Understanding Ontological Levels

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Abstract

In this paper, I defend a *multiplicative* approach that distinguishes statues from amounts of matter, political entities from physical ones, qua entities (e.g. John qua Alitalia passenger) from players (e.g. John), etc. I develop a *theory of levels* which is based on the primitive notions of *level*, *parthood*, and *grounding* (a kind of existential dependence) and that is used to characterize more specific relations like *constitution*, *inherence*, and *abstraction*. I neither aim to propose a 'definitive' theory of levels nor to commit to their ontological or conceptual nature. Hence, the adjective 'ontological' used in the title does not qualify the nature of the entities that belong to levels but the way the notion of level is characterized, i.e. in terms of general and philosophically well-founded notions. By keeping away from a purely realist attitude, I can then discuss the *adequacy* of some alternative first-order theories to account for three puzzling scenarios.

Introduction

Three examples will guide my formal analysis of the notion of *ontological level*. The first one concerns the problem of *constitution* that is discussed mainly in philosophy (Baker 2007; Rea 1996; Wasserman 2009) but also in knowledge representation (Guizzardi 2005; Vieu, Borgo, and Masolo 2008). The additional two examples are quite well known by knowledge engineers. One regards the representation of *roles* (Loebe 2007; Guizzardi 2005; Masolo et al. 2004; Steimann 2000), the other the representation of different *levels of details* (Giunchiglia and Walsh 1992; Hobbs 1985; Keet 2008; Zucker 2003).

Example 1. A sculptor creates the statue of the infant Goliath – hereby named 'Goliath' – by sculpting a lump of clay – hereby named 'Lumpl'. Lumpl, but not Goliath, would survive a squeezing; Goliath, but not Lumpl, would survive the loss of some parts, e.g. a tiny piece of the finger. Lumpl already existed before the sculptor bought it, while Goliath comes into existence only once the sculptor has completed his/her work.[1] Goliath, by a continuous and complete renovation of the clay it is made of, could survive the destruction of all parts of Lumpl. Goliath, but not Lumpl, has been created by an artist (by means of an intentional act), it costs 2000 euros, it causes you to pay a ticket to see it. One tends to conclude that Lumpl and Goliath are unquestionably two different objects that, during a period of time, are spatially co-located. More generally, 'being a statue' and 'being a lump of clay' are different properties that classify objects with different persistence conditions (e.g. parts are essential to lumps of clay while shape is essential to statues), different causal powers, etc. The example can be generalized to other cases, e.g. collection of cells vs. organ, body vs. person, piece of paper vs. money, piece of metal vs. traffic sign.

Example 2. Let us suppose that, in 2009, Alitalia carried a million passengers. If, in 2009, some persons flew Alitalia more than once, which is plausible even for Alitalia, then Alitalia served less than a million persons. The *counting problem* (Gupta 1980) states that to count the passengers of an airline one cannot simply count the persons that flew it. Passengers but not persons have a flight number and specific rights and obligations. The same person can fly different airlines or (s)he can fly several times the same airline with different destinations or simply in different days. The same schema applies in general to *roles* (e.g. 'being president', 'being student', 'being catalyst') the representation of which faces another challenging difficulty. Let us consider the following situation: Luc as passenger of Air France has the right of checking in online, while, as passenger of Alitalia, has the obligation of checking in at the airport.[2] The *conflict properties paradox* shows that if passengers reduce to persons then one obtains a contradiction: Luc cannot have both the right of checking in online and the obligation of checking in at the airport (assuming a standard view on rights and obligations).

Example 3. Parsimony is a principle that no-one, philosopher or computer scientist, renounces to. However, let us

[1]In the original example (Gibbard 1976), Goliath is instantaneously created by assembling the upper body with the lower body.

[2]This is a reformulation of the classical example 'Nixon as a Quaker is a pacifist, while Nixon as a republican is not' introduced in (Reiter and Criscuolo 1981) (that indeed considers 'John' instead of 'Nixon') and widely analyzed in the fields of non-monotonic logics and inheritance networks. Note that it is possible to find examples where the properties apply at the same time as in 'Luc as customer of Fiat spent 15K euros, while as customer of Sony just 2K euros' (assuming that the two transactions are synchronous).

suppose to “re-express truths about the global political consequences of a decline in the GNP [Gross National Product] of Eastern Europe in terms of interactions among fundamental particles” ((Heil 2005), p.31) or to plan a trip by conceiving roads as three dimensional objects composed by amount of matters that can change parts across time (Hobbs 1985). Some reductions can be so complex to make high-level patterns and relations, e.g. political decisions or social interactions, “invisible at the level of physics” (Heil 2005) or at least non manageable in reasoning terms. Humans are able to choose the relevant information necessary for a specific task and to hide (but preserve for different tasks) irrelevant details. Some properties can then apply only to objects at a given level of abstraction and *abstraction hierarchies* (see for example (Bisantz and Vicente 1994)) can be used to accommodate descriptions of complex systems at different levels of detail. For example, higher levels represent the systems in terms of purposes and functions while lower levels in terms of physical implementations. The ability to move along the abstraction hierarchy reduce complexity (saving cognitive efforts) or add detail (providing a more accurate picture of the system). High-level objects are then the result of an abstraction process that starts from basic (often physical) objects. In particular, *domain aggregation* is “a transformation that corresponds to abstraction where objects are composed to build new ones” ((Zucker 2003), p.1296).

In this paper, I will present a multiplicative approach that distinguishes statues from amounts of matter, qua entities from players, political entities from physical ones, etc. I will argue that, in knowledge representation, this multiplicative approach has some advantages with respect a purely reductionist approach.

According to a strong realist position, only amounts of matter exist in the strict ontological sense, and ‘being a statue’ is a *contingent* (non-necessary) property of amounts of matter. Goliath is then the result of a conceptual construction that collects different amounts of clay on the basis of *cognitive* criteria that can be founded on the shape of the amounts of matter, on spatio-temporal continuity, etc. Roles too are contingent properties of players while *syntactic* approaches to abstraction, e.g. (Giunchiglia and Walsh 1992), just provide links between formal languages without addressing the way the objects in the domains of quantification are connected.

Vice versa, there are philosophers and knowledge engineers that accept (spatio-temporally) co-located objects. The *constitution view* (see (Baker 2007; Wasserman 2009)) is a non-reductionist approach claiming that Lumpl constitutes, but is different from, Goliath. Constitution is a factive (and asymmetric) relation that, at a specific time, holds between two objects. It does not reduce to parthood or co-location, it just allows the constituted entity to *inherit* some properties from the constituting one and vice versa, i.e. it provides a sort of *unity*.[3] In addition to players (e.g. John) of roles (e.g. being an Alitalia passenger of flight 123 on day D) (Masolo et al. 2004) introduces *qua entitites*[4] (e.g. John *qua* Alitalia passenger of flight 123 on day D) that *inhere in* the players. To solve the counting problem and the conflicting properties paradox it is then enough to count qua entities and to apply conflicting properties to them.[5] Differently from constituted entities, a qua entity inheres, during its whole existence, in the same player, i.e. it cannot change *host*. *Semantic* approaches to abstraction, e.g. (Nayak and Levy 1995), link the objects in the domain of the ground theory with the ones in the domain of the abstract theory. The same mechanism is at the basis of some theories of *granularity* (see (Keet 2008) for a good overview).

For most philosophers it is fundamental to establish whether the nature of a given kind of entities is *ontological*, i.e. whether these entities exist in the reality, or *conceptual*, i.e. whether they are the result of cognitive processes on ontological entities to which, ultimately, they can be reduced. I think that, in knowledge representation, a more pragmatic position that does not commit neither to reductionism nor to anti-reductionim is possible. If the introduction of a new kind of entities produces a better model (in terms of conceptual clarity, reasoning performance, etc.) then, independently of the nature of these entities, the model deserves some attention. For example, if qua or constituted entities help in solving some modeling problems that are very difficult to manage otherwise, I don’t see why these entities ought to be rejected because of their conceptual nature. The clarification of what characterize different kinds of entities and how they are related seems to me more important than establishing their nature that, indeed, is a very hard task. On the other hand, the general (and foundational) point of view of philosophers is a very important input to avoid ad-hoc solutions that are difficult to generalize, re-use, and share. In this perspective, the expression ‘*ontological* level’ used in the title concerns the way levels are characterized (i.e. in terms of general and (philosophically) well-founded notions) and not the ontological nature of the entities that belong to them. Therefore, in this paper, ‘exist’ simply means ‘included in the domain of quantification’ without committing to the nature of existence.

Given these premises, in the following, I will introduce a

[3]Using the terminology of Baker, they can have properties *derivatively*. However, the properties of objects belonging to a higher level are not always reducible to the ones of their constituents, i.e. *emergent* or *supervenient* properties are possible (see (Kim 2003) for a good review on this topic).

