Characterizing Updates in Dynamic Epistemic Logic

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Abstract

Dynamic epistemic logic deals with the representation of situations in a multi-agent and dynamic setting. It allows to express in a uniform way statements about:

1. what is true about an initial situation
2. what is true about an event occurring in this situation
3. what is true about the resulting situation after the event has occurred.

We axiomatize in this framework what we can infer about (3) given (1) and (2), introducing thereby new techniques to prove completeness. We also show that this axiomatization is decidable. Besides being useful for reasoning about actions, it provides a natural characterization of the product update of dynamic epistemic logic.

1 Introduction

The theories of belief revision of (Alchourrón, Gärdenfors, and Makinson 1985) and belief update of (Katsuno and Mendelzon 1992) provide in the restricted setting of a single agent a unified framework where one can characterize revision and update operations by means of rationality postulates. This unifying approach is valuable since it allows to naturally compare, analyse and introduce new revision and update operations. However, in the multi-agent setting of dynamic epistemic logic, a similar unified framework is missing. Instead, updates are studied and compared case by case by means of examples or ad hoc properties. We propose in this paper a systematic approach allowing to naturally characterize any kind of update of dynamic epistemic logic. We characterize in particular the standard BMS product update of (Baltag, Moss, and Solecki 1998).

Our approach relies on a recent development of dynamic epistemic logic (Aucher 2009a) which allows to explicitly express statements about events. This feature turns out to be crucial in our approach to characterize updates. For the purpose of this paper, we first present a restriction of this general framework to the case where only a single event can happen at a time. Then we propose a natural characterization of the standard product update of dynamic epistemic logic by means of an axiomatization, introducing thereby new techniques of completeness proof.

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2 Dynamic Epistemic Logic: BMS Revisited

In logical formalisms for representing and reasoning about events and their effects, it is often difficult to represent situations *during* the occurrence of events.[1] The event calculus for instance (Shanahan 1997) is a narrative-based formalism, and as such is more appropriate for planing and reasoning about a given course of events (via logic programming) than for representing a situation and updating this representation as new events start or terminate. In (Aucher 2009a), we addressed this issue and others within the paradigm of dynamic epistemic logic. In this section, we are going to give an account of a simplified version of the general framework developed in (Aucher 2009a).

2.1 A Language to Talk about Events

If we want to represent a situation during the occurrence of events, we need to be able not only to express static statements about the situation but also statements about events occurring in this situation. This leads us to define two languages $\mathcal{L}$ and $\mathcal{L}'$ to express these two kinds of statements.

Let Φ and Φ' be two finite and disjoint sets of propositional letters. G is a finite set of agents.

Definition 2.1. The languages $\mathcal{L}$ and $\mathcal{L}'$ are defined inductively as follows. The propositional letters p and p' range respectively over Φ and Φ'.

$$\mathcal{L} : \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid B_j\varphi$$

$$\mathcal{L}' : \varphi' ::= p' \mid \neg\varphi' \mid \varphi' \wedge \varphi' \mid B_j\varphi'$$

In the sequel, $\langle B_j\rangle\varphi$ is an abbreviation for $\neg B_j\neg\varphi$. The propositional letters of Φ are called *atomic facts* and the propositional letters of Φ' are called *atomic events*.[2] $\lhd$

The language $\mathcal{L}$ corresponds to the classical epistemic language. The other language $\mathcal{L}'$ is used to describe events. Atomic events p' describe events, just as atomic facts p describe static properties of the world. For example r'_a = 'Ann

[1] 'Situation' means in this paper, like in the situation calculus of (McCarthy and Hayes 1969), "the complete state of the universe at an instant of time".

[2] Let φ, ψ be formulas of $\mathcal{L}$ (or $\mathcal{L}'$). $deg(\varphi)$ is defined inductively as follows. $deg(p) = 0, deg(\varphi \wedge \psi) = max\{deg(\varphi), deg(\psi)\}, deg(\neg\varphi) = deg(\varphi), deg(B_j\varphi) = deg(\varphi) + 1$.

shows her red card', p' = 'a tub of wine is being filled'. Generally, atomic events are of the form 'something is happening', 'somebody *is doing* something' whereas atomic facts are of the form 'something has this static property'. Besides, the occurrence of these atomic events might change some properties of the world, unlike atomic facts. The negation $\neg p'$ of an atomic event p' should be interpreted as 'the atomic event p' is not occurring'. However, this does not mean that another 'opposite' event is necessarily occurring.

Moreover, these atomic events might have preconditions. For example, the precondition that 'Ann shows her red card' (r'_a) is that 'Ann has the red card' (r_A): $Pre(r'_a) = r_A$. The precondition that 'a tub of wine is being filled' (p') is that 'the tub is not full' ($\neg p$): $Pre(p') = \neg p$. This leads us to introduce a precondition function which assigns to every atomic event p' a formula of $\mathcal{L}$.

Definition 2.2. $Pre : \Phi' \to \mathcal{L}$ is a function from Φ' to $\mathcal{L}$. $\lhd$

The semantics for the languages $\mathcal{L}$ and $\mathcal{L}'$ is the same as the standard one of epistemic logic:

Definition 2.3. A $\mathcal{L}$-model (resp. $\mathcal{L}'$-model) is a tuple $M = (W, R, V)$ where:

- W is a set of possible worlds (resp. possible events),
- $R : G \to 2^{W \times W}$ is a function assigning to each agent $j \in G$ an accessibility relation on W.
- $V : \Phi \to 2^W$ (resp. $V' : \Phi' \to 2^W$) is a function assigning to each propositional letter of Φ (resp. Φ') a subset of W. V is called a valuation.

We write $w \in M$ for $w \in W$, and (M, w) is called a pointed $\mathcal{L}$-model (w often represents the actual world). If $w, w' \in W$, we write wR_jw' for $R(j)(w, w')$ and $R_j(w) = \{w' \in W \mid wR_jw'\}$. $\lhd$

Intuitively, a pointed $\mathcal{L}$-model (M, w) (resp. pointed $\mathcal{L}'$-model (M', w')) represents how the actual world represented by w (resp. actual event w') is perceived by the agents. Intuitively, $w' \in R_j(w)$ means that in world w agent j considers world w' as being possibly the world w. The truth conditions are also the same as in epistemic logic:

Definition 2.4. Let M be a $\mathcal{L}$-model, $w \in M$ and $\varphi \in \mathcal{L}$. $M, w \models \varphi$ is defined inductively as follows:

$$\begin{array}{lll}
M, w \models p & \text{iff} & w \in V(p) \\
M, w \models \neg\varphi & \text{iff} & \text{not } M, w \models \varphi \\
M, w \models \varphi \wedge \psi & \text{iff} & M, w \models \varphi \text{ and } M, w \models \psi \\
M, w \models B_j\varphi & \text{iff} & \text{for all } v \in R_j(w), M, v \models \varphi
\end{array}$$

We write $M \models \varphi$ when $M, w \models \varphi$ for all $w \in M$, and $\models \varphi$ when for all $\mathcal{L}$-model M, $M \models \varphi$. The definition is identical for $\mathcal{L}'$-models and the language $\mathcal{L}'$. $\lhd$

We also introduce a particular kind of $\mathcal{L}'$-model which will be used in the next subsection. We define $M^\emptyset = (\{w^\emptyset\}, R^\emptyset, V^\emptyset)$ where $V^\emptyset(p') = \emptyset$ for all $p' \in \Phi'$, and $R^\emptyset_j(w^\emptyset) = \{w^\emptyset\}$ for all $j \in G$. So $M^\emptyset$ represents the event whereby nothing happens and this is common belief among the agents.

