neq 0)$. For $\mathcal{B}c^\circ$, use the formula $\bigwedge_{1\le i\le 5}\big(c^\circ(r_i)\wedge(r_i\neq 0)\big)\wedge\bigwedge_{1\le i<j\le 5}\big(c^\circ(r_j+r_j)\wedge(r_i\cdot r_j=0)\big)$. □

We now proceed to show (Theorems 6–8) that, over $\mathbb{R}^2$, our topological languages exhibit a different pattern of sensitivity to tameness from that which we observed over $\mathbb{R}$.

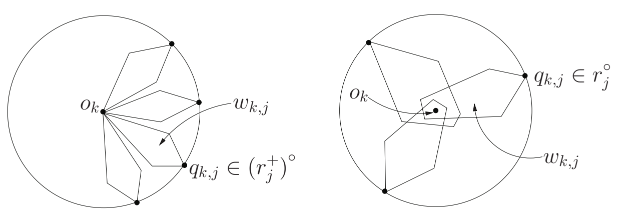
Theorem 6. *If an $\mathcal{RCC}8c$- or $\mathcal{RCC}8c^\circ$-formula is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$, then it can be satisfied over the frame of bounded regular closed polygons. In consequence:*

$$Sat(\mathcal{RCC}8c,\mathsf{RC}(\mathbb{R}^2))=Sat(\mathcal{RCC}8c,\mathsf{RCP}(\mathbb{R}^2)),$$
$$Sat(\mathcal{RCC}8c^\circ,\mathsf{RC}(\mathbb{R}^2))=Sat(\mathcal{RCC}8c^\circ,\mathsf{RCP}(\mathbb{R}^2)).$$

Proof. For the first statement, it suffices to construct, for any tuple $r_1,\dots,r_n$ in $\mathsf{RC}(\mathbb{R}^2)$, a corresponding tuple $p_1,\dots,p_n$ in $\mathsf{RCP}(\mathbb{R}^2)$ satisfying exactly the same atomic $\mathcal{RCC}8c$-formulas. We may assume that the r_i are distinct and non-empty. By reordering the variables if necessary, we can ensure that $r_i\subseteq r_j$ implies $i\le j$. For all i, j ($1\le i<j\le n$), let $R_{ij}\in\{\mathsf{DC},\mathsf{EC},\mathsf{PO},\mathsf{TPP},\mathsf{NTPP}\}$ be the unique relation such that $R_{ij}(r_i,r_j)$.

First, we construct regular closed sets $r_1^+, \ldots, r_n^+$ such that $r_j \subseteq (r_j^+)^\circ$, $(r_j^+)^\circ$ is connected whenever r_j is connected, and $r_j^+ \cap r_{j'}^+ = \emptyset$ whenever $r_j \cap r_{j'} = \emptyset$, for all j, j'. This is possible because $\mathbb{R}^2$ is a normal, locally connected topological space, and the r_i are regular closed subsets of $\mathbb{R}^2$. For all i, j $(1 \leq i < j \leq n)$, pick points o, o', o'' satisfying the conditions: (*i*) if $R_{ij} = \mathsf{EC}$, then $o \in \delta r_i \cap \delta r_j$; (*ii*) if $R_{ij} = \mathsf{PO}$, then $o \in r_i^\circ \cap r_j^\circ$, $o' \in r_i^\circ \setminus r_j$, and $o'' \in r_j^\circ \setminus r_i$; (*iii*) if $R_{ij} = \mathsf{TPP}$, then $o \in \delta r_i \cap \delta r_j$ and $o' \in r_j^\circ \setminus r_i$; (*iv*) if $R_{ij} = \mathsf{NTPP}$, then $o \in r_j^\circ \setminus r_i$. And for all i $(1 \leq i \leq n)$, pick points o, o' satisfying the condition that, if r_i is not connected, then o and o' lie in different components of r_i. Enumerate the chosen (distinct) points as $o_1, \ldots, o_m$ and call them *witness points*. We can draw disjoint closed disks $d_1, \ldots, d_m$, centred on the respective witness points o_k, such that, for all $j \leq n$ and $k \leq m$, $o_k \in r_j^\circ$ implies $d_k \subseteq r_j^\circ$; and $o_k \in (r_j^+)^\circ$ implies $d_k \subseteq (r_j^+)^\circ$. Indeed, we can ensure that none of the sets $(r_j^+)^\circ$ is disconnected by (simultaneous) removal of $d_1, \ldots, d_m$.

We now begin the construction of the $p_1, \ldots, p_n$. First, for each set r_j and each witness point o_k, we select a polygon $w_{k,j}$ inside d_k. We refer to the $w_{k,j}$ as *wedges*: for each $j \leq n$, and each $k \leq m$ we will ensure below that $w_{k,j} \subseteq p_j$. Wedges are selected as follows. (*i*) If $o_k \in \delta r_j$, pick a point $q_{k,j} \in \delta d_k \subseteq (r_j^+)^\circ$, and let $w_{k,j}$ be a lozenge with $w_{k,j} \subseteq d_k$ and $o_k, q_{k,j} \in \delta w_{k,j}$; see Fig. 6a. We may pick the $q_{k,j}$ to be distinct, and construct the $w_{k,j}$ so that no two such $w_{k,j}$ have intersecting interiors. (*ii*) If $o_k \in r_j^\circ$, pick a point $q_{k,j} \in \delta d_k \subseteq r_j^\circ$, and let $w_{k,j}$ be a lozenge such that $w_{k,j} \subseteq d_k$, $o_k \in (w_{k,j})^\circ$ and $q_{k,j} \in \delta w_{k,j}$. Again, we may pick the $q_{k,j}$ to be distinct from each other and from the $q_{k,j}$ selected in (*i*); see Fig. 6b. (*iii*) Otherwise, i.e., if $o_k \notin r_j$, let $w_{k,j} = \emptyset$. The wedges $w_{k,j}$ will ensure that $p_1, \ldots, p_n$ contain certain witness points required for the satisfaction of the relevant atomic $\mathcal{RCC}8c$-formulas. For example, if $\mathsf{PO}(r_i, r_j)$, there will exist a witness point $o_k \in r_i^\circ \cap r_j^\circ$; but then $o_k \in w_{k,i}^\circ \cap w_{k,j}^\circ$, whence $o_k \in p_i^\circ \cap p_j^\circ$, which is required to ensure that $\mathsf{PO}(p_i, p_j)$.



a) $o_k \in \delta r_j$ b) $o_k \in r_j^\circ$

Figure 6: Wedges involving the witness point o_k.