[4](Fine 1982) commits to qua entities to solve the problem of constitution too.

[5](Guizzardi 2005) proposes an alternative approach for *relational roles*, i.e. roles defined on the basis of a relation involving the players. For example, ‘being a student’ is defined on the basis of ‘enrollment’ because, to be a student, John has to be enrolled in a university. This approach relies on the existence of *relational tropes*, i.e. individualizations of relationships. If John is enrolled in the University of Trento, then there exists a relational trope, ‘the John’s being enrolled in the University of Trento’, that depends on both John and the University of Trento and that exists when the enrollment relation holds. It is then possible to count relational tropes and assign conflicting properties to them or the the sums of players and tropes. I think that the two approaches are quite close. However, here I prefer to consider qua entities because it is intuitively easier to establish a parallel with constituted entities.

multiplicative framework based on the primitive notions of *level*, *parthood*, and *grounding* (a kind of existential dependence). These notions will be used to characterize the relations involved in the previous examples: *constitution*, *inherence*, and *abstraction*. I do not aim at proposing a 'definitive' theory, instead I will discuss the conceptual and formal consequences of axioms that characterize different points of view on the primitives.

The *multiplicative* approach I adopt is called *entity staking* (Vieu, Borgo, and Masolo 2008). In this approach, constitution, inherence, and abstraction can be seen as special cases of *existential dependence*: at a specific time, Goliath depends on Lumpl, John qua passenger depends on John, the abstract entity depends on the grounding entities.[6] This dependence can be generalized to kinds. While statues, to exist, require amounts of matter, amounts of matter can exist without any statue. Passengers require persons but not vice versa and similarly in the case of abstract entities. However constitution, inherence, and abstraction do not collapse to the same relation. For example, the constituent of Goliath can change through time while John qua passenger is always anchored to John (similarly for abstract entities). Or, abstract entities are grounded on a 'plurality' of (similar) entities while constituents and hosts can be 'singular'.

The *grounding* primitive introduced in (Correia 2002) – more specifically, a temporally qualified version of it – is a good candidate to represent existential dependence. Intuitively, an object a is grounded on a (different) object b at t if the existence of b at t makes possible the existence of a at t, i.e., a owes its existence at t to b's existence at t.[7] Following Husserl and Fine[8], Correia founds his theory of dependence on the *factual primitive* of grounding. The majority of approaches to dependence has a modal nature. By defining the existential dependence of x on y by $\Box(\mathsf{E}x \rightarrow \mathsf{E}y)$ (where $\mathsf{E}x$ stands for "x exists"), modal-existential approaches can get rid of grounding. (Correia 2002) shows that purely modal approaches are inadequate to represent existential dependence (see in particular chapter 2).[9] Here I'm interested in the fact that, according to the modal definition, the existential dependence of x on y "amounts to the necessary truth of a material conditional whose antecedent is about x only and whose consequent is about y only; and given that any such material conditional fails to express any 'real' relation between the two objects, it is hard to see how prefixing it with a necessary operator could change anything in this connection" ((Correia 2002), p.58). This remark is still more relevant to 'non-rigid' dependences, like constitution. Goliath does not depend on Lumpl in this strong sense because Goliath can exist without Lumpl. An entity can then *generically* depend on a kind F of entities. This generic dependence is often defined by $\Box(\mathsf{E}x \rightarrow \exists y(\mathsf{E}y \wedge Fy))$. But this definition faces the same problems than the previous one. In particular, it does not allow to represent on which specific entity an entity depends at a given time. These observations motivate my choice to start from a temporary and factual primitive, choice that, indeed, is similar to the one of the majority of the approaches to constitution.[10]

As already suggested, two additional notions are fundamental to characterize the differences between constitution, inherence, and aggregation: time and parthood. My analysis does not rely on any specific theory of time. I will thus consider time just as a non-structured set of indexes. Things are different for parthood. How parthood and constitution are related is a highly debated issue (see (Rea 1996)). This debate is complicated by the fact that there is no consensus about the core properties of parthood (see (Casati and Varzi 1999)). In the following, I will carefully differentiate grounding from parthood by assuming a purely formal view on mereology. In this view, mereology just aims at referring to 'pluralities' of entities without committing to sets (this will be clearer later).

Once one can stack entities on the basis of grounding, then nothing prevents the existence of chains of more than two objects. For instance, at the same time, a pebble can be grounded on an amount of matter and it can ground a paperweight (an artifact with persistence properties different from the ones of the pebble). The notion of *level* is a fundamental tool to analyze and represent the world that has been used in different disciplines, e.g. cognitive science, philosophy, mathematics, computer science, engineering (see (Yao 2009) for a review). As stated in (Yao 2009), "levels and hierarchies are the result of both separation and integration". *Separation* allows to concentrate on a particular level, while *integration* allows to understand how levels interact and are organized. While integration will be analyzed in terms of grounding, separation requires a notion of '*being at the same level as*'. A recursive definition of this relation on the basis of grounding is possible but it requires bottom-level entities to stop the recursion. In addition, assuming a unique bottom level, hierarchies of levels are necessarily linear.[11] I prefer to reject this definition for two main reasons. The first one is purely technical. First-order logic with identity, the formalism adopted here, does not allow to express recursion. The second reason is more conceptual. From an applicative perspective, I don't consider the existence of bottom levels as very restrictive (infinite (down) chains of objects are impractical). Instead, the idea of reducing the notion of level to

[6]The opposite holds for none of the previous examples.

[7]This 'synchronic' grounding is a special case of the diachronic one introduced in (Correia 2002). Clearly, the 'synchronic' grounding does not allow to account for *historical* dependence (Thomasson 1999), e.g. the dependence between an event and its causes, or between a son and his parents. From a technical viewpoint, 'synchronic' grounding is enough to account for the three relations considered in the examples and, in any case, it can be, I think, quite easily extended to the diachronic case. From a more conceptual viewpoint, I find diachronic grounding as more complex because it involves the very difficult notion of *causation*.

[8]Fine's theory can be found in (Simons 1987), p.310-314.

[9]The previous definition has been largely modified to answer criticisms, see (Simons 1987) and (Correia 2002) for exhaustive discussions.

[10](Baker 2007) defines the constitution relation at a given time on the basis of more basic primitives. Admittedly I find some primitives very hard to understand and not well characterized.

[11]Assuming different bottoms one can build different linear chains of levels.

the one of 'distance' from the bottom (in terms of 'grounding steps') rules out some interesting cases. As in the case of existence, I want to minimize my commitment to the nature of levels. Levels can depend only on laws of nature (e.g. *levels of reality* (Poli 2006)) or they can be the result of a conceptualization of reality (e.g. *levels of granularity or abstraction*). Following (Wiggins 1968), levels can correspond to (natural) *kinds* of objects (i.e. objects that are at the same level satisfy the same sortal or substance concept, they have common identity criteria, common persistence conditions, etc.).[12] For some authors, e.g. (Baker 2007; Poli 2006), levels are non-linear because some comparisons do not make sense: 'are robots on a higher level than sea slugs?' But tree-hierarchies can also account for different (conceptual) points of view on reality. In the following I will thus start from a primitive notion of 'being at the same level as' that allows for non-linear hierarchies. This does not prevent the interested modeler to constrain the linearity of levels by adding appropriate axioms.

The formal characterization of the interplay among the three primitives of *being at the same level as*, (temporary) *grounding*, and (temporary) *parthood* will be the subject of the next sections. I will start by analyzing the simple case of snapshot-models of the world, i.e. theories that model the world at a given time. This step enables me to re-use some formal theories already present in the literature. I will then extend the snapshot-models by adding time and change into the picture. I conclude the paper by sketching how the proposed theory can be used to represent constitution, inherence and abstraction.

Levels

From a very abstract perspective, levels induce an order on objects and group them accordingly. From a *reflexive* and *transitive* relation $\leq$, $x \leq y$ stands for "x is at a lower level than or at the same level as y", an equivalence relation $\approx$, $x \approx y$ stands for "x is at the same level as y", can be defined as in (d1). Orders can be linear (a1), down linear (a2), up linear (a3), connected (a4) or disconnected, bounded or unbounded, discrete or dense, etc.

The theorem (t1), that follows directly from the transitivity of $\leq$,[13] shows that if $x \leq y$, then $\leq$ holds also between all the objects at the same level as x and y respectively (i.e. it orders levels intended as $\approx$-equivalence classes).

d1 $x \approx y \triangleq x \leq y \wedge y \leq x$ (*same level*)

a1 $x \leq y \vee y \leq x$ (*linearity*)

a2 $y \leq x \wedge z \leq x \rightarrow y \leq z \vee z \leq y$ (*down linearity*)

a3 $x \leq y \wedge x \leq z \rightarrow y \leq z \vee z \leq y$ (*up linearity*)

a4 $\exists z(z \leq x \wedge z \leq y)$ (*connectedness*)

t1 $z \approx x \wedge w \approx y \wedge x \leq y \rightarrow z \leq w$

Unsurprisingly, $\leq$ is inadequate to represent grounding that, as already stated, is a factual relation linking specific objects. For instance, (at a given time) Goliath is grounded on Lumpl but not on all the objects at the same level as Lumpl. Vice versa, I will show that the weaker notion of 'being at the same level as' together with grounding are enough to define $\leq$.