We get back the usual notion of precondition of a possible event (Baltag, Moss, and Solecki 1998) as follows.

Definition 2.5. Let (M', w') be a pointed $\mathcal{L}'$-model. We define $Pre(w')$ as follows.

$$Pre(w') = \bigwedge\{Pre(p') \mid M', w' \models p'\}$$

$\lhd$

This leads us to redefine equivalently in our setting the BMS product update of (Baltag, Moss, and Solecki 1998) as follows. This product update takes as argument a $\mathcal{L}$-model (M, w) and a $\mathcal{L}'$-model (M', w') representing respectively how an initial situation is perceived by the agents and how an event occurring in this situation is perceived by them and yields a new $\mathcal{L}$-model $(M, w) \otimes (M', w')$ representing how the new situation is perceived by the agents after the occurrence of the event.

Definition 2.6. Let $(M, w) = (W, R, V, w)$ be a pointed $\mathcal{L}$-model and $(M', w') = (W', R', V', w')$ be a pointed $\mathcal{L}'$-model such that $M, w \models Pre(w')$. The pointed $\mathcal{L}$-model $(M, w) \otimes (M', w') = (W^\otimes, R^\otimes, V^\otimes, (w, w'))$ is defined as follows:

- $W^\otimes = \{(v, v') \in W \times W' \mid M, v \models Pre(v')\}$
- $R^\otimes_j(v, v') = \{(u, u') \in W^\otimes \mid u \in R_j(v) \text{ and } u' \in R'_j(v')\}$
- $V^\otimes(p) = \{(v, v') \in W^\otimes \mid M, v \models p\}$.

$\lhd$

Example 2.7. We take up the card example of (van Ditmarsch 1999). Assume that agents A, B and C play a card game with three cards: a white one, a red one and a blue one. Each of them has a single card but they do not know the cards of the other players. The goal of the game is to know the cards of the other players without showing nor telling his/her own cards to other players.

1. The initial situation is represented in the pointed $\mathcal{L}$-model (M, w) of Figure 1. In this example, $\Phi = \{r_j, b_j, w_j \mid j \in \{A, B, C\}\}$ where r_j stands for 'agent j has the $r$ed card', b_j stands for 'agent j has the $b$lue card' and w_j stands for 'agent j has the $w$hite card'. The boxed possible world corresponds to the actual world. The propositional letters not mentioned in the possible worlds do not hold in these possible worlds. The accessibility relations are represented by arrows indexed by agents between possible worlds. Reflexive arrows are omitted in the figure.
2. Assume now that players A and B cheat and show their card to each other. As it turns out, C noticed that A showed her card to B but did not notice that B did so with A. Players A and B know this. This event is represented in the $\mathcal{L}'$-model (M', w') of Figure 2. The boxed possible event w' corresponds to the actual event. There, $\Phi' = \{r'_j, b'_j, w'_j \mid j \in \{A, B, C\}\}$ where r'_j stands for 'agent j shows the $r$ed card', b'_j stands for 'agent j shows her $b$lue card' and w'_j stands for 'agent j shows her $w$hite card'. Clearly, $Pre(r'_j) = r_j$, $Pre(b'_j) = b_j$ and $Pre(w'_j) = w_j$ for all agents j: an agent can show a card only if she has this card. We mention in the possible events of the model only the atomic events that hold in these possible events. So we have $M', w' \models$

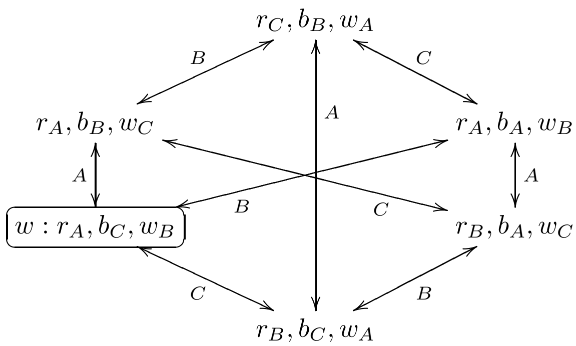

Figure 1: Cards Example

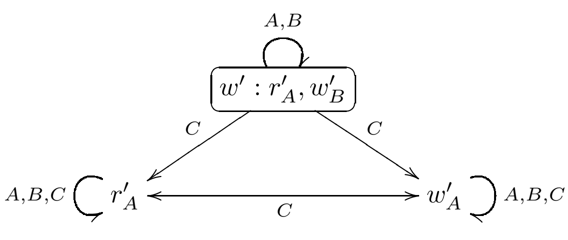

Figure 2: Ann and Bob are cheating

$w'_B \wedge B_A w'_B \wedge B_C \neg w'_B$: player A 'knows' that B shows her white card while player C believes this does not happen; $M', w' \models r'_A \wedge \langle B_C \rangle r'_A \wedge \langle B_C \rangle w'_A$: player A shows her red card but player C does not know whether she showed a red or a white card...

3. As a result of this event, the agents update their beliefs. We get the situation represented in the $\mathcal{L}$-model $(M, w) \otimes (M', w')$ of Figure 3. In this model, we have $(M, w) \otimes (M', w') \models (w_B \wedge B_A w_B) \wedge B_C \neg B_A w_B$: player A knows that player B has the white card but player C believes that it is not the case.

$\lhd$

2.2 A Language for Knowledge Representation

In this section we define a general language which can express statements about ongoing events together with static properties of the world.

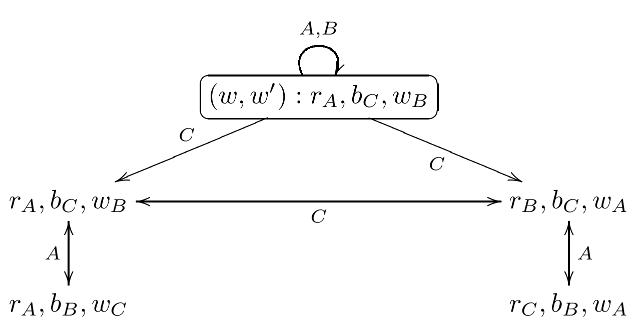

Figure 3: Situation after the cheating of Ann and Bob

Definition 2.8. The language $\mathcal{L}^{\otimes}$ is defined inductively as follows.

$$\mathcal{L}^{\otimes} : \varphi ::= \psi \mid \neg\varphi \mid \varphi \wedge \varphi \mid [start]\varphi \mid [end]\varphi$$

where $\psi \in \mathcal{L} \cup \mathcal{L}'$. $\mathcal{L}^{\otimes}_s$ is the language $\mathcal{L}^{\otimes}$ without the operators $[start]$ and $[end]$. $\lhd$

- $[end]\varphi$ reads 'φ holds after the current event ends'.
- $[start]\varphi$ reads 'φ holds when a new event starts'.