Fix any $j \leq n$. If r_j is connected, we need to connect up the various wedges $w_{k,j}$ so as to ensure that p_j is connected. Specifically, we construct a connected, regular closed polygon $a_j \subseteq (r_j^+)^\circ$ such that a_j is externally connected to all the non-empty $w_{k,j}$ (with k varying). The construction is quite elaborate, but the basic technique is illustrated in Fig. 7. The crucial point is that the a_j need only be connected—they need not have connected interiors; hence they may be crossed by regions $a_{j'}$ $(j' \neq j)$ as long as we do not have $\mathsf{DC}(r_j, r_{j'})$. The polygon a_j illustrated in Fig. 7 is crossed twice in this way. If r_j is not connected, set $a_j = 0$. For all $j \leq n$, define $b_j = a_j + \sum_{1 \leq k \leq m} w_{k,j}$. Then b_j will be connected if and only if r_j is.

Using the polygons $b_1, \ldots, b_n$, we construct the desired polygons $p_1, \ldots, p_n$, relying on the fact that we earlier ensured that $r_i \subseteq r_j$ implies $i \leq j$. Start by setting $p_1 = b_1$, and let p_1^+ be a polygon containing p_1 in its interior, but which is 'close' to p_1 (specifically: p_1^+ does not intersect any polygon involved in this construction that p_1 does not intersect). For $2 \leq j \leq n$, set

$$p_j \;=\; b_j \;+\; \sum\nolimits_{\mathsf{TPP}(r_i,r_j)} \; p_i \;+\; \sum\nolimits_{\mathsf{NTPP}(r_i,r_j)} \; p_i^+$$

and again let p_j^+ be a polygon containing p_j in its interior, and 'close' to p_j. This ensures that $r_i \subseteq r_j$ implies $p_i \subseteq p_j$, and $r_i \subseteq r_j^\circ$ implies $p_i \subseteq p_j^\circ$. We can then show that $p_1, \ldots, p_n$ satisfy exactly the same atomic $\mathcal{RCC}8c$-formulas as the $r_1, \ldots, r_n$.

A similar (but not identical) construction can be carried out for the case of $\mathcal{RCC}8c^\circ$. □

An analogous result to Theorem 6 for a more expressive spatial logic can be found in (Davis et al. 1991, Sec. 8.1).

For the languages $\mathcal{B}c$ and $\mathcal{B}c^\circ$, however, tameness does make a difference. The latter is easily dealt with:

Lemma 1. *The $\mathcal{B}c^\circ$-formula* (4) *is valid in* $\mathsf{RCP}(\mathbb{R}^n)$ *for all* $m \geq 2$ *and* $n \geq 1$.

Proof. The case $n = 1$ is trivial. See (Pratt-Hartmann 2007, p. 40) for the case $n = 2$; the proof applies almost unaltered to higher dimensions. □

Theorem 7. *Sat*$(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^2)) \subsetneq$ *Sat*$(\mathcal{B}c^\circ, \mathsf{RC}(\mathbb{R}^2))$ *and Sat*$(\mathcal{C}c^\circ, \mathsf{RCP}(\mathbb{R}^2)) \subsetneq$ *Sat*$(\mathcal{C}c^\circ, \mathsf{RC}(\mathbb{R}^2))$.

Proof. We need only show that the inclusions are proper. As observed above, (4) is invalid over $\mathsf{RC}(\mathbb{R}^2)$; but it is valid over $\mathsf{RCP}(\mathbb{R}^2)$ by Lemma 1. □

For ordinary connectedness, much more work is required.

Theorem 8. *Sat*$(\mathcal{C}c, \mathsf{RCP}(\mathbb{R}^2)) \subsetneq$ *Sat*$(\mathcal{C}c, \mathsf{RC}(\mathbb{R}^2))$. *In fact, Sat*$(\mathcal{B}c, \mathsf{RCP}(\mathbb{R}^2)) \subsetneq$ *Sat*$(\mathcal{B}c, \mathsf{RC}(\mathbb{R}^2))$.

Proof. Again, we need only show that the inclusions are proper. We begin with the language $\mathcal{C}c$; the second statement of the theorem will follow by an easy adaptation. Let

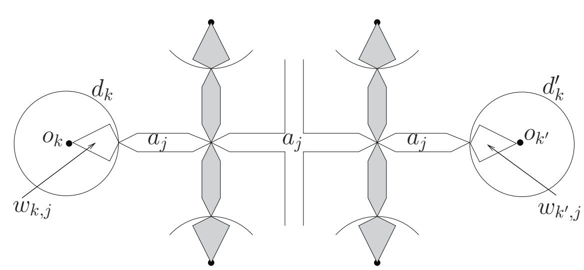


Figure 7: Connecting together wedges $w_{k,j}$ and $w_{k',j}$.

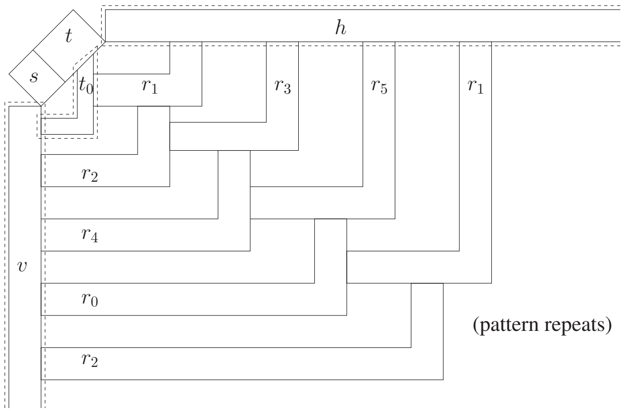


Figure 8: Satisfiability of φ over $\mathsf{RC}(\mathbb{R}^2)$.

V be the set of variables $\{v, h, s, t, t_0, r_0, r_1, r_2, r_3, r_4, r_5\}$ and, for any $x \in V$, let $\hat{x}$ be a fresh variable. Consider the assignment of elements of $\mathsf{RC}(\mathbb{R}^2)$ to the variables of V shown in Fig. 8. (We equivocate between variables and the sets assigned to them to avoid notational clutter.) Here, v and h are unbounded, connected polygons, s, t and t_0 are bounded, connected polygons, and the r_i $(0 \leq i < 6)$ all have infinitely many polygonal components. (Thus, the r_i are not themselves polygons.) Also, for all $x \in V$, $\hat{x}$ is a slightly 'enlarged' version of x: the sets $\hat{v}$, $\hat{t}_0$ and $\hat{h}$ are drawn in Fig. 8 with dotted lines; the other regions $\hat{x}$ are suppressed for clarity. Evidently, it is possible to ensure that $x \cap y = \emptyset$ implies $\hat{x} \cap \hat{y} = \emptyset$ for all $x, y \in V$.