Kamp (see (Reynolds 2002)) considers a theory based on a *strict order* $\prec$ (transitive and asymmetric) and an *equivalence relation* $\equiv$ (reflexive, symmetric, and transitive) satisfying the additional axioms (a5)-(a8). The theory is used to interpret modal logics for (branching) time: temporal operators are interpreted in terms of $\prec$ while the the standard necessity operator in terms of $\equiv$. Here I consider Kamp's theory as a starting point for representing 'being at the same level as' (with $\equiv$) and grounding (with $\prec$, where $x \prec y$ stands for "x grounds y" or "y is grounded on x").

d2 $x \leq y \triangleq x \equiv y \vee \exists u(u \equiv x \wedge u \prec y)$ (*lower/same level*)

d3 $x \preceq y \triangleq x \equiv y \vee x \prec y$

a5 $x \prec y \wedge x \prec z \rightarrow y \prec z \vee y = z \vee z \prec y$ (*up linearity*)

a6 $y \prec x \wedge z \prec x \rightarrow y \prec z \vee y = z \vee z \prec y$ (*down linearity*)

a7 $x \equiv y \rightarrow \neg x \prec y$

a8 $x \equiv y \wedge u \prec x \rightarrow \exists v(v \equiv u \wedge v \prec y)$

Note that (a5) and (a6) involve the *identity* relation while (a2) and (a3) involve just an *equivalence* relation, thus an analogue of (t1) does not hold for $\prec$. By adopting (d2)[14], it is possible to prove the reflexivity, transitivity, and down linearity of $\leq$. This makes explicit that, once levels are determined by means of $\equiv$, grounding can be used to induce an order between them. (a8) plays a central role here. It claims that, if it is possible to go deep (in the level hierarchy) from an object x, then it is possible to go deep also from all the other objects at the same level as x. In some sense, (a8) characterizes the generic dependence of higher levels on lower ones. The down linearity of $\leq$ relies on (a6). However $\leq$ is not linear because parallel and incomparable (by means of $\equiv$) hierarchies of levels are not ruled out. This also implies that $\leq$ is not connected.

While, as stated, the up linearity of $\prec$ (a5) seems too strong because, for example, it rules out the possibility of representing different conceptualizations of the same world,[15] I'm not able to clearly argue in favor or against (a6). On one hand, interpreting grounding as a purely existential dependence, (a6) seems too restrictive. For example, (i) according to (Correia 2002), if a is grounded on b then it is grounded also on all the parts of b and (ii) relational tropes can, in principle, depend on entities belonging to different levels. On the other hand, I have the strong intuitions that (i) if, at a given time, Goliath is intimately connected to Lumpl, it cannot be grounded (excluding the objects that eventually ground Lumpl) on something different; (ii) the specific plurality of objects that grounds an abstract object

[12] As far as I understand this is also the position of (Poli 2006) and (Baker 2007).

[13] The proofs of the theorems are reported in the appendix.

[14] (d2) is not equivalent to $x \equiv y \vee \exists u(u \equiv y \wedge x \prec u)$ because it is possible to have objects that do not ground any object, i.e. we don't have an upper version of (a8).

[15] Some authors reject (a5) in an explicit way. For instance, (Borgo and Vieu 2009) accepts artifacts with different functionalities to be grounded on the same physical object.

is enough to distinguish it from other abstract objects with very similar properties.

Axioms (a9) and (a10) imply the unicity of the bottom level, i.e. everything is ultimately grounded on objects belonging to the same level. The existence of a unique bottom is highly debated (see (Schaffer 2003) for a review of different positions present in the philosophical and scientific literature), therefore also in this case I don't strongly commit to (a9) and (a10). However, these axioms simplify the theory (in particular they allow to prove the connectedness of $\leq$ (t2)) without being too restrictive from the representational point of view. Basically they rule out forests of hierarchies grounded on different bottoms that, assuming a finite number of bottoms, can always be represented by introducing one grounding relation for each bottom (indeed a solution not really elegant).

d4 $\mathsf{B}x \triangleq \neg\exists y(y \prec x)$ *(bottom level)*

a9 $\mathsf{B}x \wedge \mathsf{B}y \rightarrow x \equiv y$

a10 $\mathsf{B}x \vee \exists y(\mathsf{B}y \wedge y \prec x)$

t2 $\mathcal{F} \vdash \exists z(z \leq x \wedge z \leq y)$

t3 $\mathcal{F} \vdash y \prec x \wedge z \prec x \wedge y \equiv z \rightarrow y = z$

In the following, I will consider the theory $\mathcal{F} = \{$reflexivity, symmetry, and transitivity of $\equiv$, asymmetry and transitivity of $\prec$, (a6)-(a10)$\}$. $\mathcal{F}$ includes (a6), (a9), and (a10). However, this move has to be intended just as a direction of exploration, a starting point in the study of the notions of level and grounding that offers a compromise between usefulness and complexity. As in the case of parthood (see, for example, (Casati and Varzi 1999; Simons 1987)), other options, adequate to different scenarios, deserve to be taken into account and compared.

(t3) shows that for each level, an object can be grounded only on a unique object. Later I will give more details on this aspect. For the moment note that, differently from (Correia 2002), the proper parts of an object a at the same level as a cannot ground the objects grounded on a.

Adding parthood

A *whole*, e.g. a table, can have persistence criteria and causal powers different from the ones of its *parts*, e.g. a top and four legs. To exist, the table requires the existence of the top and the legs. Is therefore parthood just another kind of grounding? Or, more strongly, can a theory of levels adopt parthood rather than grounding? Note that if Lumpl is part of Goliath, then all the parts of Lumpl, if one accepts the transitivity of parthood, are also parts of Goliath. Therefore, one has to reject transitivity or to accept that parthood captures a *partial* grounding (as in (Correia 2002)). Moreover, by accepting the antisymmetry of parthood (a property with a large consensus) if Lumpl is part of Goliath then the opposite does not hold. But, if Goliath is not part of Lumpl, then the supplementation principle (a11) assures the existence of a part of Goliath disjoint from Lumpl. Therefore, to say that Goliath is grounded only on Lumpl, one must reject (a11), yielding a *non-extensional* mereology, i.e. a theory where it is possible to have different entities with the same parts. A similar observation applies to situations in which different objects are grounded on exactly the same objects. This kind of remarks pervades the literature about parthood. The term 'part' is so general that it can be interpreted in several (often contradictory) ways. The heterogeneity of these interpretations makes very difficult the individuation of a common core of properties for parthood. In addition, note that, by reducing grounding to parthood, 'being atomic' collapses to 'being at the bottom level', or, in other terms, the parts cannot be at the same level as the whole.[16] This is particularly counter-intuitive for objects like amounts of matter for which one tends to accept that some of their parts are still amounts of matter. A change of level seems to underline a 'change in the nature' of objects that belong to them.

To address these problems, I prefer to assume both parthood and grounding to be primitive.[17] This allows me to consider a simple and well studied theory of parthood called *classical extensional mereology*. This theory has been introduced by (Lesniewski 1991) to avoid the ontological gap, typical of set-theory, between elements and sets. In classical extensional mereology, parthood is a purely *formal* relation that allows to refer to 'multitudes' of objects, called *mereological sums*, that are nothing more than their summands (Lewis 1991). In particular, mereological sums do not consider the relations that hold among the summands and therefore are not wholes in any sense. The mereological sum of specific objects is unique just because it is not possible to sum up them in different ways (by relying on intrinsic or extrinsic relations among the summands). Therefore parthood satisfies the *supplementation principle* (a11), from which it follows that two objects with the same parts are identical (*extensionality*). More formally, a classical extensional mereology with binary sums (see (Casati and Varzi 1999; Simons 1987) is a theory where P ($x\mathsf{P}y$ stands for "x is part of y") is reflexive, antisymmetric, transitive and satisfies (a11) and (a12). 'Propert parthood' (PP) and 'overlap' (O) are defined, respectively, by (d5) and (d6), while 'binary sums' ($s\mathsf{SM}ab$ stands for "s is the mereological sum of a and b") are defined as in (d7) and their existence is guaranteed by (a12). $\mathcal{M} = \{$reflexivity, antisymmetry, transitivity of P, (a11), (a12)$\}$ denotes this extensional mereology.

d5 $x\mathsf{PP}y \triangleq x\mathsf{P}y \wedge \neg y\mathsf{P}x$ *(proper part)*

d6 $x\mathsf{O}y \triangleq \exists z(z\mathsf{P}x \wedge z\mathsf{P}y)$ *(overlap)*

d7 $s\mathsf{SM}ab \triangleq \forall z(z\mathsf{P}s \leftrightarrow z\mathsf{P}a \vee z\mathsf{P}b)$ *(binary sum)*

a11 $\neg x\mathsf{P}y \rightarrow \exists z(z\mathsf{P}x \wedge \neg z\mathsf{O}y)$

a12 $\exists s(s\mathsf{SM}ab)$

Objects across levels The formal notion of parthood I assumed applies to object of any nature. In particular, mereological sums can collect objects that belong to different levels. Multi-level objects are objects that spread through levels while one-level objects have all their parts at the same

[16]Note that this way of talking is a little bit misleading. As stated in (Casati and Varzi 1999), mereology is not a theory of parts *and* wholes because to individuate wholes, in addition to parthood, a notion of *wholeness* is needed.