These two operators are analogue to the $Initiates$ and $Terminates$ predicates of the event calculus (Shanahan 1997). The semantics for this language is defined as follows. A $\mathcal{L}^{\otimes}$-model $\mathcal{M}$ below represents how the world is perceived by the agents at the current time: the $\mathcal{L}$-model (M, w) in $\mathcal{M}$ represents how the world is perceived by the agents from a static point of view, and the $\mathcal{L}'$-model (M', w') (if such a model is present in $\mathcal{M}$, i.e. if an event occurs at the current time) represents how the current event is perceived by the agents.

Definition 2.9. A $\mathcal{L}^{\otimes}$-model is either a pointed $\mathcal{L}$-model $\{(M, w)\}$ or a pair of pointed $\mathcal{L}$-model and $\mathcal{L}'$-model $\{(M, w), (M', w')\}$ such that $M, w \models Pre(w')$.

Let $\varphi \in \mathcal{L}^{\otimes}$ and $\mathcal{M}$ a $\mathcal{L}^{\otimes}$-model. $\mathcal{M} \models \varphi$ is defined inductively as follows.

$\mathcal{M} \models \varphi$	iff	$M, w \models \varphi$
$\mathcal{M} \models \varphi'$	iff	$\begin{cases} M', w' \models \varphi' \\ \text{if there is } (M', w') \in \mathcal{M} \\ M^{\emptyset}, w^{\emptyset} \models \varphi' \\ \text{otherwise} \end{cases}$
$\mathcal{M} \models [start]\varphi$	iff	for all pointed $\mathcal{L}'$-model (M', w') such that $\mathcal{M}' = \mathcal{M} \cup \{(M', w')\}$ is a $\mathcal{L}^{\otimes}$-model, $\mathcal{M}' \models \varphi$
$\mathcal{M} \models [end]\varphi$	iff	if $\mathcal{M} = \{(M, w), (M', w')\}$ then $(M, w) \otimes (M', w') \models \varphi$

The boolean cases are defined as usual. $\lhd$

Example 2.10. The scenario of Example 27 can be represented by the following sequence of $\mathcal{L}^{\otimes}$-models. These models provide a representation of the situation at every step of the scenario.

1. $\{(M, w)\}$
2. $\{(M, w), (M', w')\}$
3. $\{(M, w) \otimes (M', w')\}$

At step 2, our language allows us to express statements about the ongoing event of cheating together with static properties of the world: $\{(M, w), (M', w')\} \models (w_B \wedge \neg B_C w_B) \wedge (w'_B \wedge B_C \neg w'_B)$: player B has the white card but player C does not know it, and player B is showing his white card without player C noticing it. $\lhd$

In (Aucher 2009a), the axiomatization of the full language $\mathcal{L}^{\otimes}$ is provided. For the purpose of this paper, we only give the axiomatization of $\mathcal{L}^{\otimes}_s$.

Theorem 2.11. *(Aucher 2009a) The semantics of $\mathcal{L}^{\otimes}_s$ is sound and complete w.r.t. the logic $\mathsf{L}^{\otimes}_s$ axiomatized by the* *following axiom schemes and inference rules.*

Taut	*All propositional axiom schemes and inference rules*	
Pre	$\vdash p' \rightarrow Pre(p')$	*for all* $p' \in \Phi'$
K	$\vdash B_j(\varphi \rightarrow \psi) \rightarrow (B_j\varphi \rightarrow B_j\psi)$	*for all* $j \in G$
Nec	*If* $\vdash \varphi$ *then* $\vdash B_j\varphi$	*for all* $j \in G$

Besides, the logic $\mathsf{L}_s^{\otimes}$ *is decidable.*

2.3 Embedding BMS

We finally show that the BMS framework (Baltag and Moss 2004) can be embedded in our framework. The syntax of the BMS language $\mathcal{L}_{BMS}(A)$ is defined as follows:

$$\mathcal{L}_{BMS}(A) : \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid B_j\varphi \mid C_G\varphi \mid [A, a]\varphi$$

where $p \in \Phi$, $j \in G$ and A is an event model. The semantics for its epistemic part is the same as the one of Definition 24 (see (Fagin et al. 1995) for the definition of the operator C_G) and the one for its operator $[A, a]\varphi$ is defined as follows:

$$M, w \models [A, a]\varphi \text{ iff}$$
$$M, w \models Pre(a) \text{ implies } M \otimes A, (w, a) \models \varphi$$

We also add a common belief operator C_G to our languages $\mathcal{L}$ and $\mathcal{L}'$ and we assume as in BMS that the $\mathcal{L}'$-models are finite. Let $A = (E, R, Pre)$ be an event model, i.e. a set of possible events $E = \{a_1, \ldots, a_n\}$ together with an accessibility relation R and a precondition function $Pre : E \rightarrow \mathcal{L}$. We define the set of atomic events $\Phi' = \{p'_1, \ldots, p'_n\}$ such that $Pre(p'_i) = Pre(a_i)$. We define the pointed $\mathcal{L}'$-model $t(A, a) = (W', R', V', a)$ by $W' = E$, $R' = R$ and $V'(p'_i) = \{a_i\}$ for all $i \in \{1, \ldots, n\}$. $t(A, a)$ can be characterized[3] by a single formula $\chi(t(A, a))$ (thanks to the common belief operator C_G). We also define the operator t from $\mathcal{L}_{BMS}(A)$ to $\mathcal{L}^{\otimes}$ by $t(p) = p, t(\neg\varphi) = \neg t(\varphi), t(\varphi \wedge \varphi') = t(\varphi) \wedge t(\varphi'), t(B_j\varphi) = B_jt(\varphi), t(C_G\varphi) = C_Gt(\varphi)$ and

$$t([A, a]\varphi) = [start](\chi(t(A, a)) \rightarrow [end]t(\varphi)).$$

Theorem 2.12. *(Aucher 2009a) Let A be an event model and $\varphi \in \mathcal{L}_{BMS}(A)$. For all pointed $\mathcal{L}$-model (M, w),*

$$M, w \models \varphi \text{ iff } \{(M, w)\} \models t(\varphi).$$

3 Characterizing Updates

3.1 Mathematical Preliminaries

To axiomatize the product update, we will resort to particular kinds of formulas which capture the structure of epistemic models up to a given modal depth. These formulas are similar in spirit to Jankov-Fine formulas (Blackburn, de Rijke, and Venema 2001) and were introduced in (Balbiani and Herzig 2007). They are defined as follows.

Definition 3.1. (Balbiani and Herzig 2007) We define inductively the sets E_n as follows.

- $E_0 = \left\{ \bigwedge_{p \in S_0} p \wedge \bigwedge_{p \notin S_0} \neg p \mid S_0 \subseteq \Phi \right\}$;
- $E_{n+1} = \left\{ \delta_0 \wedge \bigwedge_{j \in G} \left(\bigwedge_{\delta_n \in S_n^j} \langle B_j \rangle \delta_n \wedge B_j \bigvee_{\delta_n \in S_n^j} \delta_n \right) \mid \delta_0 \in E_0, S_n^j \subseteq E_n \right\}$.