We construct a $\mathcal{C}c$-formula φ satisfied by the arrangement of Fig. 8, but not satisfiable over $\mathsf{RCP}(\mathbb{R}^2)$. The formula φ will be a conjunction of literals, which, for the sake of readability, we present as they are required in the proof. (Additionally, we allow ourselves some redundancy in the chosen conjuncts where this makes the argument more perspicuous.) As each group of conjuncts is introduced, a visual check suffices to ensure that they are satisfied in Fig. 8. To show that φ is not satisfiable over $\mathsf{RCP}(\mathbb{R}^2)$, we assume otherwise, and derive a contradiction. Specifically, we rely on the fact that connected polygons are arc-connected (an easy consequence of tameness) to construct an infinite sequence of arcs α_i and β_i in the plane, disjoint from certain of the regions in V. Once this sequence of arcs has been constructed, it will follow that the interiors of the regions $r_0, \ldots, r_5$ all have infinitely many components, which is impossible for elements of $\mathsf{RCP}(\mathbb{R}^2)$.

First stage: We begin with conjuncts

$$\neg C(x, -\hat{x}), \tag{6}$$

for $x \in V$, ensuring that x is included in the interior of $\hat{x}$, together with the conjuncts

$$(v \neq 0) \wedge (h \neq 0), \quad (7) \qquad \neg C(\hat{t}, \hat{v}), \quad (11)$$
$$c(v) \wedge c(h), \quad (8) \qquad \neg C(s, \hat{h}), \quad (12)$$
$$c(h + t + s + v), \quad (9) \qquad c((t + t_0) \cdot (-\hat{h}) + v), \quad (13)$$
$$\neg C(v, h), \quad (10) \qquad \neg C(t_0, s). \quad (14)$$

By (7), choose a point in v and a point in h and, by (9), connect the first to the second by an arc α_0^* in $v + s + t + h$. (Here we use the fact that connectedness implies arc-connectedness.) Let q_0 be the first point of α_0^* in h. By (10), $q_0 \notin v$, and q_0 is not the first point of α_0^*. Let p_0 be the last point of α_0^* in v and strictly before q_0. Let α_0 be the segment of α_0^* from p_0 to q_0. Hence, no interior point of α_0 lies in v or h, and so $\alpha_0 \subseteq s + t$.

By (11), some initial segment of α_0 lies in s, whence, by (12), there exists a point p_1^* on α_0 such that $p_1^* \notin s$ and $p_1^* \notin \hat{h}$. It follows that $p_1^* \in t \cdot (-\hat{h})$. By (13), draw an arc α_1^* from p_1^* to $p_0 \in v$, and lying in $(t + t_0) \cdot (-\hat{h}) + v$. Since $p_1^* \notin v$, let q_1 be the first point of v on α_1^* after p_1^*.

This arc has a last point of contact with α_0 strictly before q_1. For all points of α_1^* before q_1 are in $t + t_0$, whence, by (11), some segment of α_1^* leading to $q_1 \in v$ must lie entirely in t_0. But then, by (14), this segment does not touch some initial segment of α_0. Therefore, let p_1 be the last point of α_1^* before q_1 lying on α_0. Let α_1 be the segment of α_1^* between p_1 and q_1. Thus, no point of α_1 other than q_1 lies in v, and no point of α_1 lies in $\hat{h}$, whence $\alpha_1 \subseteq (t + t_0) \cdot (-\hat{h})$. By (8), let β_1 be an arc in v from q_1 to p_0. Thus, $\beta_1 \cap \alpha_1 = \{q_1\}$ and $\beta_1 \cap \alpha_0 = \{p_0\}$. The situation is shown in Fig. 9. Let γ_1 be the Jordan curve formed by α_1, β_1 and the segment of α_0 lying between p_0 and p_1, shown in thick lines. We denote the open set bounded by γ_1 and not containing the point q_0 by R_1. By drawing the configuration on the closed plane (with all Jordan arcs avoiding the point at infinity), we may without loss of generality regard R_1 as the 'inside' of γ_1.

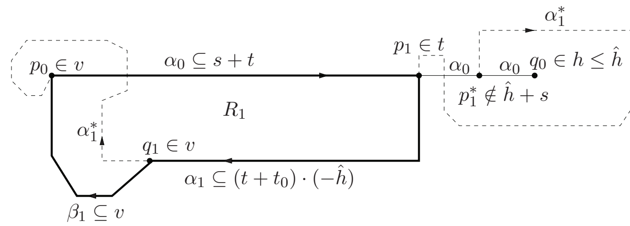


Figure 9: The first stage: the arc α_1 (solid line) is a sub-arc of α_1^*; the Jordan curve γ_1 is shown in thick lines.

Second stage: We employ the conjuncts

$$\neg C(t_0 + r_1, s), \quad (15) \qquad c((t_0 + r_1) \cdot (-\hat{t}) \cdot (-\hat{v}) + h). \quad (16)$$

Since $p_1 \in t$ and $q_1 \in v$, by (11) there exists a point p_2^* on α_1 such that $p_2^* \notin \hat{t}$ and $p_2^* \notin \hat{v}$. It follows that $p_2^* \in t_0 \cdot (-\hat{t}) \cdot (-\hat{v})$. By (16), let α_2^* be an arc from p_2^* to q_0 lying in $(t_0 + r_1) \cdot (-\hat{t}) \cdot (-\hat{v}) + h$. Let p_2 be the last point of α_2^* lying on α_1; hence, $p_2 \in t_0$. Also $p_2 \notin h$, since it lies on α_1. Let q_2 be the first point of α_2^* after p_2 lying in h; and let α_2 be the segment of α_2^* from p_2 to q_2. Thus, no point of α_2 other than q_2 lies in h, and no point of α_2 lies in $\hat{v}$; hence, $\alpha_2 \subseteq (t_0 + r_1) \cdot (-\hat{t}) \cdot (-\hat{v})$. By (8), let β_2 be an arc from q_2 to q_0 lying entirely in h. Notice that β_2 cannot intersect α_0, α_1 or α_2 except at the endpoints q_2 and q_0, because no other points of these arcs lie in h. We further claim that α_2 cannot enter region R_1, for it is impossible that any point in $(\alpha_2 \cup \beta_2) \setminus \{p_2\}$ lies on γ_1. To see this, note

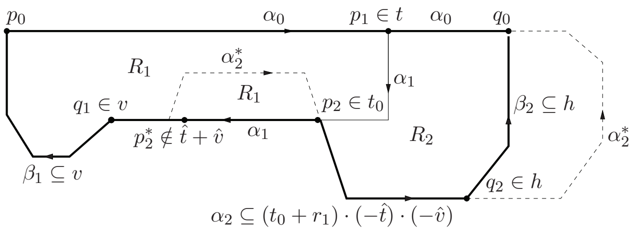


Figure 10: The second stage: the arc α_2 (solid line) is a sub-arc of α_2^*; the Jordan curve γ_2 is shown in thick lines.