[17]According to (Baker 2007) and (Gnoli and Poli 2004) this is a necessary condition for any theory of level.

level. (a13) and (a14) restrict $\equiv$ and $\prec$, respectively, to one-level objects. Together with the down linearity of $\prec$, (a14) rules out (one-level) objects grounded on different (non-comparable) levels. Hence, interesting cases of multi-dependence, e.g. the one that concerns relational roles, cannot be represented by means of $\prec$ unless one reduces relational roles to merological sums of tropes (belonging to different levels). More complex theories that avoid (a14) are left for future work.

(a15) guarantees that, if two (one-level) objects are one part of the other, then their grounding objects (at a given level) are also one part of the other. (a15) is a sort of down parthood-monotonicity assuring that lower levels are 'more detailed' than higher ones. An upper version of (a15) excludes the possibility to have different objects belonging to the same level grounded on the same object, possibility that I want to leave open as explained earlier.

d8 $\mathsf{1L}x \triangleq \forall y(y\mathsf{P}x \rightarrow y \equiv x)$ (*one-level*)

a13 $x \equiv y \rightarrow \mathsf{1L}x \wedge \mathsf{1L}y$

a14 $x \prec y \rightarrow \mathsf{1L}x \wedge \mathsf{1L}y$

a15 $x \equiv x' \wedge x \prec y \wedge x' \prec y' \wedge y\mathsf{P}y' \rightarrow x\mathsf{P}x'$

Partial grounding Let us consider the theory $\mathcal{L} = \mathcal{F} \cup \mathcal{M} \cup \{$(a13), (a14), (a15)$\}$. In $\mathcal{L}$, (one-level) objects are not grounded on their parts (t4). However, a notion of *partial grounding* can be defined as in (d9): $x \lessdot y$ stands for "x partially grounds y" or "y is partially grounded on x".[18] This relation is similar to Fine's *strong foundation* (see (Simons 1987) p.311-312). Fine's system extends an extensional mereology with *infinite* sums by assuming axioms (af1)-(af6)[19] where: $x \,\mathsf{fd}\, y$ stands for "x is strongly founded on y"[20], $\sigma' X$ is the least upper bound (in terms of parthood) of the set of objects X, and $x +' y$ is the upper bound of $\{x, y\}$. Theorems (t5)-(t10) show how $\mathcal{L}$ relates to Fine's theory. (t5) directly corresponds to (af1). (t6) shows that the 'negative conditions' about parthood present in (af2)-(af4) are not needed in $\mathcal{L}$. Therefore (t7)-(t9) correspond to (af2)-(af4), with the only addition of condition $\mathsf{1L}z$ in (t8) (due to the fact that $\prec$ but not P is defined only on one-level objects). A similar argument justifies the condition $y \equiv z$ in (t10) that can only simulate (af5) by considering binary sums. An axiom equivalent to (af6) does not hold in $\mathcal{L}$ because of (a14).

The partial grounding considered by Correia in (Correia 2002) (called "n-grounding" and represented by $x\mathsf{G}y$ that stands for "x is n-grounded on y"[21]) is irreflexive, transitive (properties that it shares with $\lessdot$), and satisfies two additional constraints: $x\mathsf{G}y \rightarrow \mathsf{E}x \wedge \mathsf{E}y$ and $\Box\forall x(\exists y(x\mathsf{G}y) \rightarrow \Box(\mathsf{E}x \rightarrow \exists z(x\mathsf{G}z)))$, where $\mathsf{E}x$ stands for "x exists". $\mathcal{L}$ cannot represent the last two constraints. However, in the next section, I will introduce a theory of time and a predicate of existence that allow to 'simulate' these constraints in temporal terms. I will come back to this point later.

d9 $x \lessdot y \triangleq \exists z(x\mathsf{P}z \wedge z \prec y)$ (*partial grounding*)

af1 $x \,\mathsf{fd}\, y \rightarrow \neg y\mathsf{P}x$

af2 $x \,\mathsf{fd}\, y \wedge y \,\mathsf{fd}\, z \wedge \neg z\mathsf{P}x \rightarrow x \,\mathsf{fd}\, z$

af3 $x \,\mathsf{fd}\, y \wedge x\mathsf{P}z \wedge \neg y\mathsf{P}z \rightarrow z \,\mathsf{fd}\, y$

af4 $x \,\mathsf{fd}\, y \wedge z\mathsf{P}y \wedge \neg z\mathsf{P}x \rightarrow x \,\mathsf{fd}\, z$

af5 $X \neq \emptyset \wedge \forall y \in X(x \,\mathsf{fd}\, y) \rightarrow x \,\mathsf{fd}\, (\sigma' X)$

af6 $x \,\mathsf{fd}\, y \rightarrow x \,\mathsf{fd}\, (x +' y)$

t4 $\mathcal{L} \vdash x\mathsf{P}y \rightarrow \neg x \prec y$

t5 $\mathcal{L} \vdash y \lessdot x \rightarrow \neg y\mathsf{P}x$

t6 $\mathcal{L} \vdash z \lessdot y \wedge y \lessdot x \rightarrow \neg z\mathsf{P}x$

t7 $\mathcal{L} \vdash z \lessdot y \wedge y \lessdot x \rightarrow z \lessdot x$

t8 $\mathcal{L} \vdash y \lessdot x \wedge x\mathsf{P}z \wedge \mathsf{1L}z \rightarrow y \lessdot z$

t9 $\mathcal{L} \vdash y \lessdot x \wedge z\mathsf{P}y \rightarrow z \lessdot x$

t10 $\mathcal{L} \vdash y \equiv z \wedge y \lessdot x \wedge z \lessdot x \wedge s\mathsf{SM}yz \rightarrow s \lessdot x$

Time and change

At different times, Goliath can be grounded on different amounts of matter, i.e. Lumpl can be substituted by another substratum. To represent this dynamic aspect, $\mathcal{L}$ must be extended to take into account time and change.

I adopt the primitive $\mathsf{E}_t x$, "x exists at time t" and a logic with two sorts, *time* and *object*, distinguished by a notational convention: variables on times are noted by t, t', t_i, etc. I focus only on objects in time (a16) without committing to a particular theory of time, i.e. time is just a non-structured set of indexes.

A temporally qualified grounding must be introduced: $x \prec_t y$ stands for "x grounds y at t", i.e., at t, the existence of x makes possible the one of y, y owes its existence at t to x. $\mathcal{F}$ must be modified to account for the temporal argument: (a20), (a21), and (a22) are direct adjustments of, respectively, (a6), (a7), and (a8), while (a23) and (a24) correspond to, respectively, (a9) and (a10) where B is now defined by (d10).[22]

d10 $\mathsf{B}x \triangleq \neg\exists yt(y \prec_t x)$ (*bottom level*)

a16 $\exists t(\mathsf{E}_t x)$

a17 $x \prec_t y \rightarrow \mathsf{E}_t x \wedge \mathsf{E}_t y$

a18 $x \prec_t y \rightarrow \neg y \prec_t x$

a19 $x \prec_t y \wedge y \prec_t z \rightarrow x \prec_t z$

a20 $y \prec_t x \wedge z \prec_t x \rightarrow y \prec_t z \vee y = z \vee z \prec_t y$

a21 $x \equiv y \rightarrow \neg\exists t(x \prec_t y)$

a22 $x \equiv y \wedge u \prec_t x \wedge \mathsf{E}_{t'} y \rightarrow \exists v(v \equiv u \wedge v \prec_{t'} y)$

a23 $\mathsf{B}x \wedge \mathsf{B}y \rightarrow x \equiv y$

a24 $\mathsf{B}x \vee \exists yt(\mathsf{B}y \wedge y \prec_t x)$

Note that $\equiv$ is not temporally qualified. A theory based on a non temporally qualified $\equiv$ presupposes a *static* notion of level: objects cannot change level through time. This static

[18] It is possible to start from $\lessdot$ and P and define $x \prec y$ as $x \lessdot y \wedge \neg\exists z(z \lessdot y \wedge x\mathsf{PP}z)$, i.e. x is the maximal (in mereological terms) object that grounds y. I prefer to start from P and $\prec$, because, as stated, $\lessdot$ is conceptually more complex than $\prec$.