We define identically the sets E'_n, except that the set of atomic facts Φ is replaced by atomic events Φ'. If $\delta'_0 \in E'_0$, we define $Pre(\delta'_0) = \bigwedge\{Pre(p') \mid\vdash \delta'_0 \rightarrow p'\}$. $\lhd$

The following proposition not only tells us that a formula δ_n completely characterizes the structure up to modal depth n of any pointed epistemic model where it holds (first item), but also that the structure of *any* epistemic modal up to modal depth n can be characterized by such a formula δ_n (second item).

Proposition 3.2. *(Balbiani and Herzig 2007) Let $n \in \mathbb{N}$, $\delta_n \in E_n$ and $\varphi \in \mathcal{L}$ such that $deg(\varphi) \leq n$.*

$$\text{Either } \delta_n \rightarrow \varphi \in \mathsf{K} \text{ or } \delta_n \rightarrow \neg\varphi \in \mathsf{K}.$$
$$\bigvee_{\delta_n \in S_n} \delta_n \in \mathsf{K}.$$

where K is the smallest normal multi-modal modal logic (note that $\mathsf{K} \subseteq \mathsf{L}_s^{\otimes}$).

The following corollary will play an important role in the axiomatization. It states that any formula (of degree n) can be reduced to a boolean combination of δ_ns.

Corollary 3.3. *Let $\varphi \in \mathcal{L}$ such that $deg(\varphi) \leq n$. Then there are $\delta_n^1, \ldots, \delta_n^k \in E_n$ such that $\varphi \leftrightarrow (\delta_n^1 \vee \ldots \vee \delta_n^k) \in \mathsf{K}$.*

Proof. Let $\delta_n^1, \ldots, \delta_n^k \in E_n$ such that $\delta_n^i \rightarrow \varphi \in \mathsf{K}$. Then $\delta_n^1 \vee \ldots \vee \delta_n^k \rightarrow \varphi \in \mathsf{K}$. Then by Proposition 32, for all $\delta'_n \neq \delta_n^1, \ldots, \delta_n^k$, $\delta'_n \rightarrow \neg\varphi \in \mathsf{K}$. Therefore $\varphi \rightarrow \neg\delta'_n \in \mathsf{K}$ for all $\delta_n \neq \delta_n^1, \ldots, \delta_n^k$. But because $\bigvee_{\delta_n \in E_n} \delta_n \in \mathsf{K}$, we have $\varphi \rightarrow \delta_n^1 \vee \ldots \vee \delta_n^k \in \mathsf{K}$. Finally $\varphi \leftrightarrow (\delta_n^1 \vee \ldots \vee \delta_n^k) \in \mathsf{K}$. $\square$

3.2 Axiomatization of the Product Update $\otimes$

So our framework allows to express in a uniform way statements about:

1. what is true about an initial situation: $\mathcal{L}$
2. what is true about an event occurring in this situation: $\mathcal{L}'$
3. what is true in the resulting situation after the occurrence of this event: $\mathcal{L}$

We are going to axiomatize what we can infer about (3) given (1) and (2). To do so, we introduce an operator $\varphi \otimes \varphi' \supset \varphi''$ taking as argument three formulas $\varphi, \varphi'' \in \mathcal{L}$ and $\varphi' \in \mathcal{L}'$. Its interpretation is 'φ'' holds after any event satisfying φ' occurred in a situation where φ held'. Because we need in the axiomatization to deal with boolean combinations of this operator, we define the following language.

Definition 3.4. The language $\mathcal{L}^{\supset}$ is defined inductively as follows.

$$\mathcal{L}^{\supset} : \varphi ::= \psi \otimes \psi' \supset \psi'' \mid \neg\varphi \mid \varphi \wedge \varphi$$

where $\psi, \psi'' \in \mathcal{L}$ and $\psi' \in \mathcal{L}'$. Let $\{(M, w), (M', w')\}$ be a $\mathcal{L}^{\otimes}$-model.

[3] A formula χ characterizes a finite and pointed $\mathcal{L}$-model (M, w) iff $M, w \models \chi$ and for all finite and pointed $\mathcal{L}$-model (M', w'), if $M', w' \models \chi$ then (M, w) is bisimilar to (M', w').

$$\{(M,w),(M',w')\} \models \psi \otimes \psi' \supset \psi''$$
iff
if $M,w \models \psi$ and $M',w' \models \psi'$ then
$(M,w) \otimes (M',w') \models \psi''$

The inductive clauses for $\neg$ and $\wedge$ are defined as usual. $\triangleleft$

The following proposition shows that our operator can actually be expressed in the general language of (Aucher 2009a).

Proposition 3.5. *Let $\varphi, \varphi'' \in \mathcal{L}$ and $\varphi' \in \mathcal{L}'$.*

$$\models \varphi \to [start](\varphi' \to [end]\varphi'') \textit{ iff } \models \varphi \otimes \varphi' \supset \varphi''$$

Proof. It suffices to apply the definitions. $\square$

Definition 3.6. The logic $\mathsf{L}^{\supset}$ is defined by the following axiom schemes and inference rules:

- A_1 $\vdash \varphi \otimes \varphi' \supset \bot$ iff $\neg(\varphi \wedge \varphi') \in \mathsf{L}_s^{\otimes}$
- A_2 $\vdash \varphi_p \otimes \varphi' \supset \varphi_p$ for all propositional formula φ_p
- A_3 If $\varphi \to \psi, \varphi' \to \psi', \psi'' \to \varphi'' \in \mathsf{L}_s^{\otimes}$ then $\vdash \psi \otimes \psi' \supset \psi'' \to \varphi \otimes \varphi' \supset \varphi''$
- A_4 $\vdash (\varphi \otimes \varphi' \supset \varphi'') \wedge (\psi \otimes \varphi' \supset \varphi'') \to (\varphi \vee \psi) \otimes \varphi' \supset \varphi''$
- A_5 $\vdash (\varphi \otimes \varphi' \supset \varphi'') \wedge (\varphi \otimes \psi' \supset \varphi'') \to \varphi \otimes (\varphi' \vee \psi') \supset \varphi''$
- A_6 $\vdash \varphi \otimes \varphi' \supset (\varphi'' \vee \psi'') \to (\varphi \otimes \varphi' \supset \varphi'') \vee (\varphi \otimes \varphi' \supset \psi'')$
- R_1 Assume $\varphi' \to \delta'_0 \in \mathsf{L}_s^{\otimes}$, for some $\delta'_0 \in E'_0$. If $\vdash \varphi \otimes \varphi' \supset \varphi''$, then $\vdash \langle B_j \rangle (\varphi \wedge Pre(\delta'_0)) \otimes \langle B_j \rangle \varphi' \supset \langle B_j \rangle \varphi''$.
- R_2 If $\vdash \varphi \otimes \varphi' \supset \varphi''$ then $\vdash B_j\varphi \otimes B_j\varphi' \supset B_j\varphi''$
- R_3 If $\vdash \varphi$ and $\vdash \varphi \to \psi$ then $\vdash \psi$

$\triangleleft$

The key axioms are A_2, and rules R_1 and R_2: A_2 can be seen as the base case, and R_1 and R_2 (together with the rest of the axioms) can be seen as the induction steps allowing to build more complex formulas. Axiom A_2 also illustrates the fact that we deal as in BMS with epistemic events, i.e. events which do not change atomic facts. Axioms A_3 to A_6 illustrate the material implication character of our operator $\varphi \otimes \varphi' \supset \varphi''$.