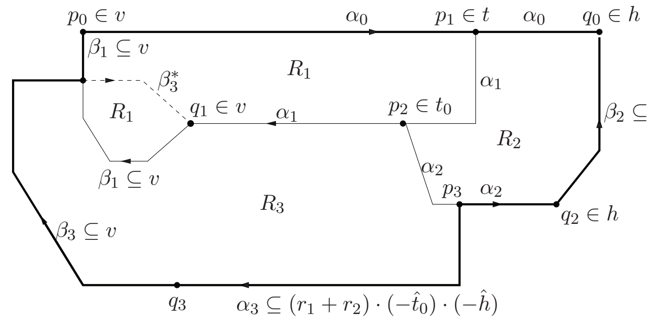


Figure 11: The third stage: the point p_3^* and arc α_3^* are not shown, for clarity; the arc β_3 (solid line) is a sub-arc of β_3^*; the Jordan curve γ_3 is shown in thick lines.

that: (*i*) by construction, no point of α_2 apart from p_2 lies on α_1, (*ii*) by (15) and the fact that $\alpha_2 \subseteq (t_0 + r_1) \cdot (-\hat{t})$, no point on α_2 apart from q_2 can lie in $s + t \supseteq \alpha_0$; (*iii*) no point of α_2 lies in $v \supseteq \beta_1$; (*iv*) no point of β_2 lies on γ_1, since no point on γ_1 lies in h. It follows that α_2 and β_2 lie on the 'outside' of γ_1 (since q_0 does). The situation is shown in Fig. 10. Thus, α_0, α_2 and β_2 divide the outside of γ_1 into two residual domains; denote that residual domain which does not have q_1 on its boundary by R_2. In addition, let γ_2 be the Jordan curve formed by (ignoring directions) α_2, β_2, α_0 and β_1, together with the segment of α_1 from q_1 to p_2. (In Fig. 10, γ_2 is drawn in thick lines.) Again, we may without loss of generality regard R_2 as lying 'inside' γ_2. Define $S_2 = (R_1 \cup R_2)^{-\circ}$. Note that S_2 is the bounded residual domain of γ_2.

Third stage: We employ the conjuncts

$$\neg C(\hat{t}_0, \hat{h}), \quad (17) \qquad c((r_1 + r_2) \cdot (-\hat{t_0}) \cdot (-\hat{h}) + v), \quad (19)$$

$$\neg C(r_1 + r_2, t), \quad (18) \qquad \neg C(r_1 + r_2, s + t). \quad (20)$$

Since $p_2 \in t_0$ and $q_2 \in h$, by (17), there exists a point p_3^* on α_2 such that $p_3^* \notin \hat{t}_0$ and $p_3^* \notin \hat{h}$. It follows that $p_3^* \in r_1 \cdot (-\hat{t}_0) \cdot (-\hat{h})$. By (19), let α_3^* be an arc from p_3^* to q_1 lying in $(r_1 + r_2) \cdot (-\hat{t}_0) \cdot (-\hat{h}) + v$. Let p_3 be the last point of α_3^* lying on α_2; hence, p_3 lies in r_1. Let q_3 be the first point of α_3^* after p_3 lying in v; and let α_3 be the segment of α_3^* from p_3 to q_3. Thus, no point of α_3 other than q_3 lies in v, and no point of α_3 lies in $\hat{h}$. By (8), let β_3^* be an arc from q_3 to q_1 lying entirely in v. Let q_3^* be the first point of β_3^* lying on $\gamma_2 \cap v$ (i.e. lying on β_1). Let β_3 be the segment of β_3^* between q_3 and q_3^*. The situation is shown in Fig. 11.

We need to show that the way in which α_3 and β_3 have been drawn is sufficiently general. Recall that R_1 is bounded by the Jordan curve γ_1, and that $S_2 = (R_1 \cup R_2)^{\circ -}$ is bounded by the Jordan curve γ_2. We first establish that the arc formed by α_3 followed by β_3 cannot enter R_1. For α_3, we have: (*i*) by (18) no point of $(r_1 + r_2) \cdot (-\hat{t}_0) \supseteq \alpha_3$ coincides with any point of $t_0 + t \supseteq \alpha_1$; (*ii*) $\alpha_3 \setminus \{q_3\}$ has no points in v, and hence none on β_1; (*iii*) by (20), no points of $(r_1 + r_2) \supseteq \alpha_3$ can coincide with any points of $s + t \supseteq \alpha_0$. For β_3, we have: (*i*) q_1 is the only point of α_1 in $v \supseteq \beta_3$; (*ii*) by construction, β_3 stops as soon as it touches β_1; (*iii*) no points of α_0 other than p_0 lie in $v \supseteq \beta_3$. We next establish that the arc formed by α_3 followed by β_3 cannot enter R_2, either. First, note that α_3 cannot *cross* the boundary of R_2, for we have: (*i*) by construction, no point of α_3, apart from the first, can lie on α_2, (*ii*) by (20) and (18), no points of $(r_1 + r_2) \supseteq \alpha_3$ can coincide with any points of $s + t \supseteq \alpha_0$; (*iii*) no point of α_3 lies in $h \supseteq \beta_2$. Second, no point on the boundary of R_2 can coincide with any point of $v \supseteq \beta_3$, and $q_1 \in v$ is outside R_2, whence v—and hence β_3—lie entirely outside R_2. This establishes that the arc formed by α_3 followed by β_3 cannot enter R_2, and so divides the exterior of γ_2 into two residual domains; denote that residual domain which does not have q_2 on its boundary by R_3. Define $S_3 = (S_2 \cup R_3)^{-\circ}$, and let γ_3 be the Jordan curve forming its boundary (drawn in Fig. 11in thick lines). Again, we may without loss of generality regard S_3 as the 'inside' of γ_3.

General stage i: We employ the following conjuncts ($0 \leq k < 6$, subscript arithmetic modulo 6):

$$\neg C(\hat{r}_{k-3}, \hat{v}), \qquad (k \text{ even}) \quad (21)$$

$$\neg C(\hat{r}_{k-3}, \hat{h}), \qquad (k \text{ odd}) \quad (22)$$

$$c((r_{k-2} + r_{k-1}) \cdot (-\hat{r}_{k-3}) \cdot (-\hat{v}) + h), \qquad (k \text{ even}) \quad (23)$$

$$c((r_{k-2} + r_{k-1}) \cdot (-\hat{r}_{k-3}) \cdot (-\hat{h}) + v), \qquad (k \text{ odd}) \quad (24)$$

$$\neg C(r_k, r_{k-2} + r_{k-3} + r_{k-4}), \quad (25)$$

$$\neg C(r_2 + r_3, t_0), \quad (26)$$

$$\neg C(r_k, s + t). \quad (27)$$

After this point, the process repeats itself through infinitely many stages; at each new stage, the numerical indices in the variables r_k are incremented (modulo 6) and h and v are transposed. In what follows we write $r_{[i-n]}$ for r_k with $i - n \equiv k \pmod 6$. The general situation (for i even), is illustrated in Fig. 12. In this stage, α_i and β_i are about to be constructed. The arc $\alpha_i \subseteq (r_{[i-2]} + r_{[i-1]}) \cdot (-\hat{r}_{[i-3]}) \cdot (-\hat{v})$ will run from a point p_i on $\alpha_{i-1} \cap r_{i-2}$ to a point q_i in h (the starting point p_i^* exists by (21) and the arc by (23)); and the arc $\beta_i \subseteq h$, will run from q_i to some point on $\gamma_{i-1} \cap h = \gamma_{i-2} \cap h$. In Fig. 12, the Jordan curve γ_{i-2}, enclosing S_{i-2}, is shown in thick lines.