[19] Note that the first four axioms define a complete semi-lattice.

[20] Note that the arguments are inverted with respect to $\lessdot$.

[21] Here too the arguments are inverted with respect to $\lessdot$.

[22] Note that (a22) applies at every time y exists. The synchronic version, $x \equiv y \wedge u \prec_t x \rightarrow \exists v(v \equiv u \wedge v \prec_t y)$, is too strong because $x \equiv y$ does not imply that y exists when x exists.

notion is assumed by some views on levels. For instance, in the view that associates natural kinds to levels, no object can survive a change in natural kind just because no object can loose one of its essential properties. *Dynamic* theories of levels would require $\equiv$ to be modified by adding two temporal arguments to diachronically compare an object as it is at t with an object as it is at t'. Again I leave this interesting alternative to future work.

Let $\mathcal{F}_\tau = \{$reflexivity, symmetry, and transitivity of $\equiv$, (a16)-(a24)$\}$. Assuming (d11), it is possible to prove that $\leq$ is a connected down-linear order. In addition (i) (t11) shows that the equivalence relation induced by $\leq$ is equivalent to $\equiv$ (a primitive of $\mathcal{F}_\tau$); (ii) (t12) generalizes the antisymmetry of $\prec$; and (iii) (t13) ensures that, at a given level and time, an object has a unique grounding.

d11 $x \leq y \triangleq x \equiv y \vee \exists zt(z \equiv x \wedge z \prec_t y)$ (*lower/same level*)

t11 $\mathcal{F}_\tau \vdash x \leq y \wedge y \leq x \leftrightarrow x \equiv y$

t12 $\mathcal{F}_\tau \vdash x \leq y \rightarrow \neg\exists t(y \prec_t x)$

t13 $\mathcal{F}_\tau \vdash y \prec_t x \wedge z \prec_t x \wedge y \equiv z \rightarrow y = z$

Parthood too needs to be temporally qualified: some parts of an object can exist only at specific times (in particular the parts of mereological sums, see the next section). *Temporary parthood*, $x\mathsf{P}_t y$ stands for "x is part of y at t", is characterized by modifying the axioms for classical extensional mereology as in (a25)-(a29). (a29) guarantees that two objects that coincide (in the sense defined by (d12)) during their whole existence are identical (see the next section for more details on this axiom).

d12 $x\mathsf{CN}_t y \triangleq x\mathsf{P}_t y \wedge y\mathsf{P}_t x$ (*temporary coincidence*)

d13 $x\mathsf{O}_t y \triangleq \exists z(z\mathsf{P}_t x \wedge z\mathsf{P}_t y)$ (*temporary overlap*)

a25 $x\mathsf{P}_t y \rightarrow \mathsf{E}_t x \wedge \mathsf{E}_t y$

a26 $\mathsf{E}_t x \rightarrow x\mathsf{P}_t x$

a27 $x\mathsf{P}_t y \wedge y\mathsf{P}_t z \rightarrow x\mathsf{P}_t z$

a28 $\mathsf{E}_t x \wedge \mathsf{E}_t y \wedge \neg x\mathsf{P}_t y \rightarrow \exists z(z\mathsf{P}_t x \wedge \neg z\mathsf{O}_t y)$

a29 $\forall t(\mathsf{E}_t x \rightarrow x\mathsf{P}_t y) \wedge \forall t(\mathsf{E}_t y \rightarrow y\mathsf{P}_t x) \rightarrow x = y$

Concerning the links between $\equiv$, $\prec$, and P, (a30) and (a31) correspond to, respectively, (a13) and (a14) where $1\mathsf{L}$ is now defined by (d14) instead of (d8), and (a32) is the temporal version of (a15).

Let $\mathcal{L}_\tau = \mathcal{F}_\tau \cup \{$(a25)-(a32)$\}$. By defining temporary partial grounding as in (d15), (t14)-(t22) can be proved. (t14)-(t18) are the temporal versions of (t5)-(t9). A theorem similar to (t10) cannot be proved because I have not introduced the mereological sums in terms of temporary parthood (I discuss this issue in the next section). In addition, (t20), (t21), and (t22) show that there are no synchronic or diachronic loops for $\lessdot$. These results make evident a close correspondence between the atemporal axioms and theorems in $\mathcal{L}$ and the temporal ones in $\mathcal{L}_\tau$.

d14 $1\mathsf{L}x \triangleq \forall yt(y\mathsf{P}_t x \rightarrow y \equiv x)$ (*one level*)

d15 $x \lessdot_t y \triangleq \exists z(x\mathsf{P}_t z \wedge z \prec_t y)$ (*temp. partial ground.*)

a30 $x \equiv y \rightarrow 1\mathsf{L}x \wedge 1\mathsf{L}y$

a31 $x \prec_t y \rightarrow 1\mathsf{L}x \wedge 1\mathsf{L}y$

a32 $x \equiv x' \wedge x \prec_t y \wedge x' \prec_t y' \wedge y\mathsf{P}_t y' \rightarrow x\mathsf{P}_t x'$

t14 $\mathcal{L}_\tau \vdash x \lessdot_t y \rightarrow \neg y\mathsf{P}_t x$

t15 $\mathcal{L}_\tau \vdash z \lessdot_t y \wedge y \lessdot_t x \rightarrow z \lessdot_t x$

t16 $\mathcal{L}_\tau \vdash z \lessdot_t y \wedge y \lessdot_t x \rightarrow \neg z\mathsf{P}_t x$

t17 $\mathcal{L}_\tau \vdash y \lessdot_t x \wedge x\mathsf{P}_t z \wedge 1\mathsf{L}z \rightarrow y \lessdot_t z$

t18 $\mathcal{L}_\tau \vdash y \lessdot_t x \wedge z\mathsf{P}_t y \rightarrow z \lessdot_t x$

t19 $\mathcal{L}_\tau \vdash x \lessdot_t y \rightarrow \mathsf{E}_t x \wedge \mathsf{E}_t y$

t20 $\mathcal{L}_\tau \vdash \neg x \lessdot_t x$

t21 $\mathcal{L}_\tau \vdash x \lessdot_t y \rightarrow x \leq y \wedge \neg x \equiv y$

t22 $\mathcal{L}_\tau \vdash x \lessdot_t y \rightarrow \neg\exists zwt'(z \equiv x \wedge w \equiv y \wedge w \lessdot_{t'} z)$

Let us come back to Correia's theory. My temporary partial grounding can be related to Correia's primitive $x_t\mathsf{G}y_{t'}$ ("x at t is grounded on y at t'")[23] by assuming $t = t'$. G is (necessarily) irreflexive, asymmetric, transitive, and it satisfies two additional constraints: $\Box\forall xytt'(x_t\mathsf{G}y_{t'} \rightarrow \mathsf{E}_t x \wedge \mathsf{E}_{t'} y)$ and $\Box\forall x(\exists ytt'(x_t\mathsf{G}y_{t'}) \rightarrow \Box(\mathsf{E}x \rightarrow \exists ztt'(x_t\mathsf{G}z_{t'})))$, where $\mathsf{E}_t x$ corresponds to my temporary predicate of existence and is related to the non-temporary E by the axiom $\Box\forall x(\exists t(\mathsf{E}_t x \rightarrow \mathsf{E}x))$. Taking apart the modality[24], the first four constraints directly correspond, respectively, to (t20), (t22), (t15), and (t19), while (considering the link between temporary and non-temporary E in Correia) the last constraint can be at least partially captured by (t23).