Proposition 3.7. *Let $\varphi, \varphi'' \in \mathcal{L}$ and $\varphi' \in \mathcal{L}'$.*
If $\vdash \varphi \otimes \varphi' \supset \varphi''$ then $\models \varphi \otimes \varphi' \supset \varphi''$.

Proof. It suffices to prove that the logic $\mathsf{L}^{\supset}$ is sound w.r.t. the semantics of $\mathcal{L}^{\supset}$. We only prove soundness of R_1 and R_2. The soundness of the other axioms is standard.

Soundness of R_2. Assume $\models \varphi \otimes \varphi' \supset \varphi''$. Let $\{(M,w),(M',w')\}$ be a $\mathcal{L}^{\otimes}$-model such that $M,w \models B_j\varphi$ and $M',w' \models B_j\varphi'$. Then for all $v \in R_j(w)$ $M,v \models \varphi$ and for all $v' \in R_j(w')$, $M',v' \models \varphi'$. So for all $v' \in R_j(w')$ and all $v \in R_j(w)$ such that $M,v \models Pre(v')$, $M,v \models \varphi$ and $M',v' \models \varphi'$. So for all $(v,v') \in R_j(w,w')$, $(M,v) \otimes (M',v') \models \varphi''$ by definition of $\varphi \otimes \varphi \supset' \varphi''$. So $(M,w) \otimes (M',w') \models B_j\varphi''$. Therefore $\models B_j\varphi \otimes B_j\varphi' \supset B_j\varphi''$.

Soundness of R_1. Assume that $\models \varphi \otimes \varphi' \supset \varphi''$ and $\models \varphi' \to \delta'_0$. Let $\{(M,w),(M',w')\}$ be a $\mathcal{L}^{\otimes}$-model such that $M,w \models \langle B_j \rangle (\varphi \wedge Pre(\delta'_0))$ and $M',w' \models \langle B_j \rangle \varphi'$. Then there is $v' \in R_j(w')$ such that $M',v' \models \varphi'$ and there is $v \in R_j(w)$ such that $M,v \models \varphi \wedge Pre(v')$. Therefore $(M,v) \otimes (M',v') \models \varphi''$ because $\models \varphi \otimes \varphi' \supset \varphi''$. So $(M,w) \otimes (M',w') \models \langle B_j \rangle \varphi''$ because $(v,v') \in R_j(w,w')$. $\square$

Proposition 3.8. *Let $\varphi, \varphi'' \in \mathcal{L}$ and $\varphi' \in \mathcal{L}'$.*
If $\models \varphi \otimes \varphi' \supset \varphi''$ then $\vdash \varphi \otimes \varphi' \supset \varphi''$.

The proof of the key Proposition 38 is based on the following rough ideas. For all $\varphi, \varphi'' \in \mathcal{L}$ and $\varphi' \in \mathcal{L}'$, $\varphi \otimes \varphi' \supset \varphi''$ can be reduced equivalently to a boolean combination of $\delta \otimes \delta' \supset \delta''$, which are always true or always false. Therefore, it suffices to show that completeness holds for this kind of formulas, which is what the following Lemma proves.

Lemma 3.9. *Let $N = max\{deg(Pre(p')) \mid p' \in \Phi'\}$. For all $\delta_{n+N} \in E_{n+N}, \delta'_n \in E'_n, \delta''_n \in E_n$ such that $\neg(\delta_{n+N} \wedge \delta'_n) \notin \mathsf{L}_s^{\otimes}$,*

$$\vdash \delta_{n+N} \otimes \delta'_n \supset \delta''_n \textit{ iff } \models \delta_{n+N} \otimes \delta'_n \supset \delta''_n$$

Proof. We prove it by induction on n. The necessary direction holds by soundness of $\mathsf{L}^{\supset}$. We only prove the sufficient direction.

- $n = 0$. Assume that $\models \delta_N \otimes \delta'_0 \supset \delta''_0$. Then $\models \delta_N \to \delta''_0$ by definition of $\otimes$ and δ_N and because $\neg(\delta_{n+N} \wedge \delta'_n) \notin \mathsf{L}_s^{\otimes}$. So $\delta_N \to \delta''_0 \in \mathsf{L}_s^{\otimes}$. But $\vdash \delta''_0 \otimes \delta'_0 \supset \delta''_0$ by Axiom A_2. So $\vdash \delta_N \otimes \delta'_0 \supset \delta''_0$ by Axiom A_3.

- $n+1$. Let $\delta_{n+N+1} \in E_{n+N+1}, \delta'_{n+1} \in E'_{n+1}$ and $\delta''_{n+1} \in E_{n+1}$ such that $\neg(\delta_{n+N+1} \wedge \delta'_{n+1}) \notin \mathsf{L}_s^{\otimes}$.

$$\delta_{n+N+1} = \delta_0 \wedge \bigwedge_{j \in G} \left(\bigwedge_{i \in I} \langle B_j \rangle \delta_{n+N,i} \wedge B_j \left(\bigvee_{i \in I} \delta_{n+N,i} \right) \right)$$

$$\delta'_{n+1} = \delta'_0 \wedge \bigwedge_{j \in G} \left(\bigwedge_{i \in I'} \langle B_j \rangle \delta'_{n,i} \wedge B_j \left(\bigvee_{i \in I'} \delta'_{n,i} \right) \right)$$

$$\delta''_{n+1} = \delta''_0 \wedge \bigwedge_{j \in G} \left(\bigwedge_{i \in I''} \langle B_j \rangle \delta''_{n,i} \wedge B_j \left(\bigvee_{i \in I''} \delta''_{n,i} \right) \right).$$

Assume $\nvdash \delta_{n+N+1} \otimes \delta'_{n+1} \supset \delta''_{n+1}$. Then

$$\text{either} \nvdash \delta_{n+N+1} \otimes \delta'_{n+1} \supset \delta''_0 \quad (1)$$
$$\text{or} \nvdash \delta_{n+N+1} \otimes \delta'_{n+1} \supset \langle B_j \rangle \delta''_{n,i} \text{ for some } i \in I'' \quad (2)$$
$$\text{or} \nvdash \delta_{n+N+1} \otimes \delta'_{n+1} \supset B_j \left(\bigwedge_{i \in I''} \delta''_{n,i} \right). \quad (3)$$

1. Assume that (3) is the case. Then by A_3, because
$\delta_{n+N+1} \to B_j \left(\bigvee_{i \in I} \delta_{n+N,i} \right) \in \mathsf{L}_s^{\otimes}$ and
$\delta_{n+1} \to B_j \left(\bigvee_{i \in I'} \delta'_{n,i} \right) \in \mathsf{L}_s^{\otimes}$,
$\nvdash B_j \left(\bigvee_{i \in I} \delta_{n+N,i} \right) \otimes B_j \left(\bigvee_{i \in I'} \delta'_{n,i} \right) \supset B_j \left(\bigvee_{i \in I''} \delta''_{n,i} \right)$.
Then $\nvdash \bigvee_{i \in I} \delta_{n+N,i} \otimes \bigvee_{i \in I'} \delta'_{n,i} \supset \bigvee_{i \in I''} \delta''_{n,i}$ by Axiom R_2.