The key observation is that the arc formed by α_i followed by β_i cannot enter $S_{i-1} = (S_{i-2} \cup R_{i-1})^{-\circ}$. We first show that α_i cannot enter S_{i-2}. To see this, we note that: (*i*) since $\alpha_i \subseteq (r_{[i-2]} + r_{[i-1]}) \cdot (-\hat{r}_{[i-3]})$ and $\alpha_{i-2} \subseteq (r_{[i-4]} + r_{[i-3]})$, by (25), $\alpha_i \cap \alpha_{i-2} = \emptyset$; (*ii*) since $\alpha_i \subseteq (r_{[i-1]} + r_{[i-2]})$ and $\alpha_{[i-3]} \subseteq (r_{[i-5]} + r_{[i-4]})$, by (25), $\alpha_i \cap \alpha_{i-3} = \emptyset$; (*iii*) since $\alpha_i \subseteq -\hat{v}$, α_i does not intersect the segment of γ_{i-2} from q_{i-3} (clockwise in Fig. 12) to p_0; (iv) since $\alpha_i \subseteq$

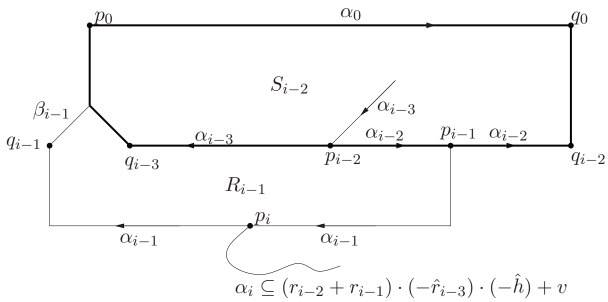


Figure 12: The general case: the arc α_i (i even); here the thick lines show the Jordan curve γ_{i-2}.

$(r_{[i-2]} + r_{[i-1]})$ and $\alpha_0 \subseteq s + t$, by (27), $\alpha_{i+1} \cap \alpha_0 = \emptyset$; (v) only the end-point q_i of α_i is in h and so α_i cannot intersect the segment of γ_{i-2} from q_0 (clockwise) to q_{i-2}. We note here that, for $i = 4$, we use (26) and (27) instead of (25) to show (*i*) and (*ii*).

We next show that α_i cannot enter R_{i-1}. To see this, we note that: (*i*) by construction, $\alpha_i \cap \alpha_{i-1} = \{p_i\}$; (*ii*) α_i cannot enter S_{i-2}, as we have just argued, and $\beta_i \subseteq h$ certainly cannot cross the boundary between R_{i-1} and S_{i-2}, none of whose points are in h by construction of α_{i-2}, α_{i-3}, and by (10); (*iii*) $\alpha_i \subseteq -\hat{v}$ and so cannot intersect the segment of the boundary of R_{i-1} from q_{i-1} (clockwise) to the point where β_{i-1} reaches γ_{i-2}; β_i cannot intersect this segment either, by (10). But this means that it is impossible for α_i and β_i to connect a point of R_{i-1} to a point of $\gamma_{i-1} \cap h$; hence α_i cannot enter R_{i-1} at all, and so must be drawn outside S_{i-1} as shown.

To complete the proof, observe that, for all $k > 0$, $p_{6k+2} \in r_0$; and since r_0 is regular closed, there exist points in the interior of r_0 arbitrarily close to p_{6k+2}. But p_{6k+2} and $p_{6(k+1)+2}$ are separated by (for example) γ_{6k+6}, which lies entirely in the set $s + t + h + v + r_3 + r_4 + r_5$, and hence contains no interior points of r_0. Therefore, r_0° has infinitely many components, as required.

For the second statement of the theorem, we replace every literal $\neg C(\tau, \tau')$ in φ with a conjunction

$$(\tau \leq p) \wedge (\tau' \leq p') \wedge c(p) \wedge c(p') \wedge \neg c(p + p'),$$

where the variables p and p' are chosen afresh for each replaced literal. Let the resulting $\mathcal{B}c$-formula be ψ. Trivially, $\mathsf{RCP}(\mathbb{R}^2) \models \psi \to \varphi$, so that ψ is not satisfiable over $\mathsf{RCP}(\mathbb{R}^2)$. That ψ is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$ is almost immediate by inspection of Fig. 8. □

We turn now to complexity-theoretic issues, employing a surprising theorem on graph-drawing. Let $\mathbf{D}$ be the frame consisting of all regular closed subsets of $\mathbb{R}^2$ homeomorphic to closed discs. (It does not matter that $\mathbf{D}$ is not a Boolean algebra.) Then *Sat*$(\mathcal{RCC}8, \mathbf{D})$ is in NP (Schaefer, Sedgwick, and Štefankovič 2003). Using Theorem 6, we can show that *Sat*$(\mathcal{RCC}8c, \mathsf{RC}(\mathbb{R}^2))$ is also in NP. The following lemma enables us to reduce *Sat*$(\mathcal{RCC}8c, \mathsf{RCP}(\mathbb{R}^2))$ non-deterministically to *Sat*$(\mathcal{RCC}8, \mathbf{D})$.