Note that the properties of $\lessdot$ are quite similar to the ones of temporary (proper) parthood. This claim is strengthened by (t24), a sort of temporal version of the *weak supplementation principle* $x\mathsf{PP}y \rightarrow \exists z(z\mathsf{PP}y \wedge \neg z\mathsf{O}x)$ (see (Casati and Varzi 1999)). In my understanding, this explains why some authors use parthood to represent constitution or, more generally, partial grounding. However, I hope to have shown that $\lessdot$ is quite different from (formal) parthood and that the differences between them can be clarified and analyzed by using a more basic theory.

t23 $\mathcal{L}_\tau \vdash \exists yt(y \lessdot_t x) \rightarrow \forall t'(\mathsf{E}_{t'} x \rightarrow \exists z(z \lessdot_{t'} x))$

t24 $\mathcal{L}_\tau \vdash \exists a(x\mathsf{PP}_t a \wedge a \prec_t y) \rightarrow \exists z(z \equiv x \wedge z \lessdot_t y \wedge \neg z\mathsf{O}_t x)$

Temporary vs. non-temporary relations

One may wonder whether $\mathcal{L}_\tau$ and $\mathcal{L}$ or, more precisely, temporary and non-temporary versions of parthood and grounding, are somewhat related. To address this topic, one has to understand whether temporary relations induce some temporal constraints. Let us start from parthood. (a33) guarantees that the temporal extension of the part is included in that of the 'whole'. (a33) seems quite intuitive but it interacts with other mereological axioms in a non-negligible way. Let us consider two objects a and b such that a exists only at t and b only at t'. By (a12), there exists s such that $s\mathsf{SM}ab$ and therefore, by (d7), both $a\mathsf{P}s$ and $b\mathsf{P}s$ hold. From (a33), one concludes that s exists at both t and t' even though only a part of it exists at each time. According to (van Invagen 2006), this shows that mereological sums can change parts through time. For other authors, e.g. (Baker 2007), sums

[23]Remember that the arguments are inverted with respect to $\lessdot$.

[24]Note however that all FOL axioms can be considered as necessary.

exist only when all the summands exist and (a12) must thus be accordingly modified. I see two problems with such a move. First, in my understanding, the formal nature of P allows SM to aggregate objects independently of their nature and, in particular, independently of their temporal extension. Second, if a theory commits to both mereological sums and differences, e.g. *extensional closure mereology* (Casati and Varzi 1999), then it is very hard to reject the difference between a and b in the situation where $b\mathsf{PP}a$, a exists at both t and t', b exists only at t, and a and b coincide at t (in the sense of (d12)). It seems that, by refusing this kind of sums and differences, one tends to accept parthood only between temporally co-located entities. On the other hand, (Masolo 2009) shows that, by defining parthood as *constant parthood* (d16), (a25)-(a29) (and the whole $\mathcal{L}_\tau$) are not enough to prove the supplementation principle (a11): the previous example given for difference is a counter-example because nothing guarantees the existence of a part of a that does not overlap b ((a28) applies only synchronically). This means that $\mathcal{L}_\tau$ is existentially less committing than $\mathcal{L}$. However, the extensionality of P (t25) is guaranteed and thus sums can be introduced via (d7) where parthood simpliciter is defined by (d16).

d16 $x\mathsf{P}y \triangleq \forall t(\mathsf{E}_t x \rightarrow x\mathsf{P}_t y)$

d17 $x \prec y \triangleq \forall t(\mathsf{E}_t x \rightarrow x \prec_t y)$

d18 $x \prec y \triangleq \forall t(\mathsf{E}_t y \rightarrow x \prec_t y)$

a33 $x\mathsf{P}y \rightarrow \forall t(\mathsf{E}_t x \rightarrow \mathsf{E}_t y)$

t25 $\forall z(z\mathsf{P}x \leftrightarrow z\mathsf{P}y) \rightarrow x = y$

Similarly, by defining $\prec$ as *constant* grounding (d17), $\mathcal{L}_\tau$ does not allow to prove all the axioms in $\mathcal{F}$. Let us consider a counter-example of (a6): at t the collectable item c is grounded on the record r that, in turn, is grounded on the piece of plastic p_1. Collectable items need to be perfect, without any mark, scratch, etc. Therefore, at t', when r is scratched (and therefore it is grounded on a different piece of plastic p_2), c does not exist anymore. It is easy to verify that this situation satisfies all the axioms in $\mathcal{L}_\tau$.[25] According to (d17), we have $p_1 \prec r$ and $p_2 \prec r$ (during their whole existence both p_1 and p_2 ground r) but $p_1 \neq p_2$ and neither $p_1 \prec p_2$ nor $p_2 \prec p_1$ because they are at the same level. By assuming a different definition of *constant* grounding (d18), the previous example is still a counter-example of (a6). According to (d18), we have $r \prec c$ and $p_1 \prec c$ (c exists only at t when it is grounded on both r and p_1) but $p_1 \neq r$ and neither $p_1 \prec r$ (r exists at t' but it is grounded on p_2 and not on p_1) nor $r \prec p_1$ (p_1 exists at t but it is not grounded on r).

Counter-examples of (a8) can also be found. Let us consider two records r_1 and r_2 that exist only at t and t'. At t', r_1 is scratched (its grounding change from the piece of plastic p_1 (at t) to p'_1 (at t')) while r_1 remains perfect (it is always grounded on the piece of plastic p_2). According to (d17), $p_2 \prec r_2$ but nothing constantly grounds r_1 even if it is at the same level as r_2. Let us now assume that, at t', r_1 ceases to exist because of a deformation of p_1 that however still exists at t' (the shape is not essential to pieces of plastic). According to (d18), $p_2 \prec r_2$ but nothing constantly grounds r_1 (at t', p_1 exists but it does not ground r_1 anymore) even though it is at the same level as r_2.

These remarks show that it is not easy to move from a static (that considers only a snapshot of the world) 'interpretation' of $\mathcal{L}$ to a dynamic one where the non-temporary relations are defined in terms of temporary ones. In particular, starting from $\mathcal{L}_\tau$, the proposed definitions are too weak to capture the whole $\mathcal{L}$.

Constitution, inherence, and abstraction

In this section, I will sketch how $\mathcal{L}_\tau$ can be extended to characterize the three kinds of grounding relations considered in the initial examples. These extensions characterize constitution, inherence, and abstraction only very partially therefore they have to be intended as suggestions and not as 'definitive' theories.

Constitution

At a given level and time, the grounding of an object is unique, therefore *constitution* can be directly represented by temporary grounding. In addition, as already observed, constitution implies spatial co-location. Even though I think that it is not too difficult to extend $\mathcal{L}_\tau$ with a theory of space and location (see, for example (Casati and Varzi 1999)) here I have not addressed this aspect. Who prefers a notion of partial constitution can use $\lessdot$.

Inherence

We introduced inherence as a relation between qua-entities and players (of roles). But inherence is also used in *trope* theory (see (Daly 1997) for an introduction) to represent the link between an *individual property* (a *trope*, e.g. 'the being red of my car') and the unique object (the *host*) it is anchored to (e.g. 'my car'). While constituted objects can change their constituents across time, both tropes and qua entities inhere in the same object during their whole existence, i.e. inherence is temporally qualified only by the temporal extension of the trope (or qua entity). By assuming (a34), if x inheres in y ($x\mathsf{IN}y$) then x is constantly grounded on y (in the sense defined by (d18)). Note that IN is here an additional primitive, therefore, in general, the opposite implication does not hold. By adopting the definition $x\mathsf{IN}y \triangleq y \prec x$ (where $\prec$ is defined by (d18)) one not only fails to capture (a6) and (a8) but also the *non-migration principle* (a35), one of the few properties stated for inherence (see (Guizzardi 2005)). Let us consider a situation in which 'John qua passenger of Alitalia' is grounded on John that, in turn, is grounded on his body. If these three entities have the same temporal extension, then, according to the assumed definition, 'John qua passenger of Alitalia' inheres both in John and his body (that are different because they belong to different levels). This problem can be solved by introducing the notion of *direct* grounding (d19) and defining inherence by (d20). (a6) trivially holds, while, unfortunately, (a8), applied to inher-

[25] Actually, it is also necessary to assume that each object is part of itself at any time at which it exists. This proves the consistency of $\mathcal{L}_\tau$.

ence, remains unsatisfied. However one can at least better characterize IN by substituting $\prec$ with $\olessthan$ in (a34).

a34 $x\mathsf{IN}y \rightarrow \forall t(\mathsf{E}_t x \rightarrow y \prec_t x)$

a35 $x\mathsf{IN}y \wedge x\mathsf{IN}z \rightarrow y = z$

d19 $x \olessthan_t y \triangleq x \prec_t y \wedge \neg\exists z(x \prec_t z \wedge z \prec_t y)$ (*direct groun.*)

d20 $x\mathsf{IN}y \triangleq \forall t(\mathsf{E}_t x \rightarrow y \olessthan_t x)$

Abstraction

The distinction between parthood and grounding allows to address the notion of *granularity* by relying on the fact that *atoms* (objects without proper parts) can be grounded on non-atomic objects. In this section, I will formally restrict grounding to capture (an approximate notion of) abstraction. In the following, $x \prec_t y$ must then be read as "at t, y is an abstraction of x".