Then $\nvdash \delta_{n+N,i_0} \otimes \delta'_{n,k_0} \supset \bigvee_{i\in I''} \delta''_{n,i}$ for some $\delta_{n+N,i_0}, \delta'_{n,k_0}$ such that $\nvdash \neg(\delta_{n+N,i_0} \wedge \delta'_{n,k_0})$ by $\mathsf{A}_1, \mathsf{A}_4$ and A_5.

Indeed, if for all $\delta_{n+N,i}, \delta'_{n,k}$ such that $\nvdash \neg(\delta_{n+N,i} \wedge \delta'_{n,k})$, $\vdash \delta_{n+N,i} \otimes \delta'_{n,k} \supset \bigvee_{i\in I''} \delta''_{n,i}$, then for all $\delta_{n+N,i}, \delta_{n,i}$, $\vdash \delta_{n+N,i} \otimes \delta'_{n,k} \supset \bigvee_{i\in I''} \delta''_{n,i}$ by Axiom A_1.

So $\vdash \bigvee_{i\in I} \delta_{n+N,i} \otimes \bigvee_{i\in I'} \delta_{n,i} \supset \bigvee_{i\in I''} \delta''_{n,i}$ by Axioms A_4 and A_5, which is impossible.

So for all $i \in I''$, $\nvdash \delta_{n+N,i_0} \otimes \delta'_{n,k_0} \supset \delta''_{n,i}$ by Axiom A_3. So for all $i \in I''$, $\nvDash \delta_{n+N,i_0} \otimes \delta'_{n,k_0} \supset \delta''_{n,i}$ by Induction Hypothesis. So for all $i \in I''$, there is $\{(M_i, w_i), (M'_i, w'_i)\}$ a $\mathcal{L}^{\otimes}$-model such that $M_i, w_i \models \delta'_{n,k_0}$ and $(M_i, w_i) \otimes (M'_i, w'_i) \models \neg\delta''_{n,i}$.

Lemma 3.10. *Assume there is a $\mathcal{L}_s^{\otimes}$-model $\{'M, w), (M', w')\}$ such that $M, w \models \delta_{n+N}$ and $M', w' \models \delta'_n$ and $(M, w) \otimes (M', w') \models \varphi$ for some $\delta_{n+N} \in E_{n+N}, \delta'_n \in E'_n$ and $deg(\varphi) \leq n$.*
Then for all $\mathcal{L}_s^{\otimes}$-model $\{(M, w), (M', w')\}$ such that $M, w \models \delta_{n+N}$ and $M', w' \models \delta'_n$, $(M, w) \otimes (M', w') \models \varphi$.

Proof sketch. It is due to the fact that the structure of $(M, w) \otimes (M', w')$ up to modal depth n is determined only by the structure of (M, w) up to modal depth $n + N$ and the modal structure of (M', w') up to modal depth n, which are themselves completely determined by some δ_{n+N} and δ_n respectively. □

Then by Lemma 310, for all $i \in I''$, for all $\mathcal{L}_s^{\otimes}$-model $\{(M, w), (M', w')\}$, if $M, w \models \delta_{n+N,i_0}$ and $M', w' \models \delta'_{n,k_0}$ then $(M, w) \otimes (M', w') \models \neg\delta''_{n,i}$.

So for all $i \in I''$, $\models \delta_{n+N,i_0} \otimes \delta'_{n,k_0} \supset \neg\delta''_{n,i}$. So $\models \delta_{n+N,i_0} \otimes \delta'_{n,k_0} \supset \bigwedge_{i\in I''} \neg\delta''_{n,i}$. But $\nvdash \neg(\delta_{n+N,i_0} \wedge \delta'_{n,k_0})$, so $\delta_{n+N,i_0} \rightarrow Pre(\delta'_0) \in \mathsf{L}_s^{\otimes}$. Indeed, otherwise $\delta_{n+N,i_0} \rightarrow \neg Pre(\delta'_0) \in \mathsf{L}_s^{\otimes}$ because $deg(Pre(\delta'_0)) \leq N$, and then $\delta_{n+N,i_0} \rightarrow \neg\delta'_0 \in \mathsf{L}_s^{\otimes}$ because $\delta'_0 \rightarrow Pre(\delta'_0) \in \mathsf{L}_s^{\otimes}$, so $\neg(\delta_{n+N,i_0} \wedge \delta'_{n,k_0}) \in \mathsf{L}_s^{\otimes}$.

But $\delta'_{n,k_0} \rightarrow \delta'_0 \in \mathsf{L}_s^{\otimes}$. So $\models \langle B_j\rangle\delta_{n+N,i_0} \otimes \langle B_j\rangle\delta'_{n,k_0} \supset \langle B_j\rangle \bigwedge_{i\in I''} \neg\delta''_{n,i}$ by Rule R_1 and soundness.

i.e. $\models \langle B_j\rangle\delta_{n+N,i_0} \otimes \langle B_j\rangle\delta'_{n,k_0} \supset \neg B_j \left(\bigvee_{i\in I''} \delta''_{n,i}\right)$

then $\models \delta_{n+N+1} \otimes \delta'_{n+1} \supset \neg\delta''_{n+1}$ by Axiom A_3

then $\nvDash \delta_{n+N+1} \otimes \delta'_{n+1} \supset \delta''_{n+1}$.

Indeed, otherwise $\models \delta_{n+N+1} \otimes \delta'_{n+1} \supset \bot$, which entails that $\vdash \neg(\delta_{n+N+1} \wedge \delta'_{n+1})$. This is impossible by assumption.