Lemma 2. *Let φ be an $\mathcal{RCC}8c$-formula, and suppose φ is satisfied by bounded polygons $r_1, \ldots, r_n$. Then φ is satisfied by bounded polygons $r'_1, \ldots, r'_n$ such that, for all $i \leq n$, (a) if u is a connected component of $(r'_i)^\circ$, then the topological closure u^- of u is in $\mathbf{D}$; and (b) $(r'_i)^\circ$ has at most $O(n^3)$ components.*

Theorem 9. *The problems Sat$(\mathcal{RCC}8c, \mathsf{RCP}(\mathbb{R}^2))$ and Sat$(\mathcal{RCC}8c^\circ, \mathsf{RCP}(\mathbb{R}^2))$ are both* NP*-complete.*

Proof. Suppose φ is an $\mathcal{RCC}8c$-formula with n variables. We describe an NP procedure for determining whether φ is satisfiable over $\mathsf{RCP}(\mathbb{R}^2)$. For each i ($1 \leq i \leq n$), take up to $O(n^3)$ fresh variables $t_{i,1}, \ldots, t_{i,m_i}$, and list all these variables as $t_1, \ldots, t_m$. For all i, j ($1 \leq i < j \leq m$), guess an $\mathcal{RCC}8$-relation R_{ij}, and let ψ be the conjunction of all the formulas $R_{ij}(t_i, t_j)$. By the result of (Schaefer, Sedgwick, and Štefankovič 2003), we check, in NP, that ψ is satisfiable over $\mathbf{D}$. Finally, we check, in deterministic polynomial time, that, if ψ is satisfied by the $t_1, \ldots, t_m$, then φ is satisfied by the regions $r_1, \ldots, r_n$, where $r_i = \sum_{1 \leq j \leq m_i} t_{i,j}$. If both of these tests succeed, we report that ψ is satisfiable. It follows from Theorem 8 and Lemma 2 this procedure has a successful run if and only if φ is satisfied over $\mathsf{RCP}(\mathbb{R}^2)$. The case of $\mathcal{RCC}8c^\circ$ is handled similarly. □

The precise computational complexity of the languages $\mathcal{B}c$, $\mathcal{C}c$ and $\mathcal{C}c^\circ$ is not known; we have only the following:

Theorem 10. *Sat$(\mathcal{L}, \mathsf{RC}(\mathbb{R}^2))$ and Sat$(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^2))$ are all* ExpTime*-hard, for $\mathcal{L} \in \{\mathcal{B}c, \mathcal{C}c, \mathcal{C}c^\circ\}$.*

Proof. The proof proceeds using the technique developed in (Kontchakov et al. 2009, Theorem 5.9). We omit the details. □

The only other complexity bound regarding logics over $\mathbb{R}^2$ reported in this paper is that *Sat*$(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^2))$ is ExpTime-hard (Theorem 14). At the time of writing, no non-trivial lower complexity bound is known for *Sat*$(\mathcal{B}c^\circ, \mathsf{RC}(\mathbb{R}^2))$. Moreover, no upper bound at all is known for these problems. Thus, we do not even know whether *Sat*$(\mathcal{B}c, \mathsf{RC}(\mathbb{R}^2))$, *Sat*$(\mathcal{B}c, \mathsf{RCP}(\mathbb{R}^2))$, etc. are decidable.

Three-dimensional Euclidean Space

Languages based on $\mathcal{RCC}8$ cannot distinguish between Euclidean spaces of more than 3 dimensions. Indeed, they are even insensitive to the tameness of sets, and to the distinction between connectedness and interior-connectedness.

Theorem 11. *The problems Sat$(\mathcal{RCC}8c, \mathfrak{F})$ are identical, where $\mathfrak{F}$ is any of* RC, $\mathsf{RC}(\mathbb{R}^n)$ *or* $\mathsf{RCP}(\mathbb{R}^n)$ *for any $n \geq 3$. Further, for an $\mathcal{RCC}8c$-formula φ, let φ° be the result of replacing all occurrences of c with c°. Then φ is satisfiable over $\mathfrak{F}$ if and only if φ° is.*

Proof. Follows from the observation (Renz 1998) that any satisfiable $\mathcal{RCC}8$-formula is satisfied by (interior-) connected polyhedra in $\mathbb{R}^n$, for $n \geq 3$. □

It is open whether *Sat*$(\mathcal{L}, \mathsf{RC}(\mathbb{R}^n)) = $ *Sat*$(\mathcal{L}, \mathsf{RC}(\mathbb{R}^3))$, for $n > 3$, where $\mathcal{L}$ is any of $\mathcal{B}c$, $\mathcal{C}c$ or $\mathcal{C}c^\circ$. The best result we have is:

Theorem 12. *Sat*$(\mathcal{C}c^\circ, \mathsf{RC}(\mathbb{R}^n)) \subsetneqq$ *Sat*$(\mathcal{C}c^\circ, \mathsf{RC})$ *for* $n \geq 1$.

Proof. Recall that (3) can be invalidated in the torus (Fig. 3). To show it is valid over Euclidean spaces, we use the following fact (Newman 1951, p. 137): if $r_1, r_2 \in \mathsf{RC}(\mathbb{R}^n)$ are non-intersecting, and points p_1 and p_2 lie in the same component of $\mathbb{R}^n \setminus r_i = (-r_i)^\circ$ for $i = 1, 2$, then p_1 and p_2 lie in the same component of $\mathbb{R}^n \setminus (r_1 \cup r_2) = (-(r_1 + r_2))^\circ$. Thus, (3) is valid over $\mathsf{RC}(\mathbb{R}^n)$. □

In the case $\mathcal{L} = \mathcal{B}c^\circ$, however, we can give a fuller answer. A *connected partition in* $\mathsf{RCP}(\mathbb{R}^n)$ is a tuple of non-empty, interior-connected, pairwise disjoint elements of $\mathsf{RCP}(\mathbb{R}^n)$ which sum to the entire space. If $r_1, \ldots, r_n$ is a connected partition, its *neighbourhood graph* is the graph (V, E) with vertices $V = \{r_1, \ldots, r_n\}$ and edges $E = \{(r_i, r_j) \mid i \neq j \text{ and } (r_i + r_j)^\circ \text{ is connected}\}$.

Lemma 3. *Let G be a connected graph. Then G is* (*isomorphic to*) *the neighbourhood graph of some connected partition in $\mathbb{R}^n$, $n \geq 3$. If G is also planar, it is the neighbourhood graph of some connected partition in $\mathbb{R}^2$.*

Proof. To prove the second statement, take a plane embedding H of G, and let H^* be its geometric dual (which we may draw with piecewise linear edges). The faces of H^* then form a connected partition in $\mathsf{RCP}(\mathbb{R}^2)$, and the geometric dual H^{**} of H^* is a drawing of the neighbourhood graph of this connected partition. Since H is connected, H^{**} is isomorphic to H, and hence to G. For the first statement, we proceed by induction on the number k of vertices of G. The case $k = 1$ is trivial. If $k > 1$, let $G' = G/e$ be the minor of G formed by collapsing some edge e of G into a single node. By inductive hypothesis, let $r_1, \ldots, r_{k-2}, r'_{k-1}$ be a connected partition in $\mathbb{R}^n$ whose neighbourhood graph is G', with the interior-connected polyhedron r'_{k-1} corresponding to the node e. It is routine to decompose r'_{k-1} into two interior-connected polyhedra r_{k-1} and r_k so that the neighbourhood graph of $r_1, \ldots, r_{k-1}, r_k$ is G. □

A *graph model* is a pair $\mathfrak{G} = (G, \sigma)$, where $G = (V, E)$ is a graph and σ is a function mapping any variable of $\mathcal{B}c^\circ$ to a subset of V. The function symbols $+$, $\cdot$ and $-$ are interpreted, respectively, as union, intersection and complement in the power-set algebra on V, and c° is interpreted as the property of graph-theoretic connectedness.