I consider the following assumptions: (i) at any given time, all the objects are ultimately composed (in mereological terms) by atoms (a36) (see (Masolo and Vieu 1999) for a deep discussion of atomicity); (ii) higher levels are coarser than lower ones, i.e., abstract atoms are grounded on non-atomic objects (a37); (iii) at any given time, higher atoms *partition* lower ones (a38) and (a39) (where the *direct partial grounding* $\dot{\olessthan}$ is defined by (d22)).[26] My theory of granularity is naive and it considers 'synchronic' abstraction in the sense that both the set of atoms and the way they are abstracted can change across time. Alternatively, it is possible to consider a theory where both the set of atoms and the way they are abstracted remain constant (this seems to go in the direction of a non temporally qualified abstraction relation) and thus mereological changes reduce to substitution, loss, or gain of atoms.

d21 $\mathsf{A}_t x \triangleq \mathsf{E}_t x \wedge \neg\exists y(y\mathsf{PP}_t x)$ (*atom*)

d22 $x \dot{\olessthan}_t y \triangleq \exists z(x\mathsf{P}_t z \wedge z \olessthan_t y)$ (*partial direct ground.*)

d23 $\mathsf{T}x \triangleq \neg\exists yzt(y \equiv x \wedge y \prec_t z)$ (*top level*)

a36 $\forall z(\mathsf{A}_t z \rightarrow (z\mathsf{P}_t x \rightarrow z\mathsf{P}_t y)) \rightarrow x\mathsf{P}_t y$

a37 $\mathsf{A}_t x \wedge \neg\mathsf{B}x \rightarrow \exists y(y \olessthan_t x \wedge \neg\mathsf{A}_t y)$

a38 $\mathsf{A}_t x \wedge \neg\mathsf{T}x \rightarrow \exists y(x \dot{\olessthan}_t y \wedge \mathsf{A}_t y)$

a39 $\mathsf{A}_t x \wedge \mathsf{A}_t x' \wedge y \olessthan_t x \wedge y' \olessthan_t x' \rightarrow (x\mathsf{CN}_t x' \leftrightarrow y\mathsf{O}_t y')$

t26 $\mathsf{A}_t x \wedge \mathsf{A}_t x' \wedge y \olessthan_t x \wedge y \olessthan_t x' \rightarrow x\mathsf{CN}_t x'$

t27 $\mathsf{A}_t x \wedge \mathsf{A}_t x' \wedge y \dot{\olessthan}_t x \wedge y \dot{\olessthan}_t x' \rightarrow x\mathsf{CN}_t x'$

Abstract objects are often assumed to correspond to equivalence classes of (lower level) objects. This correspondence is often characterized by $\mathsf{f}(x) = \mathsf{f}(y) \leftrightarrow x\mathsf{R}y$, where R is an equivalence relation on objects and $\mathsf{f}(x)$ denotes the abstraction of the object x. If both f and R depend on time, then it is necessary to consider a temporally qualified version of the previous constraint: $\mathsf{f}(x,t) = \mathsf{f}(y,t) \leftrightarrow x\mathsf{R}_t y$. In my theory of granularity, such constraint can be represented by assuming that the elements in the equivalence classes generated by R at a given time t correspond to the atomic parts of objects that, at t, *directly* ground atoms (i.e. to atoms x such that $x \dot{\olessthan} y$ where y is an atom). This means that: (i) at t, the equivalence classes generated by R correspond to the objects that ground (in terms of $\olessthan$) atoms, and (ii) that the abstract object associated (via f) to an equivalence class corresponds to 'the' atom (it is unique, in the sense of coincidence, (t26)) directly grounded on the object that corresponds to such equivalence class. By (t27), f can then be simulated by direct partial grounding.[27]

Clearly, a much deeper analysis is necessary to treat these aspects in a satisfactory way. Nevertheless, I hope that, by showing how $\mathcal{L}_\tau$ can be *used* to characterize different notions of granularity (as well as constitution and inherence), the previous discussion can provide some arguments in favour of the generality and usefulness of $\mathcal{L}_\tau$.

[26] Theories of granularity often adopt similar constraints that in general are not temporally qualified (see (Keet 2008)).

[27] Note that (t26) and (t27) follow directly from (a39) and the reflexivity of temporary parthood.

Conclusion

In the previous sections, I discussed a theory that carefully distinguishes the notions of level, parthood, and grounding. I mentioned alternative characterizations of these primitives that reflect different points of view on levels. As in the case of mereotopologies (see (Casati and Varzi 1999)) or theories of time (see (Hajnicz 1996; Van Benthem 1983)), a deeper study that formalizes and compares these alternative views would highly improve our understanding of levels. However this is a slow and complex process to which, I hope, this paper can contribute.

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Appendix: proofs

In this section, I will sketch the proofs of the theorems previously stated. In the text, I will not explicitly refer to the use of the reflexivity, symmetry, and transitivity of $\equiv$.

Proofs in $\mathcal{F}$

$\mathcal{F} = \{$reflexivity, symmetry, and transitivity of $\equiv$, asymmetry and transitivity of $\prec$, (a6)-(a10)$\}$.

The proofs of the reflexivity of $\leq$, transitivity of $\leq$, down linearity of $\leq$, (t3) and (t2) are straightforward adjustments of the proofs of the corresponding properties in $\mathcal{F}_\tau$. The reader can refer to these proofs just paying attention to the fact that in $\mathcal{F}_\tau$, $\leq$ is defined by (d11) instead than (d2).

Proofs in $\mathcal{L}$

$\mathcal{L} = \mathcal{F} \cup \mathcal{M} \cup \{$(a13), (a14), (a15)$\}$, where $\mathcal{M} = \{$reflexivity, antisymmetry, transitivity of P, (a11), (a12)$\}$.

t4 $\mathcal{L} \vdash x\mathsf{P}y \rightarrow \neg x \prec y$
If $\neg 1\mathsf{L}y$ then the thesis follows from (a14). Let $1\mathsf{L}y$, then $x \equiv y$. The thesis follows from (a7). □

t5 $\mathcal{L} \vdash y \lessdot x \rightarrow \neg y\mathsf{P}x$
From $\exists z(y\mathsf{P}z \wedge z \prec x)$, by (a14), we have $1\mathsf{L}x \wedge 1\mathsf{L}z$ and therefore $y \equiv z$. By contradiction, assume $y\mathsf{P}x$. It follows that $x \equiv y \equiv z$ that, together with $z \prec x$, contradicts (a7). □

t6 $\mathcal{L} \vdash z \lessdot y \wedge y \lessdot x \rightarrow \neg z\mathsf{P}x$
From the hypotheses, by (t7) (see below), we have $z \lessdot x$. The thesis follows directly from (t5). □

t7 $\mathcal{L} \vdash z \lessdot y \wedge y \lessdot x \rightarrow z \lessdot x$
From $\exists u(z\mathsf{P}u \wedge u \prec y) \wedge \exists v(y\mathsf{P}v \wedge v \prec x)$, by (a14), we have $1\mathsf{L}v \wedge 1\mathsf{L}u$ and, therefore, $y \equiv v \wedge z \equiv u$. From $y \equiv v \wedge u \prec y$, by (a8), we have $\exists w(w \equiv u \equiv z \wedge w \prec v)$ and, by (a15), $u\mathsf{P}w$ follows. By the transitivity of P and of $\prec$ we conclude that $\exists w(z\mathsf{P}w \wedge w \prec x)$. □

t8 $\mathcal{L} \vdash y \lessdot x \wedge x\mathsf{P}z \wedge 1\mathsf{L}z \rightarrow y \lessdot z$
The proofs is very similar to the one of (t7). □

t9 $\mathcal{L} \vdash y \lessdot x \wedge z\mathsf{P}y \rightarrow z \lessdot x$
It trivially follows from the transitivity of P. □

t10 $\mathcal{L} \vdash y \equiv z \wedge y \lessdot x \wedge z \lessdot x \wedge s\mathsf{SM}yz \rightarrow s \lessdot x$
By the down linearity of $\leq$, the object u, at the same level as y and z, that grounds x is unique. From the hypotheses and (d9) we conclude that both y and z are part of u. Thus, from (d7) (and the properties of P) $s\mathsf{P}u$ and the thesis follow. □

Proofs in $\mathcal{F}_\tau$

$\mathcal{F}_\tau = \{$reflexivity, symmetry, and transitivity of $\equiv$, (a16)-(a24)$\}$.

The proof of the reflexivity of $\leq$ is trivial, while the proof of the transitivity of $\leq$ is very similar to the one of the down linearity of $\leq$.