2. Assume that (2) is the case: $\nvdash \delta_{n+N+1} \otimes \delta'_{n+1} \supset \langle B_j\rangle\delta''_{n,l}$ for some $l \in I''$. So $\nvdash \langle B_j\rangle\delta_{n+N,i} \otimes \langle B_j\rangle\delta'_{n,k} \supset \langle B_j\rangle\delta''_{n,l}$ for all $i \in I$ and $k \in I'$.
   (a) Assume that for all $i \in I$, $k \in I'$ $\neg(\delta_{n+N,i} \wedge \delta'_{n,k}) \in \mathsf{L}_s^{\otimes}$. Then for all $i \in I, k \in I'$, $\delta_{n+N,i} \otimes \delta_{n,k} \supset \bot$ by Axiom A_1. Therefore, $\bigvee_{i\in I} \delta_{n+N,i} \otimes \bigvee_{i\in I'} \delta_{n,i} \supset \bot$ by Axioms A_4 and A_5.
   So $\vdash B_j \left(\bigvee_{i\in I} \delta_{n+N,i}\right) \otimes B_j \left(\bigvee_{i\in I'} \delta_{n,i}\right) \supset B_j\bot$ by Rule R_2. So $\vdash \delta_{n+N+1} \otimes \delta'_{n+1} \supset B_j\bot$ by Axiom A_3. Then $\vdash \delta_{n+N+1} \otimes \delta'_{n+1} \supset \neg\delta''_{n+1}$ by Axiom A_3. So $\models \delta_{n+N+1} \otimes \delta'_{n+1} \supset \neg\delta''_{n+1}$ by soundness. Then $\nvDash \delta_{n+N+1} \otimes \delta'_{n+1} \supset \delta''_{n+1}$ because otherwise $\models \delta_{n+N+1} \otimes \delta'_{n+1} \supset \bot$, which would entail that $\models \neg(\delta_{n+N+1} \wedge \delta'_{n+1})$ which is impossible.
   (b) Assume that there is $i_0 \in I, k_0 \in I'$ such that $\neg(\delta_{n+N,i_0} \wedge \delta'_{n,k_0}) \notin \mathsf{L}_s^{\otimes}$. Then $\delta_{n+N,i_0} \rightarrow Pre(\delta'_0) \in \mathsf{L}_s^{\otimes}$. Indeed, otherwise $\delta_{n+N,i_0} \rightarrow \neg Pre(\delta'_0) \in \mathsf{L}_s^{\otimes}$ because $deg(Pre(\delta_0)) \leq N \leq n + N$. In that case, $\delta_{n+N,i_0} \rightarrow \neg\delta'_0 \in \mathsf{L}_s^{\otimes}$ because $\delta'_0 \rightarrow Pre(\delta'_0) \in \mathsf{L}_s^{\otimes}$ by Axiom scheme Pre. So $\delta_{n+N,i_0} \rightarrow \neg\delta'_n \in \mathsf{L}_s^{\otimes}$, i.e. $\neg(\delta_{n+N,i_0} \wedge \delta'_n) \in \mathsf{L}_s^{\otimes}$ which is impossible. Then $\nvdash \delta_{n+N,i_0} \otimes \delta'_{n,k_0} \supset \delta''_{n,l}$ by Rule R_1.
   So $\nvDash \delta_{n+N,i_0} \otimes \delta'_{n,k_0} \supset \delta''_{n,l}$ by Induction Hypothesis, and so for all $i_0 \in I, k_0 \in I'$ such that $\neg(\delta_{n+N,i_0} \wedge \delta'_{n,k_0}) \notin \mathsf{L}_s^{\otimes}$.
   Then $\models \delta_{n+N,i_0} \otimes \delta'_{n,k_0} \supset \neg\delta''_{n,l}$ by Lemma 310, for all $i_0 \in I, k_0 \in I'$ such that $\neg(\delta_{n+N,i_0} \wedge \delta'_{n,k_0}) \notin \mathsf{L}_s^{\otimes}$.
   Then $\models \delta_{n+N,i} \otimes \delta'_{n,k} \supset \neg\delta''_{n,l}$ for all $i \in I$, $k \in I'$ by soundness and Axiom A_1.
   So $\models \bigvee_{i\in I} \delta_{n+N,i} \otimes \bigvee_{k\in I'} \delta'_{n,k} \supset \neg\delta''_{n,l}$ for some $l \in I''$ by Axioms A_4 and A_5.
   Then $\models B_j \left(\bigvee_{i\in I} \delta_{n+N,i}\right) \otimes B_j \left(\bigvee_{k\in I'} \delta'_{n,k}\right) \supset \neg\langle B_j\rangle\delta''_{n,l}$ by Rule R_2. So $\models \delta_{n+N+1} \otimes \delta'_{n+1} \supset \neg\delta''_{n+1}$ by Axiom A_3. Therefore $\nvDash \delta_{n+N+1} \otimes \delta'_{n+1} \supset \delta''_{n+1}$ because $\neg(\delta_{n+N+1} \wedge \delta'_{n+1}) \notin \mathsf{L}_s^{\otimes}$ by assumption.
3. Assume that (1) is the case: $\nvdash \delta_{n+N+1} \otimes \delta'_{n+1} \supset \delta''_0$. Then $\delta_{n+N+1} \rightarrow \delta''_0 \notin \mathsf{L}_s^{\otimes}$. Indeed, otherwise, because $\vdash \delta''_0 \otimes \delta'_{n+1} \supset \delta''_0$ by Axiom A_2, we would have $\vdash \delta_{n+N+1} \otimes \delta'_{n+1} \supset \delta''_0$ by Axiom A_3, which is impossible.
   Therefore $\delta_{n+N+1} \rightarrow \neg\delta''_0 \in \mathsf{L}_s^{\otimes}$ by Proposition 3. .2. Therefore, because $\models \neg\delta''_0 \otimes \delta'_{n+1} \supset \neg\delta''_0$ by Axiom A_2, $\models \delta_{n+N+1} \otimes \delta'_{n+1} \supset \neg\delta_0$ by Axiom A_3. So $\nvDash \delta_{n+N+1} \otimes \delta'_{n+1} \supset \delta''_0$ because $\neg(\delta_{n+N+1} \wedge \delta'_{n+1}) \notin \mathsf{L}_s^{\otimes}$ by assumption.

□

Proof of Proposition 38. Let $\varphi, \varphi'' \in \mathcal{L}$, $\varphi' \in \mathcal{L}'$ such that $max\{deg(\varphi), deg(\varphi'), deg(\varphi'')\} \leq n$.

Assume that $\models \varphi \otimes \varphi' \supset \varphi''$. By Corollary 33, there are $\delta_{n+N,i} \in E_n, \delta'_{n,k} \in E'_n, \delta''_{n,l} \in E_n$ such that $\models \varphi \leftrightarrow \bigvee_{i\in I} \delta_{n+N,i}$, $\models \varphi' \leftrightarrow \bigvee_{k\in I'} \delta'_{n,k}$ and $\models \varphi'' \leftrightarrow \bigvee_{l\in I''} \delta''_{n,l}$. Therefore, by soundness of Axiom A_3,

$\models \varphi \otimes \varphi' \supset \varphi'' \leftrightarrow \bigvee_{i \in I} \delta_{n+N,i} \otimes \bigvee_{k \in I'} \delta'_{n,k} \supset \bigvee_{l \in I''} \delta''_{n,l}$. So

$$\models \varphi \otimes \varphi' \supset \varphi'' \leftrightarrow \bigvee_{l \in I''} \left(\bigvee_{i \in I} \delta_{n+N,i} \otimes \bigvee_{k \in I'} \delta'_{n,k} \supset \delta''_{n,l} \right)$$

by Axiom A_6. Then by axioms A_4 and A_5,

$$\models \varphi \otimes \varphi' \supset \varphi'' \leftrightarrow \bigvee_{l \in I''} \left(\bigwedge_{i \in I} \bigwedge_{k \in I'} \delta_{n+N,i} \otimes \delta'_{n,k} \supset \delta''_{n,l} \right).$$

So

$$\models \varphi \otimes \varphi' \supset \varphi'' \leftrightarrow \bigvee_{l \in I''} \left(\bigwedge_{i \in I_0} \bigwedge_{k \in I'_0} \delta_{n+N,i} \otimes \delta'_{n,k} \supset \delta''_{n,l} \right)$$

where $I_0 = \{i \in I \mid \exists k \in I' \not\vdash \neg(\delta_{n+N,i} \wedge \delta'_{n,k})\}$
and $I'_0 = \{k \in I' \mid \exists i \in I \not\vdash \neg(\delta_{n+N,i} \wedge \delta'_{n,k})\}$,
because of the soundness of Axiom A_1.

Assume that for all $l \in I''$ there are $i_0 \in I_0, k_0 \in I'_0$ such that $\not\models \delta_{n+N,i_0} \otimes \delta'_{n,k_0} \supset \delta''_{n,l}$.

Then by Lemma 39, we have $\not\models \delta_{n+N,i_0} \otimes \delta'_{n,k_0} \supset \delta'_{n,l}$.