Lemma 4. (*i*) *A $\mathcal{B}c^\circ$-formula φ is satisfiable over* $\mathsf{RCP}(\mathbb{R}^2)$ *if and only if it is true in a connected planar graph model.*
(*ii*) *A $\mathcal{B}c^\circ$-formula φ is satisfiable over* $\mathsf{RCP}(\mathbb{R}^n)$*, $n \geq 3$, if and only if it is true in a connected graph model.*

Proof. We prove (*ii*); the proof of (*i*) is similar. Suppose φ is satisfiable over $\mathsf{RCP}(\mathbb{R}^n)$, $n \geq 3$. Let $\bar{s} = s_1, \ldots, s_k$ be a tuple of polyhedra satisfying φ, and $\bar{t}$ a connected partition in $\mathsf{RCP}(\mathbb{R}^2)$ such that, for all $i \leq k$, there exists a subset $R_i \subseteq \bar{t}$ of these elements such that $s_i = \sum R_i$. (Such a $\bar{t}$ always exists as long as the $\bar{s}$ are polyhedra.) Let G be the neighbourhood graph of $\bar{t}$; and make G into a graph model $\mathfrak{G}$ by assigning to each variable x_i the set of nodes R_i. Using Lemma 1, we can show that φ is true in $\mathfrak{G}$. Conversely, suppose φ is true in a connected graph model $\mathfrak{G} = (G, \sigma)$. By Lemma 3, let $\bar{r}$ be a connected partition in $\mathsf{RCP}(\mathbb{R}^n)$ whose neighbourhood graph is isomorphic to G. Taking the nodes of G to be the elements $\bar{r}$, we define a model over $\mathsf{RCP}(\mathbb{R}^n)$ as follows: if $\mathfrak{G}$ maps a variable x to the set of elements $R \subseteq \bar{r}$, interpret x as $\sum R$. One can check that φ is true in this model. □

Turning next to tameness, Theorem 11 has already shown that $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$ are not sensitive to the difference between $\mathsf{RC}(\mathbb{R}^n)$ and $\mathsf{RCP}(\mathbb{R}^n)$ for $n \geq 3$. By contrast:

Theorem 13. *Sat*$(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^n)) \subsetneqq$ *Sat*$(\mathcal{B}c^\circ, \mathsf{RC}(\mathbb{R}^n))$ *and Sat*$(\mathcal{C}c^\circ, \mathsf{RCP}(\mathbb{R}^n)) \subsetneqq$ *Sat*$(\mathcal{C}c^\circ, \mathsf{RC}(\mathbb{R}^n))$*, for all $n \geq 3$.*

Proof. By cylindrification of Fig. 4 and Lemma 1. □

For $n \geq 3$, the question of whether *Sat*$(\mathcal{L}, \mathsf{RC}(\mathbb{R}^n)) =$ *Sat*$(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^n))$, where $\mathcal{L}$ is either $\mathcal{B}c$ or $\mathcal{C}c$, is open.

Finally, we address the complexity of satisfiability. As well as settling the insensitivity of $\mathcal{B}c^\circ$ to dimension ≥ 3 in Euclidean spaces, Lemma 4 gives us some complexity-theoretic information.

Lemma 5. *The problem of determining whether a $\mathcal{B}c^\circ$-formula has a graph model is* EXPTIME*-complete. It is* EXPTIME*-hard to decide whether a $\mathcal{B}c^\circ$-formula has a planar graph model.*

Proof. The proof proceeds using the technique developed in (Kontchakov et al. 2009, Theorems 5.3 and 5.19). We omit the details. □

Theorem 14. *Sat*$(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^3)) =$ *Sat*$(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^n))$*, for $n > 3$ and the problem is* EXPTIME*-complete.*
Sat$(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^2))$ *is* EXPTIME*-hard.*

Proof. Follows from Lemmas 4 and 5. □

This result is, however, in stark contrast to the following:

Theorem 15. *Sat*$(\mathcal{B}c^\circ, \mathsf{RC})$ *is* NP*-complete.*

Proof. We need only show membership in NP. Let φ be a $\mathcal{B}c^\circ$-formula. We may assume without loss of generality that φ is of the form

$$(\rho = 0) \wedge \bigwedge_{1 \leq j \leq m} (\sigma_j \neq 0) \wedge \bigwedge_{1 \leq i \leq n} \big(c^\circ(\pi_i) \wedge (\pi_i \neq 0)\big) \wedge \bigwedge_{1 \leq k \leq p} \neg c^\circ(\tau_k).$$

We proceed in two stages: (*i*) We show that, if φ is satisfiable over RC, then it is satisfiable in a model $\mathfrak{A}$ over an Aleksandrov space with at most $2 \cdot 2^\ell + n$ points, where ℓ is the number of variables in φ; (*ii*) we identify a sub-model $\mathfrak{B} \subseteq \mathfrak{A}$ of size at most $m + 2n + 2p + n \cdot p$, such that $\mathfrak{B} \models \varphi$. Thus, we establish a polynomial model property for $\mathcal{B}c^\circ$ over RC, which gives us NP membership.

(*i*) Suppose $\mathfrak{M} \models \varphi$. We may safely assume that $\mathfrak{M}$ is finite. By a *type* ξ for φ we mean any term of the form $\prod_{1 \leq i \leq \ell} \pm r_i$, where $r_1, \ldots, r_\ell$ are all the variables of φ. Let Θ be the set of all formulas of the forms $\xi = 0$ and $\xi \neq 0$ true in $\mathfrak{M}$, where ξ is a type. Let W_0 contain a pair of distinct points x_ξ and x'_ξ, for each type ξ such that $\mathfrak{M} \models (\xi \neq 0)$ (we need two points to make some regions disconnected).

Let W_1 contain a distinct point z_{π_i} for each i ($1 \le i \le n$), and let $W = W_0 \cup W_1$. Let R be the reflexive closure of

$$\{(z_{\pi_i}, x_\xi), (z_{\pi_i}, x'_\xi) \mid \xi \le \pi_i\}.$$

Let $\mathfrak{A}$ be a model over the Aleksandrov space (W, R) with the valuation $\cdot^{\mathfrak{A}}$ defined by taking $x_\xi, x'_\xi \in r_i^{\mathfrak{A}}$ if and only if $\xi \le r_i$ (the valuation on W_1 needs no definition because the $r_i^{\mathfrak{A}}$ are to be regular closed sets, cf. the proof of Theorem 4).