- $\mathcal{F}_\tau \vdash y \leq x \wedge z \leq x \rightarrow y \leq z \vee z \leq y$ (**down linearity**)
We have four cases. **(a)** $y \equiv x \wedge z \equiv x$. Trivial.
(b) $y \equiv x \wedge \exists wt(w \equiv z \wedge w \prec_t x)$. From (a16) we have $\exists t'(\mathsf{E}_{t'}y)$ and from (a22) we have $\exists ut'(u \equiv w \equiv z \wedge u \prec_{t'} y)$.
(c) $\exists wt(w \equiv y \wedge w \prec_t x) \wedge z \equiv x$. See case (b).
(d) $\exists ut(u \equiv y \wedge u \prec_t x) \wedge \exists wt'(w \equiv z \wedge w \prec_{t'} x)$.
From $w \prec_{t'} x$, by (a17), we obtain $\mathsf{E}_{t'}x$. From $x \equiv x \wedge u \prec_t x \wedge \mathsf{E}_{t'}x$, by (a22), we have $\exists v(v \equiv u \wedge v \prec_{t'} x)$. From $v \prec_{t'} x \wedge w \prec_{t'} x$, by (a20), we conclude $v \prec_{t'} w \vee v = w \vee w \prec_{t'} v$.
(i) If $v = w$, then $v \equiv w \equiv u \equiv y \equiv z$, from which $z \leq y$ follows. (ii) Assume $v \prec_{t'} w$. From (a16) we have $\exists t''(\mathsf{E}_{t''}z)$. From $w \equiv z \wedge v \prec_{t'} w \wedge \mathsf{E}_{t''}z$, by (a22), we obtain $\exists at''(a \equiv v \wedge a \prec_{t''} z)$. But $v \equiv y$ therefore $a \equiv y$ and finally $y \leq z$. (iii) Assume $w \prec_{t'} v$. The proof is analogous of the one of the case (ii). □
- $\mathcal{F}_\tau \vdash \exists z(z \leq x \wedge z \leq y)$ (**connectedness**)
(a) If $\mathsf{B}x \wedge \mathsf{B}y$ the thesis follows trivially. **(b)** Assume $\mathsf{B}x \wedge \neg\mathsf{B}y$. By (a24), $\exists wt(\mathsf{B}w \wedge w \prec_t y)$ and, by (a23), $w \equiv x$, therefore $x \leq y$. It is enough to take $z = x$. **(c)** Assume $\neg\mathsf{B}x \wedge \mathsf{B}y$. See (b). **(d)** Assume $\neg\mathsf{B}x \wedge \neg\mathsf{B}y$. From (a24) we have $\exists ut(\mathsf{B}u \wedge u \prec_t x)$ and $\exists wt'(\mathsf{B}w \wedge w \prec_{t'} y)$. From (a23) we have $u \equiv w$. Therefore $u \leq x \wedge u \leq y$. It is enough to take $z = u$. □

t11 $\mathcal{F}_\tau \vdash x \leq y \wedge y \leq x \leftrightarrow x \equiv y$
The direction $\leftarrow$ is trivial. For the other direction we have four cases: **(a)** $x \equiv y \wedge y \equiv x$. Trivial. I will show that the other cases lead to a contradiction.
(b) From $x \equiv y \wedge \exists zt(z \equiv y \wedge z \prec_t x)$ we directly have $\exists zt(z \equiv x \wedge z \prec_t x)$ that contradicts (a21).
(c) $y \equiv x \wedge \exists zt(z \equiv x \wedge z \prec_t y)$. See (b).
(d) $\exists zt(z \equiv y \wedge z \prec_t x) \wedge \exists z't'(z' \equiv x \wedge z' \prec_{t'} y)$. From $z \prec_t x$, by (a17), we have $\mathsf{E}_t z$. From $z \equiv y \wedge z' \prec_{t'} y \wedge \mathsf{E}_t z$, by (a22), we deduce $\exists u(u \equiv z' \wedge u \prec_t z)$. From $u \prec_t z \wedge z \prec_t x$, by (a19), we have $u \prec_t x$ while from $u \equiv z' \wedge z' \equiv x$ we have $u \equiv x$. These two fact contradict (a21). □

t12 $\mathcal{F}_\tau \vdash x \leq y \rightarrow \neg\exists t(y \prec_t x)$
If $x \equiv y$ the thesis follows directly from (a21). Otherwise, by contradiction, assume $\exists ztt'(z \equiv x \wedge z \prec_{t'} y \wedge y \prec_t x)$. $\mathsf{E}_t y$ follows from (a17). From $y \equiv y \wedge z \prec_{t'} y \wedge \mathsf{E}_t y$, by (a22), $\exists u(u \equiv z \wedge u \prec_t y)$. From $u \prec_t y \wedge y \prec_t x \wedge u \equiv z \wedge z \equiv x$, by (a19), we have $u \prec_t x \wedge u \equiv x$ that contradicts (a21). □

t13 $\mathcal{F}_\tau \vdash y \prec_t x \wedge z \prec_t x \wedge y \equiv z \rightarrow y = z$
From the hypotheses, by (a20), we obtain $y \prec_t z \vee y = z \vee z \prec_t y$. By (a21), $y \prec_t z$ and $z \prec_t y$ both imply $\neg y \equiv z$ that contradicts the hypotheses. The only remaining option is $y = z$. □

Proofs in $\mathcal{L}_\tau$

$\mathcal{L}_\tau = \mathcal{F}_\tau \cup \{$(a25)-(a32)$\}$.
(t18)-(t22), and (t24) have quite straightforward proofs.

t14 $\mathcal{L}_\tau \vdash x \lessdot_t y \rightarrow \neg y\mathsf{P}_t x$
From $\exists z(x\mathsf{P}_t z \wedge z \prec_t y)$, by (a31), we have $1\mathsf{L}z$ and (using the properties of P) $1\mathsf{L}x$. By contradiction, assume $y\mathsf{P}_t x$. Because $1\mathsf{L}x$ then $x \equiv y$. From $1\mathsf{L}z \wedge x\mathsf{P}_t z$ we have $z \equiv x \equiv y$ that, by (a21), implies $\neg z \prec_t y$. Contradiction. □

t15 $\mathcal{L}_\tau \vdash z \lessdot_t y \wedge y \lessdot_t x \rightarrow z \lessdot_t x$
From $\exists u(z\mathsf{P}_t u \wedge u \prec_t y)$ and $\exists w(y\mathsf{P}_t w \wedge w \prec_t x)$, by (a31) and (a25), we have $1\mathsf{L}u \wedge 1\mathsf{L}w \wedge \mathsf{E}_t u \wedge \mathsf{E}_t w$. From $1\mathsf{L}w \wedge y\mathsf{P}_t w$ we obtain $w \equiv y$. From $w \equiv y \wedge u \prec_t y \wedge \mathsf{E}_t w$, by (a22), we have $\exists a(a \equiv u \wedge a \prec_t w)$ that, by (a19), implies $\exists a(a \equiv u \wedge a \prec_t x)$. From $u \equiv a \wedge u \prec_t y \wedge a \prec_t w \wedge y\mathsf{P}_t w$, by (a32), we obtain $u\mathsf{P}_t a$. From $z\mathsf{P}_t t \wedge u\mathsf{P}_t a$, by (a27), we have $z\mathsf{P}_t a$ that, together with $a \prec_t x$, conclude the proof. □

t16 $\mathcal{L}_\tau \vdash z \lessdot_t y \wedge y \lessdot_t x \rightarrow \neg z\mathsf{P}_t x$
From (t15) we have $z \lessdot_t x$, i.e. $\exists a(z \equiv a \wedge a \prec_t x)$. By contradiction, assume $z\mathsf{P}_t x$. By (a31) we have $1\mathsf{L}x$ and therefore $z \equiv x \equiv a$. But $x \equiv a \wedge a \prec_t x$ contradicts (a21). □

t17 $\mathcal{L}_\tau \vdash y \lessdot_t x \wedge x\mathsf{P}_t z \wedge 1\mathsf{L}z \rightarrow y \lessdot_t z$
The proof is very similar to the one of (t15). □

t23 $\mathcal{L}_\tau \vdash \exists yt(y \lessdot_t x) \rightarrow \forall t'(\mathsf{E}_{t'}x \rightarrow \exists z(z \lessdot_{t'} x))$
From (d15) we have $\exists u(y\mathsf{P}_t u \wedge u \prec_t x)$. Suppose $\mathsf{E}_{t'}x$. (a22) and (a17) imply $\exists z(z \equiv u \wedge z \prec_{t'} x \wedge \mathsf{E}_{t'} z)$. Therefore, by (a26), $\exists z(z\mathsf{P}_{t'} z \wedge z \prec_{t'} x)$ follows. □

t25 $\forall z(z\mathsf{P}x \leftrightarrow z\mathsf{P}y) \rightarrow x = y$
If $x \neq y$, by (a29), $\exists t(\mathsf{E}_t x \wedge \neg x\mathsf{P}_t y)$ or $\exists t(\mathsf{E}_t y \wedge \neg y\mathsf{P}_t x)$. In the first case, by (d16), we have $\neg x\mathsf{P}y$. Therefore we have $x\mathsf{P}x \wedge \neg x\mathsf{P}y$ ($x\mathsf{P}x$ follows from (a26)). Similarly in the second case. □