By Lemma 310, we have $\models \delta_{n+N,i_0} \otimes \delta'_{n,k_0} \supset \neg\delta''_{n,l}$ $(*)$.

But by assumption there is $l_0 \in I''$ such that for all $i \in I_0, k \in I'_0$, $\models \delta_{n+N,i} \otimes \delta'_{n,k} \supset \delta''_{n,l_0}$. In particular, $\models \delta_{n+N,i_0} \otimes \delta'_{n,k_0} \supset \delta''_{n,k_0}$ $(**)$.

But $(*)$ and $(**)$ together entail that $\models \neg(\delta_{n+N,i_0} \wedge \delta_{n,k_0})$ by soundness of Axiom A_1. This is impossible.

Therefore there is $l_0 \in I''$ such that for all $i \in I_0, k \in I'_0$, $\vdash \delta_{n+N,i} \otimes \delta'_{n,k} \supset \delta''_{n,l_0}$

Then $\vdash \bigwedge_{i \in I} \bigwedge_{k \in I'} \delta_{n+N,i} \otimes \delta'_{n,k} \supset \delta''_{n,l_0}$.

So $\vdash \varphi \otimes \varphi' \supset \varphi''$ by Axioms A_3, A_4 and A_5. ☐

Putting Propositions 37 and 38 together, we obtain the following theorem.

Theorem 3.11. *Let $\varphi, \varphi'' \in \mathcal{L}$ and $\varphi' \in \mathcal{L}'$.*

$$\models \varphi \otimes \varphi' \supset \varphi'' \text{ iff } \vdash \varphi \otimes \varphi' \supset \varphi''$$

Theorem 3.12. *Given some formulas $\varphi, \varphi'' \in \mathcal{L}$ and $\varphi' \in \mathcal{L}'$, the problem of determining whether $\vdash \varphi \otimes \varphi' \supset \varphi''$ holds is decidable.*

Proof. Let $\varphi, \varphi'' \in \mathcal{L}$ and $\varphi' \in \mathcal{L}'$. Determining whether $\vdash \varphi \otimes \varphi' \supset \varphi''$ holds amounts to determine whether $\models \varphi \otimes \varphi' \supset \varphi''$ holds by Theorem 311. But by Proposition 3. .5, this amounts to determine whether $\models \varphi \to [start](\varphi' \to [end]\varphi'')$ holds, which is a decidable problem as proved in (Aucher 2009a). ☐

Example 3.13. Let us take up our card example. We formalize syntactically the general setting of the game as follows. $S = \{Card_j \to \neg Card_i, r_j \vee b_j \vee w_j, Card_j \to B_j Card_j \mid i, j \in \{A, B, C\}, i \neq j$ and $Card \in \{r, b, w\}\}$: players A, B, C have a unique card and they 'know' their card. If we assume that $\varphi \in \mathsf{L}_s^{\otimes}$ for all $\varphi \in S$ then we can prove the following, for all epistemic formulas φ and $j \in \{A, B, C\}$.

1. $\vdash \varphi \otimes B_j r'_A \supset B_j r_A$: player j believes that player A has the red card after any event during which player j believed that player A showed her $r$ed card.
2. $\vdash w_B \otimes B_B r'_A \supset B_B b_C$: if player B has the $w$hite card then after an event where he believes that player A shows her $r$ed card, he believes that player C has the $b$lue card.
3. $\vdash (w_B \wedge \neg B_B r_A) \otimes (\neg B_B r'_A \wedge \neg B_B b'_A \wedge \neg B_B r'_C \wedge \neg B_B b'_C) \supset \neg B_B r_A$.

(1) is proved as follows. $\vdash \neg r_A \otimes r'_A \supset \bot$ because $\neg(r'_A \wedge \neg r_A) \in \mathsf{L}_s^{\otimes}$ and A_1. Then $\vdash \neg r_A \otimes r'_A \supset r_A$ by R_1. Besides, $\vdash r_A \otimes r'_A \supset r_A$ by A_2. So $\top \otimes r'_A \supset r_A$ by A_6. Then $B_j \top \otimes B_j r'_A \supset B_j r_A$ by A_5. But because $B_j \top \in \mathsf{L}_s^{\otimes}$, we have $\vdash \varphi \otimes B_j r'_A \supset B_j r_A$ for all epistemic formulas φ. ◁

3.3 Ontic Events

In (Aucher 2009a), the treatment of ontic events, that is events that change truth values of propositional facts during an update (as opposed to epistemic events), is inspired from Reiter's solution to the frame problem (Reiter 2001). We introduce a function $Post : \Phi \times \Phi' \to \mathcal{L}$. $Post(p, p')$ is a sufficient condition *before* the occurrence of p' for p to be true after the occurrence of p'. Just as for preconditions, we transfer this notion to the case of possible events:

$$Post(p, w') = \begin{cases} \bigvee\{Post(p, p') \mid M', w' \models p'\} \\ \text{if } M', w' \models p' \text{ for some } p' \in \Phi' \\ p \text{ otherwise.} \end{cases}$$

where (M', w') is a pointed $\mathcal{L}'$-model. Then the new valuation in Definition 26 is defined as follows:

$$V^{\otimes}(p) = \{(v, v') \in W^{\otimes} \mid M, v \models Post(p, v')\}.$$

One can easily show that this new definition of the product update is axiomatized by replacing axiom A_2 by the following three Axiom schemes:

$\mathsf{A}_{2.1}$ $\quad Post(p, p') \otimes p' \supset p$

$\mathsf{A}_{2.2}$ $\quad \varphi_p \otimes \delta_0^* \supset \varphi_p$ for all propositional formulae φ_p, where $\delta_0^* = \bigwedge\{\neg p' \mid p' \in \Phi'\}$

$\mathsf{A}_{2.3}$ $\quad \left(\bigwedge_{p' \in S'} \neg Post(p, p') \right) \otimes \delta'_0 \supset \neg p$ where $S' = \{p' \in \Phi' \mid \vdash \delta'_0 \to p'\}$

4 Conclusion

We have proposed a natural characterization of the standard BMS update by means of an axiomatization. Our general approach paves the way not only to a systematic study of other updates (such as the ones dealing with plausibility or probability) but also to the introduction of new updates: other axiomatizations than the one mentioned here could be proposed and given a corresponding semantics.

Our axiomatization is also obviously of interest for any artificial agent so that she can plan her behaviour and reason about effects of actions in an uncertain environment. However, note that we followed in this paper the external approach, representing situations from an external and omniscient point of view. But it would be more appropriate in this context to follow the internal approach (Aucher 2009b) and represent situations from the point of view of the artificial agent. This might yield a different axiomatization. Besides, artificial agents should also be able to reason about the effects of a sequence of events (for planing for instance).

So our language should be extended with constructs of the form $\varphi \otimes \varphi'_1 \otimes \ldots \otimes \varphi'_n \supset \varphi''$. We leave these issues for future work.

Finally, in this paper we were interested in the axiomatization of what we can infer about (3) given (1) and (2) (see abstract). Other natural questions that we could ask are: what can we infer about (2) given (1) and (3)? or what can we infer about (1) given (2) and (3)? Answering these questions might give us some insights on the notions of causality and abduction. We leave these axiomatizations for future work.

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