We show now that $\mathfrak{A} \models \varphi \wedge \bigwedge \Theta$. This is trivial for Θ and for the first three conjuncts of φ. So, it remains to show that $\mathfrak{A} \not\models c^\circ(\tau_k)$, for each k ($1 \le k \le p$). Suppose to the contrary that τ_k is interior-connected in $\mathfrak{A}$. Since $\mathfrak{M}$ is finite, there is a sequence $x_1, z_1, x_2, z_2, \cdots, x_s$ such that $z_i R x_i$, $z_i R x_{i+1}$ and the x_i and the z_i are *all* the points in $\tau_k^{\mathfrak{A}}$. Then, by the definition of $\mathfrak{A}$, there are $\pi_{i_1}, \dots, \pi_{i_{s-1}}$ such that $x_j, z_j, x_{j+1} \in \pi_{i_j}^{\mathfrak{A}} \cap \tau_k^{\mathfrak{A}}$, for all j ($1 \le j < s$). It follows then that $z_{i_j} \in (\pi_{i_j}^\circ)^{\mathfrak{A}} \cap (\tau_k^\circ)^{\mathfrak{A}}$ and, as z_{i_j} is the R-predecessor of all points in $\pi_{i_j}^{\mathfrak{A}}$, we obtain $\pi_{i_j}^{\mathfrak{A}} \le \tau_k^{\mathfrak{A}}$. So, we have $(\sum_{1 \le j < s} \pi_{i_j})^{\mathfrak{A}} \le \tau_k^{\mathfrak{A}}$. On the other hand, as the path contains all points in $\tau_k^{\mathfrak{A}}$, we obtain $\tau_k^{\mathfrak{A}} \le (\sum_{1 \le j < s} \pi_{i_j})^{\mathfrak{A}}$. Thus, $\mathfrak{A} \models (\tau_k = \sum_{1 \le j < s} \pi_{i_j})$, and further, $\mathfrak{A} \models (\pi_{i_j} \cdot \pi_{i_{j+1}} \neq 0)$ for all j ($1 \le j < s$). But Θ determines the truth of all equations and inequations involving terms in the variables of φ; hence $\mathfrak{M} \models (\tau_k = \sum_{1 \le j < s} \pi_{i_j})$ and $\mathfrak{M} \models (\pi_{i_j} \cdot \pi_{i_{j+1}} \neq 0)$ for all j ($1 \le j < s$). This is easily seen to contradict $\mathfrak{M} \models \bigwedge_{1 \le j < s} c^\circ(\pi_{i_j}) \wedge \neg c^\circ(\tau_k)$.

(*ii*) We select the following points:

– for each j ($1 \le j \le m$), pick $x \in W_0 \cap \sigma_j^{\mathfrak{A}}$;

– for each i ($1 \le i \le n$), pick $x \in W_0 \cap \pi_i^{\mathfrak{A}}$ and $z_{\pi_i} \in W_1$;

– for each k ($1 \le k \le p$), pick two points $x_{\tau_k}, x'_{\tau_k} \in W_0 \cap \tau_k^{\mathfrak{A}}$ form distinct components of $\tau_k^{\mathfrak{A}}$ and up to n points $y_{\overline{\tau_k}, \pi_i} \in W_0 \cap \pi_i^{\mathfrak{A}} \cap (-\tau_k)^{\mathfrak{A}}$, for $1 \le i \le n$ (no point is picked if the set is empty).

As φ is true in $\mathfrak{A}$, all the points mentioned above necessarily exist (apart from the $y_{\overline{\tau_k}, \pi_i}$, some of which may not exist). Let V be the set of all these points and $\mathfrak{B}$ the restriction of $\mathfrak{A}$ onto V. We claim that φ is true in $\mathfrak{B}$. Indeed, this is clearly the case for the first three conjuncts of φ. So, it remains to show that $\mathfrak{B} \not\models c^\circ(\tau_k)$, for each k ($1 \le k \le p$). This fact follows from the observation that $z_{\pi_i} \in (\tau_k^\circ)^{\mathfrak{A}}$ if and only if $z_{\pi_i} \in (\tau_k^\circ)^{\mathfrak{B}}$, for each i ($1 \le i \le n$) and therefore, τ_k is interior-connected in $\mathfrak{B}$ if and only if it is interior-connected in $\mathfrak{A}$. □

As shown in (Kontchakov et al. 2009), *Sat*$(\mathcal{L}, \mathsf{RC}(\mathbb{R}^n))$ and *Sat*$(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^n))$ are ExpTime-hard, for $\mathcal{L}$ any of $\mathcal{B}c$, $\mathcal{C}c$ or $\mathcal{C}c^\circ$ and $n \ge 3$. The upper complexity bounds for these problems are open.

Conclusions

In this paper, we have investigated the six languages $\mathcal{RCC}8c$, $\mathcal{B}c$, $\mathcal{C}c$, $\mathcal{RCC}8c^\circ$, $\mathcal{B}c^\circ$ and $\mathcal{C}c^\circ$, obtained by variously extending $\mathcal{RCC}8$ with the connectedness and interior-connectedness predicates c and c°, and the function symbols $+$, $\cdot$ and $-$, representing operations on regular closed sets. We paid particular regard to the interpretation of these languages over low-dimensional Euclidean spaces. We showed that—in contrast to the less expressive $\mathcal{RCC}8$—the dimensionality of the space is important for all of our languages, and that, in addition, these languages exhibit varying patterns of sensitivity to tameness in different dimensions. Thus, $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$ both distinguish between $\mathsf{RC}(\mathbb{R}^n)$ and $\mathsf{RCP}(\mathbb{R}^n)$ for $n = 1$, but do not for $n \ge 2$; $\mathcal{B}c$ does for $n = 2$, but not for $n = 1$; $\mathcal{B}c^\circ$ does for $n \ge 2$, but not for $n = 1$; $\mathcal{C}c$ does for $n = 1$ and 2; $\mathcal{C}c^\circ$ does in all dimensions. We also obtained results on the complexity of reasoning in these logics. For example, the satisfiability problems for $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$, under all the interpretations considered here, are NP-complete, whereas the corresponding problems for $\mathcal{B}c$, $\mathcal{B}c^\circ$, $\mathcal{C}c$ and $\mathcal{C}c^\circ$ in $\mathbb{R}^n$, for $n \ge 2$, are generally ExpTime-hard. (Two cases are still open). A matching ExpTime upper bound was proved for $\mathcal{B}c^\circ$ over polyhedra in $\mathbb{R}^n$, $n \ge 3$. The expressiveness and complexity problems that remain open are indicated in Fig. 2.

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